Abstract. This paper is concerned with the space $K_{w^*}(X^*, Y)$ of weak$^*$ to weak continuous compact operators from the dual space $X^*$ of a Banach space $X$ to a Banach space $Y$. We show that if $X^*$ or $Y^*$ has the Radon-Nikodým property, $C$ is a convex subset of $K_{w^*}(X^*, Y)$ with $0 \in C$ and $T$ is a bounded linear operator from $X^*$ into $Y$, then $T \in C_{\tau_c}$ if and only if $T \in \{S \in C : \|S\| \leq \|T\\}_\tau$, where $\tau_c$ is the topology of uniform convergence on each compact subset of $X$, moreover, if $T \in K_{w^*}(X^*, Y)$, here $C$ need not to contain 0, then $T \in C_{\tau_c}$ if and only if $T \in C$ in the topology of the operator norm. Some properties of $K_{w^*}(X^*, Y)$ are presented.

1. Introduction and the main result

Representations of dual spaces of operator spaces provide a useful tool to study approximation properties of operators. Grothendieck [8] established a representation of the dual space of $\mathcal{L}(X, Y)$, the space of bounded linear operators between Banach spaces $X$ and $Y$, when endowed with the topology $\tau_c$ of uniform convergence on each compact subset of $X$ and the representation was applied to study the approximation property. A Banach space $X$ is said to have the approximation property (AP) if the identity operator $id_X \in \mathcal{F}(X, X)^\tau_c$, where $\mathcal{F}(X, X)$ is the space of finite rank operators on $X$, and we say that $X$ has the metric approximation property (MAP) if $id_X \in \{T \in \mathcal{F}(X, X) : \|T\| \leq 1\}_\tau$. The AP is formally weaker than the MAP, in fact Figiel and Johnson [6] showed that the AP is strictly weaker than the MAP; more precisely, they constructed a separable Banach space having the AP but failing to have the MAP. Grothendieck [8] applied the representation of the dual space of $(\mathcal{L}(X, Y), \tau_c)$ to show that for separable dual spaces, the AP and MAP are equivalent. But it is a long-standing famous problem whether
the AP and MAP are equivalent for general dual spaces (cf. [2, Problem 3.8]). The main purpose of the paper is to establish an approximation theorem in $K_{w^*}(X^*, Y^*)$, the space of weak$^*$ to weak continuous compact operators from $X^*$ to $Y$, using the bidual space of $K_{w^*}(X^*, Y)$ endowed with the topology of the operator norm. The main result is originated from the following result of Godefroy and Saphar [7].

**Theorem 1.1** ([7, Theorem 1.5]). Suppose that $X^*$ or $Y^*$ has the Radon-Nikodým property. Let $C$ be a convex subset of $K_{w^*}(X^*, Y^*)$ and let $T \in \mathcal{L}(X^*, Y^*)$. Then $T \in \overline{C}_{\tau^c}$ if and only if for every $\varepsilon > 0$,

$$T \in \{S \in C : \|S\| \leq \|T\| + \varepsilon\}^c.$$

Note that the above mentioned passage from the AP to the MAP for separable dual spaces easily follows from Theorem 1.1. Recently, Choi and Kim [3] used a representation of the dual space of $K_{w^*}(X^*, Y)$, endowed with the topology of the operator norm, to obtain the following.

**Theorem 1.2** ([3, Theorem 2.3]). Suppose that $X^*$ or $Y^*$ has the Radon-Nikodým property. Let $Y$ be a subspace of $K_{w^*}(X^*, Y)$ and let $T \in \mathcal{L}(X^*, Y)$. Then $T \in \overline{Y}^c$ if and only if $T \in \{S \in Y : \|S\| \leq \|T\|\}^c$.

In this paper, we adjust arguments of Feder, Godefroy and Saphar ([5, Theorem 1], [7, Theorem 1.5]) to extend Theorem 1.1:

**Theorem 1.3.** Suppose that $X^*$ or $Y^*$ has the Radon-Nikodým property. Let $C$ be a convex subset of $K_{w^*}(X^*, Y)$ and let $T \in \mathcal{L}(X^*, Y)$. Then $T \in \overline{C}_{\tau^c}$ if and only if for every $\varepsilon > 0$, $T \in \{S \in C : \|S\| < \|T\| + \varepsilon\}^c$.

The following corollary extends Theorem 1.2.

**Corollary 1.4.** Suppose that $X^*$ or $Y^*$ has the Radon-Nikodým property. Let $C$ be a convex subset of $K_{w^*}(X^*, Y)$ with $0 \in C$ and let $T \in \mathcal{L}(X^*, Y)$. Then $T \in \overline{C}_{\tau^c}$ if and only if $T \in \{S \in C : \|S\| \leq \|T\|\}^c$.

**Proof.** Suppose $T \in \overline{C}_{\tau^c}$. Let $K$ be a compact subset of $X^*$ and let $\varepsilon > 0$. Choose $\delta > 0$ so that $(\delta/(\|T\| + \delta))$ sup$_{x^* \in K} \|Tx^*\| < \varepsilon/2$. Then by Theorem 1.3 there exists an $S \in \{S \in C : \|S\| < \|T\| + \delta\}$ such that sup$_{x^* \in K} \|Sx^* - Tx^*\| < \varepsilon/2$. Consider $(\|T\|/(\|T\| + \delta))S \in C$ with $\|(\|T\|/(\|T\| + \delta))S\| \leq \|T\|$. Then

$$\sup_{x^* \in K} \frac{\|T\|}{\|T\| + \delta} Sx^* - Tx^* \leq \frac{\|T\|}{\|T\| + \delta} \sup_{x^* \in K} \|Sx^* - Tx^*\| + \frac{\delta}{\|T\| + \delta} \sup_{x^* \in K} \|Tx^*\| < \varepsilon.

Hence $T \in \{S \in C : \|S\| \leq \|T\|\}^c$. □

We end the paper by a section collecting some results concerning the space $K_{w^*}(X^*, Y)$. First we give a simple characterization of elements in $K_{w^*}(X^*, Y)$.
[Proposition 3.1]. Then we describe \( \mathcal{K}_{w^*}(X^*, Y)^* \) in general and look at the particular case when \( X^* \) is separable. We end the section by showing how one can simplify the proof of a factorization result for \( \mathcal{K}_{w^*}(X^*, Y) \) from [1] and [14]. We use standard Banach space notation as can be found e.g. in [13].

2. A representation of the bidual space of \( \mathcal{K}_{w^*}(X^*, Y) \) and a proof of Theorem 1.3

Godefroy and Saphar [7, Proposition 1.1] established a representation of \( \mathcal{K}(X, Y)^{**} \) under the assumption that \( X^{**} \) or \( Y^{**} \) has the RNP. In this section, we adopt the factorization argument of Feder, Godefroy and Saphar [5, 7] to represent \( \mathcal{K}_{w^*}(X^*, Y)^{**} \) under the assumption that \( X^* \) or \( Y^* \) has the RNP, and then the representation will be a main tool of the proof of Theorem 1.3.

For Banach spaces \( Z \) and \( W \) we denote the projective and injective tensor product by \( Z \otimes_{\pi} W \) and \( Z \otimes_{\varepsilon} W \), respectively (cf. see [15, Chapters 2 and 3]). Recall that \( L(Z, W^*) \) is isometrically isomorphic to \( (Z \otimes_{\pi} W^*)^* \) and that for a net \( \{T_n\} \) in \( L(Z, W^*) \) and \( T \in L(Z, W^*) \)

\[
T_n \xrightarrow{w^*} T \quad \text{if and only if} \quad \sum_n (T_n z_n)(w_n) \to \sum_n (T z_n)(w_n)
\]

for every \( (z_n) \) in \( Z \) and \( (w_n) \) in \( W \) with \( \sum_n \|z_n\|\|w_n\| < \infty \) (see [15, p. 24]).

We now have:

**Theorem 2.1.** Suppose that \( X^* \) or \( Y^* \) has the Radon-Nikodým property. Then there exists a \( w^* \) to \( w^* \) homeomorphic linear isometry \( \Phi \) from \( \mathcal{K}^*_w(X^*, Y)^{w^*} \) (in \( \mathcal{L}(Y^*, X^{**}) \)) onto \( \mathcal{K}_{w^*}(X^*, Y)^{**} \) such that

\[
\Phi(\mathcal{K}^*_w(X^*, Y)) = j(\mathcal{K}_{w^*}(X^*, Y)),
\]

where \( \mathcal{K}^*_w(X^*, Y) = \{T^* : T \in \mathcal{K}_w(X^*, Y)\} \) and \( j : \mathcal{K}_{w^*}(X^*, Y) \to \mathcal{K}_{w^*}(X^*, Y)^{**} \) is the natural isometry.

**Proof.** Suppose that \( X^* \) has the Radon-Nikodym property. We define the map \( V : Y^* \otimes_{\pi} X^* \to \mathcal{K}_{w^*}(X^*, Y)^* \) by

\[
V_v(T) = \sum_n y_n^* (T x_n^*)
\]

for \( v = \sum_n y_n^* \otimes x_n^* \in Y^* \otimes_{\pi} X^* \). Then \( V \) is well defined, linear and \( \|V\| \leq 1 \).

First we use the proof of [5, Theorem 1] to show that \( V \) is a quotient map and so \( V^* \) is an isometry. Let the map \( i : Y \to l^\infty(B Y^*) \) be defined by \( i(y)(y^*) = y^*(y) \) for every \( y^* \in B Y^* \). Then \( i \) is an isometry and so the map \( J_1 : \mathcal{K}_{w^*}(X^*, Y) \to \mathcal{K}_{w^*}(X^*, l^\infty(B Y^*)) \) defined by \( J_1(T) = iT \) is an isometry. Since \( l^\infty(B Y^*) \) has the approximation property, \( \mathcal{K}_{w^*}(X^*, l^\infty(B Y^*)) \) is isometrically isomorphic to \( X \otimes_{\varepsilon} l^\infty(B Y^*) \) by the isometry \( J_2 \).

\[
\mathcal{K}_{w^*}(X^*, Y) \xrightarrow{J_1} \mathcal{K}_{w^*}(X^*, l^\infty(B Y^*)) \xrightarrow{J_2} X \otimes_{\varepsilon} l^\infty(B Y^*).
\]
Since $l^\infty(B_Y\cdot)^*$ has the approximation property and $X^*$ has the Radon-Nikodým property, $l^\infty(B_Y\cdot)^* \otimes_\pi X^*$ is isometrically isomorphic to $(X \otimes l^\infty(B_Y\cdot))^*$ by the isometry $J_3$ (see [15, Theorem 5.33]).

$$l^\infty(B_Y\cdot)^* \otimes_\pi X^* \xrightarrow{J_3} (X \otimes l^\infty(B_Y\cdot))^* \xrightarrow{(J_2J_1)^*} K_{w^*}(X^*, Y)^*.$$

Let $J = (J_2J_1)^*J_3$. We show that the following diagram is commutative:

\[
\begin{array}{ccc}
   l^\infty(B_Y\cdot)^* \otimes_\pi X^* & \xrightarrow{id_X^*} & Y^* \otimes_\pi X^* \\
   J \downarrow & & \downarrow V \\
   K_{w^*}(X^*, Y)^* & & 
\end{array}
\]

Let $\mu \in l^\infty(B_Y\cdot)^*$, $x^* \in X^*$ and $T \in K_{w^*}(X^*, Y)$. Then

$$J(\mu \otimes x^*)(T) = (J_2J_1)^*J_3(\mu \otimes x^*)(T) = J_3(\mu \otimes x^*)(J_2J_1(T)) = J_3(\mu \otimes x^*)(J_2(iT)) = \mu(iT x^*) = i^*(\mu)(T x^*) = V(i^*(\mu) \otimes x^*)(T) = V(i^* \otimes id_{X^*})(\mu \otimes x^*)(T).$$

It follows that the diagram is commutative. Now let $\varphi \in K_{w^*}(X^*, Y)^*$. Since $J_2J_1$ is an isometry, we see that there exists $u \in l^\infty(B_Y\cdot)^* \otimes_\pi X^*$ so that $J(u) = \varphi$ and $\|u\|_\pi = \|\varphi\|$. Let $v = i^* \otimes id_{X^*}(u)$. Then by the above diagram $\varphi = V(v)$ and we have

$$\|\varphi\| \leq \|V\|\|i^* \otimes id_{X^*}(u)\|_\pi \leq \|i^*\|\|id_{X^*}\|\|u\|_\pi \leq \|u\|_\pi = \|\varphi\|.$$ 

Thus $\|\varphi\| = \|v\|_\pi$ and so $V$ is a quotient map.

Now we use the proof of [7, Proposition 1.1]. Let the map $W : K_{w^*}(X^*, Y) \to \mathcal{L}(Y^*, X^{**})$ be defined by $W(T) = T^*$, let $i_1 : \mathcal{L}(Y^*, X^{**}) \to (Y^* \otimes_\pi X^*)^{**}$ be the isometry and let $i_2 : Y^* \otimes_\pi X^* \to (Y^* \otimes_\pi X^*)^{**}$ be the natural isometry.

\[
\begin{array}{ccc}
   Y^* \otimes_\pi X^* & \xrightarrow{V} & K_{w^*}(X^*, Y)^* \\
   i_2 \downarrow & & \downarrow W^* \\
   (Y^* \otimes_\pi X^*)^{**} & \xrightarrow{i_1^*} & \mathcal{L}(Y^*, X^{**}) \\
\end{array}
\]

Then for every $v = \sum_n y_n^* \otimes x_n^* \in Y^* \otimes_\pi X^*$ and $T \in K_{w^*}(X^*, Y)$,

$$W^*i_1i_2(v)(T) = i_2(v)i_1 W(T) = i_1 W(T)(v) = i_1(T^*)(v) = \sum_n (T^* y_n^*)(x_n^*) = \sum_n y_n^* (T x_n^*) = (Vv)(T).$$
Thus $W^*i_1^*i_2 = V$. Now consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{K}_{w^*}(X^*, Y) & \xrightarrow{W} & \mathcal{L}(Y^*, X^{**}) \\
\downarrow & & \downarrow \iota_{w^*}^{-1} \\
\mathcal{K}_{w^*}(X^*, Y)^{**} & \xrightarrow{v^*} & (Y^* \otimes \pi X^*)^* \\
\end{array}
\]

Let $i_3 : \mathcal{L}(Y^*, X^{**}) \to \mathcal{L}(Y^*, X^{**})^{**}$ be the natural isometry. Then for every $T \in \mathcal{K}_{w^*}(X^*, Y)$ and $v \in Y^* \otimes \pi X^*$,

\[
i_3(i_1^{-1}V^*j(T))(v) = (W^*i_1^*i_2)^*j(T)(v)
= (i_1^*i_2)^*W^*j(T)(v)
= W^*j(T)i_1^*i_2(v)
= i_1^*i_2(v)i_3^1W^*j(T)
= i_2(v)i_1(W(T))
= i_1(W(T))(v).
\]

Thus the above diagram is commutative and so $i_1^{-1}V^*j(T) = T^*$ for every $T \in \mathcal{K}_{w^*}(X^*, Y)$. Recall that, if the range of an adjoint operator is norm closed, then the range is $w^*$ closed. Thus we have

\[
i_1^{-1}V^*(\mathcal{K}_{w^*}(X^*, Y)^{**}) = i_1^{-1}V^*(j(\mathcal{K}_{w^*}(X^*, Y)))^{w^*} = \mathcal{K}_{w^*}(X^*, Y)^{w^*}.
\]

We have shown that $i_1^{-1}V^* : \mathcal{K}_{w^*}(X^*, Y)^{**} \to \mathcal{K}_{w^*}(X^*, Y)^{w^*}$ is a surjective linear isometry. Put

\[
\Phi = (i_1^{-1}V^*)^{-1} : \mathcal{K}_{w^*}(X^*, Y)^{w^*} \to \mathcal{K}_{w^*}(X^*, Y)^{**}.
\]

Note that if an adjoint operator is an isomorphism, then the inverse of this adjoint operator is $w^*$ to $w^*$ continuous on its range. Hence $\Phi$ is a $w^*$ to $w^*$ homemorphic linear isometry and for every $T \in \mathcal{K}_{w^*}(X^*, Y)$

\[
\Phi(T^*) = (i_1^{-1}V^*)^{-1}(T^*) = (i_1^{-1}V^*)^{-1}i_1^{-1}V^*j(T) = j(T).
\]

This completes the proof for the case that $X^*$ has the Radon-Nikodým property.

Now suppose that $Y^*$ has the Radon-Nikodým property. Define the map $\psi : \mathcal{L}(Y^*, X^{**}) \to \mathcal{L}(X^*, Y^{**})$ by $\psi(T) = T^*jX^*$. Then it is easy to check that $\psi$ is a surjective linear isometry with the inverse $\psi^{-1}(R) = R^*jY^*$. Let $(T_n)$ be a net in $\mathcal{L}(Y^*, X^{**})$ and $T \in \mathcal{L}(Y^*, X^{**})$ with $T_n \xrightarrow{w^*} T$. Let $v = \sum_n y_n^* \otimes x_n^* \in X^* \otimes \pi Y^*$. Since $\sum_n y_n^* \otimes x_n^* \in Y^* \otimes \pi X^*$, we have

\[
\sum_n (T_n y_n^*)(x_n^*) \rightarrow \sum_n (Ty_n^*)(x_n^*).
\]
Thus
\[ \psi(T_\alpha)(v) = \sum_n (T_\alpha j x \cdot x_n^\ast)(y_n^\ast) = \sum_n j x \cdot (x_n^\ast)(T_\alpha y_n^\ast) \]
\[ = \sum_n (T_\alpha y_n^\ast)(x_n^\ast) \quad \longrightarrow \quad \sum_n (Ty_n^\ast)(x_n^\ast) = \psi(T)(v). \]
Hence \( \psi \) is \( w^\ast \) to \( w^\ast \) continuous and, similarly, so is \( \psi^{-1} \). Let \( S \in K_{w^\ast}(X^*, Y) \) and let \( x^* \in X^* \) and \( y^* \in Y^* \). Then \( S^*(y^*) = j_X(x) \) for some \( x \in X \) and so we have
\[ \psi(S^*)(x^*)(y^*) = S^{**} j_X \cdot (x^*)(y^*) = S^*(y^*)(x^*) = j_X(x)(x^*) \]
\[ = x^*(x) = x^*(j_X^{-1}S^*(y^*)) = (j_X^{-1}S^*)(x^*)(y^*). \]
Thus \( \psi(S^*) = (j_X^{-1}S^*)^* \in K_{w^\ast}(Y^*, X) \). Similarly, for every \( U \in K_{w^\ast}(Y^*, X) \)
\( \psi^{-1}(U^*) = (j_Y^{-1}U^*)^* \in K_{w^\ast}(X^*, Y) \). Therefore \( \psi(K_{w^\ast}(X^*, Y)) = K_{w^\ast}(Y^*, X) \)
and so
\[ \psi(K_{w^\ast}(X^*, Y) \to K_{w^\ast}(Y^*, X) \to K_{w^\ast}(Y^*, X)^{**}) \]
Since \( Y^* \) has the Radon-Nikodým property, we can find the map
\[ \Psi : K_{w^\ast}(Y^*, X)^{**} \to K_{w^\ast}(Y^*, X)^{**} \]
in the first case. Define the map \( \phi : K_{w^\ast}(Y^*, X) \to K_{w^\ast}(X^*, Y) \) by \( \phi(T) = j_Y^{-1}T^* \). Then we see that \( \phi \) is a surjective linear isometry. Then \( \phi^{**} \) is a \( w^* \) to \( w^* \) homeomorphic isometry from \( K_{w^\ast}(Y^*, X)^{**} \) onto \( K_{w^\ast}(X^*, Y)^{**} \). Put
\[ \Phi = \phi^{**}\Psi \Psi : K_{w^\ast}(Y^*, X)^{**} \to K_{w^\ast}(X^*, Y)^{**}. \]
Then \( \Phi \) is a \( w^* \) to \( w^* \) homeomorphic and surjective linear isometry, and
\[ \Phi(K_{w^\ast}(X^*, Y)) = \phi^{**}\Psi(K_{w^\ast}(X^*, Y)) \]
\[ = \phi^{**}(j(K_{w^\ast}(Y^*, X))) = j(K_{w^\ast}(X^*, Y)). \]
\( \Box \)

**Remark 2.2.** Suppose that \( X^{**} \) or \( Y^* \) has the Radon-Nikodým property. Let \( i : \mathcal{K}(X, Y) \to K_{w^\ast}(Y^*, X^*) \) be the surjective linear isometry defined by \( i(T) = T^* \). Then \( i^{**} : \mathcal{K}(X, Y)^{**} \to K_{w^\ast}(Y^*, X^*)^{**} \) is a \( w^* \) to \( w^* \) homeomorphic and surjective isometry. We can find the map \( \Phi : K_{w^\ast}(Y^*, X^*)^{**} \to K_{w^\ast}(Y^*, X^*)^{**} \)
in Theorem 2.1. Here note that \( K_{w^\ast}(X^*, Y^*) = \{ T^{**} : T \in \mathcal{K}(X, Y) \} \). Hence \( \Phi^{-1}i^{**} : \mathcal{K}(X, Y)^{**} \to K_{w^\ast}(Y^*, X^*)^{**} \) is a \( w^* \) to \( w^* \) homeomorphic isometry and \( \Phi^{-1}i^{**}(j(\mathcal{K}(X, Y))) = K_{w^\ast}(X^*, Y^*) \). Consequently Theorem 2.1 extends [7, Proposition 1.1].

To show Theorem 1.3 we need the following simple but useful lemma which is contained in the proof of [7, Theorem 1.5]. For the sake of completeness we provide the concrete proof.
Lemma 2.3. Let $C$ be a convex subset of a Banach space $B$ and let $x^{**} \in B^{**}$. If $x^{**} \in \overline{j_B(C)}^{w^*}$ in $B^{**}$, then for every $\varepsilon > 0$,

$$x^{**} \in \{x \in j_B(C) : ||x|| < ||x^{**}|| + \varepsilon\}^{w^*}.$$ 

Moreover, if $x^{**} \in j_B(B)$ and $x^{**} \in \overline{j_B(C)}^{w^*}$, then $x^{**} \in \overline{j_B(C)}$ in the topology of the norm.

Proof. Let $\varepsilon > 0$ and let $U$ be a convex set in $w^*$ closed neighborhood of $x^{**}$. Then $U \cap j_B(C)$ is not empty. Define the map $\psi : B^{**} \oplus B^{**} \to B^{**}$ by $\psi(x_1^{**}, x_2^{**}) = x_1^{**} - x_2^{**}$. Then $\psi$ is clearly linear and $w^*$ to $w^*$ continuous. Put $V = U \cap j_B(C)$ and $W = \{x \in V : ||x|| < ||x^{**}|| + \varepsilon/2\}$. Note that $x^{**} \in V^{w^*}$ and $x^{**} \in W^{w^*}$ by Goldstine’s theorem. Thus

$$0 = x^{**} - x^{**} = \psi(x^{**}, x^{**}) \in \psi(V^{w^*} \times W^{w^*}) = \psi(V \times W^{w^*}) = \psi(V^{w^*} \times W^{w^*}).$$

Thus there exists a net $(x_\alpha, y_\alpha) \in V \times W$ so that $j_B(x_\alpha) - j_B(y_\alpha) \xrightarrow{w^*} 0$ in $B^{**}$ and so $x_\alpha - y_\alpha \xrightarrow{w^*} 0$ in $B$. Define the map $\tilde{\psi} : B \oplus B \to B$ by $\tilde{\psi}(x_1, x_2) = x_1 - x_2$. Then $\tilde{\psi}(x_\alpha, y_\alpha) \xrightarrow{w^*} 0$ in $B$ and so $0 \in \tilde{\psi}(j_B(V^{w^*}) \times j_B(W^{w^*})) = \tilde{\psi}(j_B(V) \times j_B(W))$ in the topology of the norm, for $\tilde{\psi}(j_B(V) \times j_B(W))$ is a convex set in $B$. Thus there exist $j_B(x_1) \in V$ and $j_B(x_2) \in W$ so that $||x_1 - x_2|| < \varepsilon/2$. Then $||x_1|| \leq ||x_2|| + ||x_1 - x_2|| < ||x^{**}|| + \varepsilon$. We have shown that $j_B(x_1) \in U \cap \{x \in j_B(C) : ||x|| < ||x^{**}|| + \varepsilon\}$. Hence $x^{**} \in \overline{j_B(C)}^{w^*}$. 

The remaining part follows from convexity of $C$ and that $j_B$ is $w^*$ homeomorphic from $B$ onto $j_B(B)$. \qed

Grothendieck [8] obtained that the dual space $(\mathcal{L}(X, Y), \tau_\omega)$ consists of all functionals $f$ of the form $f(T) = \sum_n y_n^*(T x_n)$, where $(x_n)$ in $X$, $(y_n^*)$ in $Y^*$, and $\sum_n ||x_n|| ||y_n^*|| < \infty$. The summable weak operator topology (swo) on $\mathcal{L}(X, Y)$ is the topology induced by $(\mathcal{L}(X, Y), \tau_\omega)^*$ (see [4]). Then, for a net $(T_n)$ in $\mathcal{L}(X, Y)$ and $T \in \mathcal{L}(X, Y)$, $T_n \xrightarrow{\text{aws}} T$ if and only if $\sum_n y_n^*(T_n x_n) \xrightarrow{\text{aws}} \sum_n y_n^*(T x_n)$ for every $(x_n)$ in $X$ and $(y_n^*)$ in $Y^*$ with $\sum_n ||x_n|| ||y_n^*|| < \infty$, and $\mathcal{C}^* = \mathcal{C}_w^{\omega^*}$ for every convex subset $C$ of $\mathcal{L}(X, Y)$ (cf. see [4, Proposition 3.6]).

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose $T \in \mathcal{C}^{\omega^*}$ and let $\varepsilon > 0$. By the above note there exists a net $(T_n)$ in $\mathcal{C}$ such that

$$\sum_n (T_n y_n^*)(x_n^*) = \sum_n y_n^*(T_n x_n^*) \xrightarrow{\text{aws}} \sum_n y_n^*(T x_n^*) = \sum_n (T y_n^*)(x_n^*)$$

for every $(x_n^*)$ in $X^*$ and $(y_n^*)$ in $Y^*$ with $\sum_n ||x_n^*|| ||y_n^*|| < \infty$. Thus $T^* \in \{S^* : S \in \mathcal{C}\}^{\omega^*}$ in $\mathcal{L}(Y^*, X^{**})$. Let $\Phi : \mathcal{K}_w^{\omega^*}(X^*, Y)^{\omega^*} \to \mathcal{K}_w^{\omega^*}(X^*, Y)^{\omega^*}$ be the
map in Theorem 2.1. Then $\Phi(T^*) \in \Phi(\{S^* : S \in C\})$ in $K_{w^*}(X^*, Y)^{**}$. Now by Lemma 2.3,
\[\Phi(T^*) \in \{\Phi(S^*) \in \Phi(\{S^* : S \in C\}) : ||S|| < ||\Phi(T^*)|| + \varepsilon\}\].
Thus there exists a net $(S_\beta)$ in $C$ so that $\Phi(S_\beta) \overset{w^*}{\to} \Phi(T^*)$ and $||S_\beta|| < ||T|| + \varepsilon$ for every $\beta$. Then $S_\beta \overset{w^*}{\to} T^*$ in $\mathcal{L}(Y^*, X^{**})$, which is equivalent to $S_\beta \overset{w^*}{\to} T$ in $\mathcal{L}(X^*, Y)$. Hence, by the above note,
\[T \in \{S \in C : ||S|| < ||T|| + \varepsilon\}\].

**Corollary 2.4.** Suppose that $X^*$ or $Y^*$ has the Radon-Nikodým property. Let $C$ be a convex subset of $K_{w^*}(X^*, Y)$ and let $T \in K_{w^*}(X^*, Y)$. Then $T \in \mathcal{C}$ if and only if $T \in \mathcal{C}$ in the topology of the operator norm.

**Proof.** If $T \in \mathcal{C}$, then by the proof of Theorem 1.3, $\Phi(T^*) \in \Phi(\{S^* : S \in C\})$ in $K_{w^*}(X^*, Y)^{**}$. Then $\Phi(T^*) \in \mathcal{C}$ if and only if $T \in \mathcal{C}$ in the topology of the operator norm. A Banach space $X$ is reflexive if and only if the space $K(X^*, Y)$ of compact operators and $K_{w^*}(X^*, Y)$ are the same. Indeed, if $X$ is nonreflexive, then there exists an $x_0^* \in X^{**}$ so that $x_0^*$ is not a $w^*$ continuous linear functional. Then the operator $x_0^* \overset{w^*}{\to} Y^*$ for every $y \in Y$ but $x_0^* \overset{w^*}{\to} \mathcal{C}$ in $K(X^*, Y)$. Also $\mathcal{K}(X^*)$ is isometrically isomorphic to $K_{w^*}(X^{**}, Y)$ by the map $T \mapsto j_Y^{-1}T^*$. The $bw^*$ topology is strictly stronger than the $w^*$ topology (cf. see [13, Corollary 2.7.7]). For $T \in \mathcal{L}(X^*, Y)$, $T$ is $w^*$ to $w$ continuous if and only if $T$ is $bw^*$ to $w$ continuous. Indeed, if $T$ is $bw^*$ to $w$ continuous, then for every net $(x_\alpha)$ in $X^*$ and $x^* \in X^*$ with $x_\alpha \overset{bw^*}{\to} x^*$
\[(T^* y)x_\alpha = y^*(Tx_\alpha) \to y^*(Tx) = (T^* y)x^*\]
for every $y^* \in Y^*$, which shows $T^{**}y^* \in (X^*, bw^*)^*$. Since $(X^*, bw^*)^* = (X^*, w^*)^*$ (see [13, Theorem 2.7.8]), $T^{**}(Y^*) \subset j_Y(X)$. Hence $T$ is $w^*$ to $w$ continuous because $T$ is $w^*$ to $w$ continuous if and only if $T^{**}(Y^*) \subset j_X(X)$.

We now establish some criteria of $w^*$ to $w$ continuous compact operators.

**Proposition 3.1.** For $T \in \mathcal{L}(X^*, Y)$ the following assertions are equivalent.
(a) $T$ is $bw^*$ to norm continuous.
(b) $T$ is $w^*$ to $w$ continuous compact.
(c) $T$ is $bw^*$ to $w$ continuous.
(d) $Tx_\alpha \overset{norm}{\to} Tx^*$ whenever $x_\alpha \overset{w^*}{\to} x^*$ in $B_{X^*}$.
Proof. From the above note we only need to show (a)⇒(c)⇒(d)⇒(a).

(a)⇒(c) Let \((x^*_\alpha)\) be a net in \(B_{X^*}\). Then there exists a subnet \((x^*_\beta)\) of \((x^*_\alpha)\) and \(x^* \in B_{X^*}\), so that \(x^*_\beta \overset{w^*}{\to} x^*\) because the \(bw^*\) and \(w^*\) topology are the same on \(B_{X^*}\) (see [13, Theorem 2.7.2]) and \(B_{X^*}\) is \(w^*\) compact. Thus by the assumption (a)
\[
T x^*_\beta \overset{\text{norm}}{\to} T x^*,
\]
which shows that \(T(B_{X^*})\) is norm compact in \(Y\). Hence \(T\) is \(bw^*\) to \(w\) continuous compact.

(c)⇒(d) Let \((x^*_\alpha)\) be a net in \(B_{X^*}\), and \(x^*_\epsilon \in B_{X^*}\) with \(x^*_\alpha \overset{w^*}{\to} x^*_\epsilon\). Then \(x^*_\alpha \overset{bw^*}{\to} x^*\) and so \(T x^*_\alpha \overset{w}{\to} T x^*\) by the assumption (c). Since the norm closure \(\bar{T}(B_{X^*})\) in \(Y\) is norm compact, the norm and \(w\) topology are the same on \(\bar{T}(B_{X^*})\). Hence
\[
T x^*_\alpha \overset{\text{norm}}{\to} T x^*.
\]

(d)⇒(a) If \(T x^*_\alpha \overset{\text{norm}}{\to} T x^*\) whenever \(x^*_\alpha \overset{w^*}{\to} x^*\) in \(B_{X^*}\), then \(T x^*_\alpha \overset{\text{norm}}{\to} T x^*\) whenever \(t > 0\) and \(x^*_\alpha \overset{w^*}{\to} x^*\) in \(tB_{X^*}\). Therefore \(T\) is \(w^*\) to norm continuous with respect to the relative \(w^*\) topology of \(tB_{X^*}\) whenever \(t > 0\). Let \(V\) be a norm open set in \(Y\). Then for every \(t > 0\), \(T^{-1}(V) \cap tB_{X^*}\) is a relatively \(w^*\) open set in \(tB_{X^*}\). By [13, Corollary 2.7.4] \(T^{-1}(V)\) is a \(bw^*\) open set in \(X^*\). Hence \(T\) is \(bw^*\) to norm continuous.

Now we summarize some results for the space \(K_{w^*}(X^*, Y)\). First, we comment on the dual space of \(K_{w^*}(X^*, Y)\) (see P. Harmand, D. Werner and W. Werner [9, pp. 265, 266]). We say that a linear functional \(\varphi\) on \(K_{w^*}(X^*, Y)\) is an integral linear functional if there exists a regular Borel measure \(\mu\) on \(B_{X^*} \times B_{Y^*}\), where \(B_{X^*}\) and \(B_{Y^*}\) are equipped with the \(w^*\) topology, so that
\[
\varphi(T) = \int_{B_{X^*} \times B_{Y^*}} y^*(T x^*) d\mu
\]
for all \(T \in K_{w^*}(X^*, Y)\). We denote the space of integral functionals on \(K_{w^*}(X^*, Y)\) by \(\mathcal{I}_{w^*}\) and define the norm on \(\mathcal{I}_{w^*}\) by
\[
\|\varphi\|_1 = \inf \{\|\mu\| : \mu\text{ represents }\varphi\}.
\]

Let \(C(B_{X^*} \times B_{Y^*})\) be the Banach space of scalar valued continuous functions on \(B_{X^*} \times B_{Y^*}\). Our first application of Proposition 3.1 is for the proof of the following well-known and very useful observation.

Lemma 3.2. \(K_{w^*}(X^*, Y)\) is isometrically isomorphic to a subspace of \(C(B_{X^*} \times B_{Y^*})\).

Proof. We consider the map \(\Lambda : K_{w^*}(X^*, Y) \to C(B_{X^*} \times B_{Y^*})\) defined by
\[
\Lambda(T)(x^*, y^*) = y^*(T x^*).
\]
From Proposition 3.1(d), it is easy to check that \(\Lambda(T) \in C(B_{X^*} \times B_{Y^*})\) for all \(T \in K_{w^*}(X^*, Y)\) and \(\Lambda\) is a linear isometry. Hence the conclusion follows. \(\square\)
We are now ready to represent the dual space of $\mathcal{K}_{w^*}(X^*, Y)$.

**Theorem 3.3.** $\mathcal{K}_{w^*}(X^*, Y)^*$ is isometrically isomorphic to $\mathcal{I}_{w^*}$.

**Proof.** If $\psi \in \mathcal{K}_{w^*}(X^*, Y)^*$, then by Lemma 3.2, Hahn-Banach extension and Riesz representation theorem, there exists a regular Borel measure $\mu$ on $B_{X^*} \times B_{Y^*}$ such that

$$
\psi(T) = \int_{B_{X^*} \times B_{Y^*}} y^*(Tx^*)d\mu
$$

for all $T \in \mathcal{K}_{w^*}(X^*, Y)$ and $\|\psi\| = \|\mu\|$ and so $\|\psi\| \geq \|\psi\|_I$. Also for every such representation $\nu$ of $\psi$, we see $\|\psi\| \leq \|\nu\|$. Hence $\|\psi\| = \|\psi\|_I$. Since for every $\varphi \in \mathcal{I}_{w^*}$ clearly $\varphi \in \mathcal{K}_{w^*}(X^*, Y)^*$, the conclusion follows. □

**Remark 3.4.** Under the assumption that $X^*$ or $Y^*$ has the Radon-Nikodým property, elements of $\mathcal{K}_{w^*}(X^*, Y)^*$ can be represented by a series form, more precisely, for every $\varphi \in \mathcal{K}_{w^*}(X^*, Y)^*$ and $\varepsilon > 0$ there exist $(x_i^*)$ in $X^*$ and $(y_i^*)$ in $Y^*$ with $\sum_i \|x_i^*\||y_i^*\| < \|\varphi\| + \varepsilon$ such that $\varphi(T) = \sum_i y_i^*(Tx_i^*)$ for all $T \in \mathcal{K}_{w^*}(X^*, Y)$. Indeed, if $X^*$ or $Y^*$ has the Radon-Nikodým property, then the map $V : Y^* \otimes \pi X^* \to \mathcal{K}_{w^*}(X^*, Y)^*$, in the proof of Theorem 2.1, is a quotient map. Thus for every $\varphi \in \mathcal{K}_{w^*}(X^*, Y)^*$ there exists $v = \sum_i y_i^* \otimes x_i^* \in Y^* \otimes \pi X^*$ with $\|v\|_x = \|\varphi\|$ such that $\varphi(T) = \sum_i y_i^*(Tx_i^*)$ for all $T \in \mathcal{K}_{w^*}(X^*, Y)$. Another proof of this was presented in [3, Theorem 1.2].

We need the following lemma to obtain a more concrete representation of $\mathcal{K}_{w^*}(X^*, Y)^*$ than the one in Remark 3.4 when $X^*$ is separable.

**Lemma 3.5** ([12, Lemma 1.e.16]). Let $X$ be a separable Banach space and $\varepsilon > 0$. Then there exists a sequence $(f_i)_{i=1}^\infty$ of functions on $B_X$ so that $x = \sum_{i=1}^\infty f_i(x)$, for every $x \in B_X$, each $f_i(x)$ is of the form $\sum_{j=1}^\infty \chi_{E_{i,j}}(x)x_{i,j}$, where $\{E_{i,j}\}_{j=1}^\infty$ are disjoint Borel subsets of $B_X$, $\{x_{i,j}\}_{j=1}^\infty \subset B_X$ and

$$
\sum_{i=1}^\infty \|f_i\|_\infty < 1 + \varepsilon \text{ with } \|f_i\|_\infty = \sup_x \|f_i(x)\| = \sup_j \|x_{i,j}\|.
$$

We now have:

**Corollary 3.6.** Suppose that $X^*$ is separable. If $\varphi \in \mathcal{K}_{w^*}(X^*, Y)^*$, then for $\varepsilon > 0$ there exist sequences $(x_{i,j}^*)$ in $X^*$ and $(y_{i,j}^*)$ in $Y^*$ with $\sum_{i,j}^\infty \sup_i \|x_{i,j}^*\| < 1 + \varepsilon$ and $\sum_{j=1}^\infty \|y_{i,j}^*\| \leq \|\varphi\|$ for every $i$ so that

$$
\varphi(T) = \sum_{i=1}^\infty \sum_{j=1}^\infty y_{i,j}^*(Tx_{i,j}^*)
$$

for every $T \in \mathcal{K}_{w^*}(X^*, Y)$.

**Proof.** Let $\varphi \in \mathcal{K}_{w^*}(X^*, Y)^*$ and let $\varepsilon > 0$. Then by Theorem 3.3 there exists a regular Borel measure $\mu$ on $B_{X^*} \times B_{Y^*}$ with $\|\varphi\| = \|\mu\|$ so that

$$
\varphi(T) = \int_{B_{X^*} \times B_{Y^*}} y^*(Tx^*)d\mu
$$
for every $T \in \mathcal{K}_{\text{w}^*}(X^*, Y)$. Then by Lemma 3.5, for every $T \in \mathcal{K}_{\text{w}^*}(X^*, Y)$,

$$
\varphi(T) = \int_{B_{X^*} \times B_Y} y^*T\left(\sum_{i=1}^{\infty} f_i(x^*)\right) d\mu
$$

$$
= \sum_{i=1}^{\infty} \int_{B_{X^*} \times B_Y} y^* T\left(\sum_{j=1}^{\infty} \chi_{E_{i,j}}(x^*)x^*_{i,j}\right) d\mu
$$

$$
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{E_{i,j} \times B_Y} y^* (Tx^*_{i,j}) d\mu
$$

$$
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} y^*_{i,j}(Tx^*_{i,j}),
$$

where $y^*_{i,j}$ is the functional on $Y$ defined by $y^*_{i,j}(y) = \int_{E_{i,j} \times B_Y} y^*(y) d\mu$. Since for every $i$, $j$, and $y \in B_Y$, $|y^*_{i,j}(y)| \leq \int_{E_{i,j} \times B_Y} |y^*(y)| d|\mu| \leq |\mu|(E_{i,j} \times B_Y)$, $|y^*_{i,j}| \leq |\mu|(E_{i,j} \times B_Y)$ for every $i$ and $j$. Hence for every $i$, we have $\sum_{j=1}^{\infty} ||y^*_{i,j}|| \leq ||\mu|| = ||\varphi||$ and $\sum_{i=1}^{\infty} \sup_{j} ||x^*_{i,j}|| < 1 + \varepsilon$. \hfill \Box

Next we present a variant of a result of Kalton [10]. Recall the weak operator topology (wo) on $\mathcal{L}(X, Y)$. For a net $(T_\alpha)$ in $\mathcal{L}(X, Y)$ and $T \in \mathcal{L}(X, Y)$, $T_\alpha \xrightarrow{\text{wo}} T$ if and only if $y^*(T_\alpha x) \xrightarrow{\text{wo}} y^* (Tx)$ for every $x \in X$ and $y^* \in Y^*$. The following are the $\mathcal{K}_{\text{w}^*}(X^*, Y)$ versions of [10, Theorem 1] and [10, Corollary 3], respectively.

**Proposition 3.7.** Let $\mathcal{A}$ be a subset of $\mathcal{K}_{\text{w}^*}(X^*, Y)$. Then $\mathcal{A}$ is w compact if and only if $\mathcal{A}$ is weakly compact.

**Corollary 3.8.** Let $(T_\alpha)$ be a sequence in $\mathcal{K}_{\text{w}^*}(X^*, Y)$ and $T \in \mathcal{K}_{\text{w}^*}(X^*, Y)$. Then $T_\alpha \xrightarrow{\text{wo}} T$ if and only if $T_\alpha \xrightarrow{\text{we}} T$.

Finally we consider a factorization of elements in $\mathcal{K}_{\text{w}^*}(X^*, Y)$.

**Lemma 3.9** ([11, Lemma 1.1 and Theorem 2.2]). If $T \in \mathcal{K}(X, Y)$, then there exist a separable reflexive Banach space $Z$ with $T(B_X)/\|T\| \subset B_Z \subset B_Y$, $S \in \mathcal{K}(X, Z)$, and the inclusion map $J \in \mathcal{K}(Z, Y)$ such that $\|J\| = 1$, $T = JS$, and $\|S\| = \|T\|$.

The following theorem is essentially contained in Aron, Lindström, Ruess, Ryan [1], and Mikko, Oja [14]. But we use Proposition 3.1 to slightly simplify the existing proof.

**Proposition 3.10.** If $T \in \mathcal{K}_{\text{w}^*}(X^*, Y)$, then there exist a separable reflexive Banach space $Z$, $R \in \mathcal{K}_{\text{w}^*}(X^*, Z^{**})$ with $\|R\| = \|T\|$, $U \in \mathcal{K}_{\text{w}^*}(Z^{**}, Y)$ with $\|U\| = 1$ such that $T = UR$.

**Proof.** Let $T \in \mathcal{K}_{\text{w}^*}(X^*, Y)$. Then by Lemma 3.9, there exist a separable reflexive Banach space $Z$ with $T(B_X)/\|T\| \subset B_Z \subset B_Y$, $S \in \mathcal{K}(X^*, Z)$, and
the inclusion map \( J \in \mathcal{K}(Z,Y) \) such that \( \| J \| = 1 \), \( T = JS \), and \( \| S \| = \| T \| \). Let \( R = jzS \in \mathcal{K}(X^*, Z^{**}) \) and \( U = J^{-1}jzS \in \mathcal{K}(Z^{**}, Y) \). Then \( \| R \| = \| T \| \), \( \| U \| = 1 \), and \( T = UR \). If \( (x^*_\alpha) \) in \( B_{X^*} \), and \( x^* \in B_{X^*} \) with \( x^*_\alpha \xrightarrow{w^*} x^* \), then by Proposition 3.1(d)

\[
T x^*_\alpha \xrightarrow{\| \cdot \|_Y} T x^*.
\]

Since \( (T x^*_\alpha/\| T \|) \) and \( T x^*/\| T \| \) in \( T(B_{X^*})/\| T \| \), by [11, Lemma 2.1(ii)]

\[
T x^*_\alpha/\| T \| \xrightarrow{\| \cdot \|_Y} T x^*/\| T \|.
\]

Consequently \( T x^*_\alpha \|_Y \xrightarrow{\| \cdot \|} T x^* \) and so \( Sx^*_\alpha \xrightarrow{\| \cdot \|_Y} Sx^* \) because \( Sx^* =Tx^* \) for all \( x^* \in X^* \) (see [11, Theorem 2.2]). Therefore

\[
Rx^*_\alpha = jzSx^*_\alpha \xrightarrow{\| \cdot \|_Y} jzSx^* = Rx^*.
\]

Hence \( R \in \mathcal{K}_{w^*}(X^*, Z^{**}) \) by Proposition 3.1(d). Since \( Z \) is reflexive, \( U \in \mathcal{K}_{w^*}(Z^{**}, Y). \)

\( \square \)

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References

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