CONGRUENCES OF THE WEIERSTRASS $\wp(x)$ AND $\wp''(x)(x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2})$-FUNCTIONS ON DIVISORS

DAEYEOL KIM, AERAN KIM, AND HWASIN PARK

Abstract. In this paper, we find the coefficients for the Weierstrass $\wp(x)$ and $\wp''(x)(x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2})$-functions in terms of the arithmetic identities appearing in divisor functions which are proved by Ramanujan ([23]). Finally, we reprove congruences for the functions $\mu(n)$ and $\nu(n)$ in Hahn’s article [11, Theorems 6.1 and 6.2].

1. Introduction

In a series of articles [18] Liouville stated many identities for general functions satisfying certain parity conditions. When specialized these yield results of number-theoretic interest. The Liouville identities are equivalent to identities among elliptic functions. In this article we considered the Weierstrass $\wp$-functions and identities of the basic hypergeometric series. Let $\sigma_s(N)$ denote the sum of $s$th power of the positive divisors of $N$, and let $\sigma_s(0) = \frac{1}{2}\zeta(-s)$, where $\zeta(s)$ is the Riemann Zeta-function. Ramanujan ([23]) wrote several formulas for

$$
\sigma_r(0)\sigma_s(N) + \sigma_r(1)\sigma_s(N - 1) + \cdots + \sigma_r(N)\sigma_s(0).
$$

Some of these convolution sums involving divisor functions had been considered earlier by Glaisher [8], [9], MacMahon [20, pp. 303–341], Melfi [21], Huard, Ou, Spearman and Williams [12], etc.

For $N, m, r, s, d \in \mathbb{Z}$ with $d, s > 0$ and $r \geq 0$, we define some necessary divisor functions and infinite products for later use, which appear in many areas of number theory:

$$
\sigma_{s,r}(N; m) = \sum_{d | N \text{ and } r \mod m} d^s,
$$

$$
\sigma(N) := \sigma_1(N) = \sum_{d | N} d,
$$

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\[ S_1 := \sum_{N \text{ odd}} \sigma_{1,1}(N;2)q^N, \]
\[ S_2 := \sum_{N \geq 2 \text{ even}} \sigma_{1,1}(N;2)q^N, \]
\[(a; q)_\infty := (a)_\infty := \prod_{n \geq 0} (1 - aq^n).\]

In Section 2, we state the coefficients for \( \wp(\tau^2) \), \( \wp(\tau^2 + 1) \) and \( \wp(1/2) \) dealing with the summation of odd divisors and even divisors. They permit us to obtain \( \Delta(\tau) \) from \( g_2(\tau) \) and \( g_3(\tau) \) and to get the differences between roots.

In Section 3, it will be shown that the derivatives of the Weierstrass \( \wp \)-functions have the infinite \( q \)-series. We can retrieve the actual values of the coefficients belonging to \( q \)-series. Also, we introduce the tables about the coefficients of \( \wp'(x) \) for \( x = 1/2, \tau, \tau^2, \tau^3 \). Using the function \( p_r(n) \) by
\[
\sum_{n=0}^{\infty} p_r(n)q^n := \prod_{n=1}^{\infty} (1 - q^n)^r.
\]

Note that \( p_{-1}(n) = p(n) \), the ordinary partition function. A positive integer \( n \) has \( k \) colors if there are \( k \) copies of \( n \) available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called colored partitions. For example, if 1 is allowed to have 2 colors, say \( r \) (red), and \( g \) (green), then all colored partitions of 2 are \( 2, 1_r + 1_g, 1_r + 1_g, 1_r + 1_g \). Setting \( p_{e,r}(n) \) and \( p_{o,r}(n) \) denote the number of \( r \)-colored partitions into an even (respectively, odd) number of distinct parts, it is easy to see that
\[
p_r(n) = p_{e,r}(n) - p_{o,r}(n),
\]
when \( r \) is a positive integer. In [11, Theorems 6.1 and 6.2], Hahn considered congruences for the function \( \mu(n) \) and \( \nu(n) \) which was defined by
\[
\sum_{n=1}^{\infty} \mu(n)q^n := \prod_{n=1}^{\infty} (1 - q^n)^8(1 - q^{2n})^8 \quad \text{and} \quad \sum_{n=0}^{\infty} \nu(n)q^n := \prod_{n=1}^{\infty} (1 - q^{2n})^8(1 + q^n)^8.\]

Using the function \( \wp''(\frac{\tau}{2}, \tau) \) and \( \wp''(\frac{\tau}{2}, \tau) \times (\wp(\tau) - \wp(\frac{\tau}{2}))^2 \), we prove
\[
\nu(n - 1) \equiv \sigma_3(n) \pmod{3} \quad \text{and} \quad \mu(3n - 1) \equiv 0 \pmod{3}.
\]
(Please see Remarks 3.9 and 3.12).

2. Divisor functions

N. J. Fine’s list of identities of the basic hypergeometric series type appeared in [7]. While studying these identities, we found that some identities appeared more than once on the list, usually in similar form (see [3], [4]). In this section,
we state two identities that appeared in [7, pp. 78–79]:

\[
(2) \quad \frac{(q^2; q^4)_{\infty}^8}{(q; q^4)_{\infty}^8} = 1 + 8 \sum_{N=1}^{\infty} q^N (2 + (-1)^N) \sum_{\omega | N \omega \text{ odd}} \omega,
\]

\[
(3) \quad \frac{q(q^4; q^4)_{\infty}^8}{(q^2; q^2)_{\infty}^8} = \sum_{N \text{ odd}} \sigma(N) q^N.
\]

Let \( \Lambda_\tau = \mathbb{Z} + \tau \mathbb{Z} (\tau \in \mathfrak{H}) \) the complex upper half plane be a lattice and \( z \in \mathbb{C} \). The Weierstrass \( \wp \) function relative to \( \Lambda_\tau \) is defined by the series

\[
\wp(z; \Lambda_\tau) = \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau, \omega \neq 0} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),
\]

and the Eisenstein series of weight 2\( k \) for \( \Lambda_\tau \) with \( k > 1 \) is the series

\[
G_{2k}(\Lambda_\tau) = \sum_{\omega \in \Lambda_\tau, \omega \neq 0} \omega^{-2k}.
\]

We used the notations \( \wp(z) \) and \( G_{2k} \) instead of \( \wp(z; \Lambda_\tau) \) and \( G_{2k}(\Lambda_\tau) \), respectively, when the lattice \( \Lambda_\tau \) has been fixed. Now, Laurent series for \( \wp(z) \) about \( z = 0 \) is given by

\[
\wp(z) = z^{-2} + \sum_{k=1}^{\infty} (2k + 1) G_{2k+2} z^{2k}.
\]

As is customary, by setting

\[
g_2(\tau) = g_2(\Lambda_\tau) = 60 G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140 G_6,
\]

the algebraic relation between \( \wp(z) \) and \( \wp'(z) \) becomes

\[
\wp'(z)^2 = 4 \wp(z)^3 - g_2(\tau) \wp(z) - g_3(\tau).
\]

**Proposition 2.1** ([15, p. 251]). Let \( e_1 = \wp(\frac{1}{2}), \ e_2 = \wp(\frac{1}{3}) \) and \( e_3 = \wp(\frac{\tau + 1}{2}) \), where \( P_0 = \prod_{n=1}^{\infty} (1 - q^{2n}), \ P_1 = \prod_{n=1}^{\infty} (1 - q^{2n-1}), \ P_2 = \prod_{n=1}^{\infty} (1 + q^{2n}) \) and \( P_3 = \prod_{n=1}^{\infty} (1 + q^{2n-1}) \). Then,

\[\begin{align*}
(a) \quad e_2 - e_1 &= \pi^2 P_0^4 P_3^8, \\
(b) \quad e_2 - e_3 &= \pi^2 P_0^4 P_1^8, \\
(c) \quad e_3 - e_1 &= 2^4 \pi^2 q P_0^4 P_2^8.
\end{align*}\]

Let us recall

\[
\prod_{n=1}^{\infty} (1 - q^{2n-1}) = \prod_{n=1}^{\infty} \left( \frac{1 - q^n}{1 - q^{2n}} \right),
\]

\[
\prod_{n=1}^{\infty} (1 + q^{2n-1}) = \prod_{n=1}^{\infty} \left( \frac{1 - q^{2n}}{1 - q^n} \right) \left( \frac{1 - q^{2n}}{1 - q^{2n}} \right).
\]
\[
\prod_{n=1}^{\infty} (1 + q^{2n}) = \prod_{n=1}^{\infty} \frac{(1 - q^{4n})}{(1 - q^{2n})}.
\]

Equations (2), (3) and (4) suggest that

\[
\wp(\tau/2) = -\frac{\pi^2}{3} \left( \frac{(q^2; q^2)_\infty^{20}}{(q^8; q^8)_\infty} + 16 \frac{q(q^4; q^4)_\infty^{8}}{(q^2; q^2)_\infty^{4}} \right)
\]

\[
= -\frac{\pi^2}{3} \left( 1 + 8 \sum_{N=1}^{\infty} q^N (2 + (-1)^N) \sum_{\omega | N \text{ odd}} \omega + 16 \sum_{N \text{ odd}} \sigma(N) q^N \right)
\]

\[
= -\frac{\pi^2}{3} \left( 1 + 24 \sum_{N=1}^{\infty} q^N \sum_{\omega | N \text{ odd}} \omega \right)
\]

\[
= -\frac{\pi^2}{3} \left( 1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) q^N \right)
\]

\[
= -\frac{\pi^2}{3} (1 + 24S_1 + 24S_2).
\]

Similarly, the relations (2) and (3) yield the following arithmetic results [13], [14]:

\[
\wp(\tau + 1/2) = -\frac{\pi^2}{3} \left( \frac{(q^2; q^2)_\infty^{20}}{(q^8; q^8)_\infty} - 32 \frac{q(q^4; q^4)_\infty^{8}}{(q^2; q^2)_\infty^{4}} \right)
\]

\[
= -\frac{\pi^2}{3} (1 - 24S_1 + 24S_2),
\]

\[
\wp(1/2) = 2\frac{\pi^2}{3} \left( \frac{(q^2; q^2)_\infty^{20}}{(q^8; q^8)_\infty} - 8 \frac{q(q^4; q^4)_\infty^{8}}{(q^2; q^2)_\infty^{4}} \right)
\]

\[
= 2\frac{\pi^2}{3} (1 + 24S_2),
\]

\[
g_2(\tau) = 4\frac{\pi^4}{9} [3(1 + 24S_2)^2 + 24^2 S_1^2],
\]

and

\[
g_3(\tau) = \frac{8\pi^6}{27} [(1 + 24S_2)^3 - 24^3 S_1^3 (1 + 24S_2)].
\]

We consider the formula for the modular discriminant \(\Delta(\tau) = (2\pi)^{12} \eta(\tau)^{24} = g_2(\tau)^3 - 27g_3(\tau)^2\), where the Dedekind \(\eta\)-function is given by the infinite
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The product $\eta(\tau) = q^{\frac{1}{24}}(q^2; q^2)_{\infty}$ ([26]). From (8) and (9) we see that

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$$

$$= \left\{ \frac{4\pi^4}{9} [3(1 + 24S_2)^2 + 24^2S_1^2] \right\}^3 - 27 \left\{ \frac{8\pi^6}{27} [(1 + 24S_2)^3 - 24^2S_1^2(1 + 24S_2)] \right\}^2$$

$$= 4096\pi^4S_1^2(-1 + 8S_1 - 24S_2)^2(1 + 8S_1 + 24S_2)^2.$$  

Calculating the differences between roots, writing them in terms of infinite sums, and using (5), (6) and (7), we get

(11)  
$$\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau}{2}\right) := \pi^2 \sum_{n=0}^{\infty} a_n q^n$$

$$= \frac{2\pi^2}{3} (1 + 24S_2) + \frac{\pi^2}{3} (1 + 24S_1 + 24S_2)$$

$$= \pi^2 \left( 1 + 8 \sum_{N \text{ odd}} \sigma_{1,1}(N; 2)q^N + 24 \sum_{N \geq 2 \text{ even}} \sigma_{1,1}(N; 2)q^N \right),$$

(12)  
$$\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau + 1}{2}\right) := \pi^2 \sum_{n=0}^{\infty} b_n q^n$$

$$= \frac{2\pi^2}{3} (1 + 24S_2) + \frac{\pi^2}{3} (1 - 24S_1 + 24S_2)$$

$$= \pi^2 \left( 1 - 8 \sum_{N \text{ odd}} \sigma_{1,1}(N; 2)q^N + 24 \sum_{N \geq 2 \text{ even}} \sigma_{1,1}(N; 2)q^N \right),$$

and

(13)  
$$\wp\left(\frac{\tau + 1}{2}\right) - \wp\left(\frac{\tau}{2}\right) := \pi^2 \sum_{n=0}^{\infty} c_n q^n$$

$$= -\frac{\pi^2}{3} (1 - 24S_1 + 24S_2) + \frac{\pi^2}{3} (1 + 24S_1 + 24S_2)$$

$$= \pi^2 \sum_{N \text{ odd}} 16\sigma_{1,1}(N; 2)q^N.$$  

If we define

$$H(q) := \frac{1}{16} \left( \prod_{n=1}^{\infty} \left( \frac{1 - q^n}{1 + q^n} \right)^8 - 1 \right) := \sum_{n=0}^{\infty} h(n)q^n,$$

then by

(14)  
$$\prod_{n=1}^{\infty} \left( \frac{1 - q^n}{1 + q^n} \right)^8 = 1 + 16 \sum_{N \geq 1} q^N \sum_{d | N} (-1)^d d^3$$
as in [7, p. 77], we find that

\[
H(q) = \sum_{N \geq 1} (\sigma_{3,0}(N; 2) - \sigma_{3,1}(N; 2))q^N.
\]

Thus, we can obtain the table below for the coefficients appearing in the infinite sum with positive integers for \(0 \leq N \leq 51\).

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Therefore, we summarize the above results obtained from (11) to (14) as follows.

**Proposition 2.2.** Let \( n \) be non-negative integers and let \( \varphi(\frac{1}{2}) - \varphi(\frac{\tau}{2}) := \pi^2 \sum_{n=0}^\infty a_n q^n \), \( \varphi(\frac{1}{2}) - \varphi(\frac{\tau+1}{2}) := \pi^2 \sum_{n=0}^\infty b_n q^n \), \( \varphi(\frac{\tau}{2}) - \varphi(\frac{\tau+1}{2}) := \pi^2 \sum_{n=0}^\infty c_n q^n \) and \( H(q) := \frac{1}{16} \left( \prod_{n=1}^\infty \left( \frac{1-q^n}{1+q^n} \right)^8 - 1 \right) := \sum_{n=0}^\infty h(n) q^n \). The following assertions hold:

(a) \( a_n - b_n = c_n \).
(b) \( a_{2n-1} = -b_{2n-1} \) and \( a_{2n} = b_{2n} \) and \( c_{2n} = 0 \).

(c) \( h(0) = 0 \) and \( h(n) = \sigma_{3,1}(n; 2) - \sigma_{3,0}(n; 2) \) \((n > 0)\).

By (11), (12), (13), (15) and Proposition 2.2, we see the following result.

**Corollary 2.3.** Let \( p \) be any odd positive integer and \( n > 0 \).

(a) \( a_{2^n} = 24 \) and \( a_{2^np} = 24(p + 1) \).

(b) \( a_p = 8(p + 1) \) and \( c_p = 16(p + 1) \).

Ramanujan’s theta functions \( \varphi(q) \), \( \psi(q) \) and \( f(-q) \) [1, Entry 22, p. 36] are defined, for \(|q| < 1\), by

\[
\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; q^2)_{\infty}(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}(-q^2; q^2)_{\infty}}
\]

\[
\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}
\]

and

\[
f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} = (q; q)_{\infty}.
\]

Here, the product representations arise from the Jacobi triple product identity.

From (16), (17), (18), Proposition 2.1 and Proposition 2.2, we can deduce the following.

**Remark 2.4.**

(a) \( e_2 - e_1 = \pi^2 \varphi^4(q) = \pi^2 \sum_{n=0} a_n q^n \).

(b) \( e_2 - e_3 = \pi^2 \varphi^4(-q) = \pi^2 \sum_{n=0} b_n q^n \).

Let us introduce the triangular numbers. It is immediate from the definitions of \( \psi(q) \) and \( \varphi(q) \) in (16) and (17), respectively, that if

\[
\varphi^s(q) := \sum_{n=0}^{\infty} r_s(n) q^n
\]

and

\[
\phi^s(q) := \sum_{n=0}^{\infty} \delta_s(n) q^n,
\]

then \( r_s(n) \) and \( \delta_s(n) \) are the number of representations of \( n \) as a sum of \( s \) square and \( s \) triangular numbers, respectively. Clearly, \( r_s(0) = \delta_s(0) = 1 \). Here, for each nonnegative integer \( n \), the triangular number \( T_n \) is defined by

\[
T_n := \frac{n(n + 1)}{2}.
\]
Proposition 2.5. In [11, Theorem 5.1(5.3)], for each positive integer \( n \), we have
\[
r_4(n) = 16\hat{\sigma}(\frac{n}{2}) + 8\hat{\sigma}(n),
\]
with \( \hat{\sigma}(n) = \sum_{d|n}(-1)^{\frac{n}{d}-1}d^n \).

Combining (11) with Remark 2.4(a) and applying (16) to (15), we can get the following.

Corollary 2.6. If \( r_4(n), r_8(n) \) are defined by (19), then
(a) \[
r_4(n) = \begin{cases} 8\sigma_1(n;2) & n \text{ odd} \\ 24\sigma_1(n;2) & n \text{ even} \end{cases}
\]
(b) \[
r_8(n) = \begin{cases} 16\sigma_3(n;2) & n \text{ odd} \\ 16(\sigma_3(n;2) - \sigma_3(n;2)) & n \text{ even} \end{cases}
\]
which is also described in [11, p. 17].

3. The coefficients of \( \wp'' \)-functions

Formula (1) is appeared in several articles ([2], [6, p. 338], [9], [10], [16, p. 678], [17, p. 106], [19, p. 81], [22, p. 146], [23], [24, p. 115], and [27]). Using (5), (6), (7), and (8), we can obtain
\[
5\sigma_3(M) = \sigma_1,1(M;2) + 12\sum_{k=1}^{M-1}\sigma_1,1(k;2)\sigma_1,1(M-k;2)
\]
\[
+ 4\sum_{k=1}^{M-1}\sigma_1,1(2(k-1);2)\sigma_1,1(2M-2k+1;2)
\]
(21)
(see [14]).

Glaisher [5, p. 300] and Ramanujan [23] proved that
\[
\sigma(1)\sigma(2n-1) + \sigma(3)\sigma(2n-3) + \cdots + \sigma(2n-1)\sigma(1) = \frac{1}{8}[\sigma_3(2n) - \sigma_3(n)].
\]
(22)

From (21) and (22), we find that
\[
5\sum_{k=1}^{M-1}\sigma_1,1(k;2)\sigma_1,1(M-k;2) = \frac{1}{24}[11\sigma_3(M) - \sigma_3(2M) - 2\sigma_1,1(M;2)]
\]
(23)
(see [14]).

In this section, we will find the coefficients of the Weierstrass \( \wp''(z) \)-functions using (23). Recall ([25], p. 63) that we have expressed \( \wp''(\frac{1}{7}, \tau), \wp''(\frac{5}{7}, \tau) \) and \( \wp''(\frac{3}{7}, \tau) \).
We see from (5), (6) and (7) that
\[
\wp''(\frac{\tau}{2}, \tau) = 2(e_1 - e_2)(e_1 - e_3)
\]
\[
= 2\left[-\frac{\pi^2}{3}(1 + 24S_1 + 24S_2) - \frac{2\pi^2}{3}(1 + 24S_2)\right]
\times \left[-\frac{\pi^2}{3}(1 + 24S_1 + 24S_2) + \frac{\pi^2}{3}(1 - 24S_1 + 24S_2)\right]
\]
\[
= 32\pi^4 S_1(1 + 8S_1 + 24S_2)
\]
\[
(24)
\]
\[
= 32\pi^4 \sum_{M=1}^{\infty} \sigma_{1,1}(2M - 1; 2)q^{2M-1}
\]
\[
+ 256\pi^4 \sum_{M,N=1}^{\infty} \sigma_{1,1}(2M - 1; 2)\sigma_{1,1}(2N - 1; 2)q^{2(M+N-1)}
\]
\[
+ 768\pi^4 \sum_{M,N=1}^{\infty} \sigma_{1,1}(2M - 1; 2)\sigma_{1,1}(2N; 2)q^{2(M+N)-1}.
\]

Then, using (23), we replace \( n \) by \( n = 2L - 1 \) to obtain
\[
\sum_{k=1}^{L-1} \sigma_{1,1}(2k - 1)\sigma_{1,1}(2L - 1 - (2k - 1))
\]
\[
= \frac{1}{2} \sum_{k=1}^{2L-2} \sigma_{1,1}(k)\sigma_{1,1}(2L - 1 - k)
\]
\[
= \frac{1}{48}(11\sigma_3(2L - 1) - \sigma_3(4L - 2) - 2\sigma_{1,1}(2L - 1; 2))
\]
\[
= \frac{1}{24}(\sigma_3(2L - 1) - \sigma(2L - 1)).
\]

We observe that \( \sigma_3 \) is multiplicative, that is,
\[
11\sigma_3(2L+1) - \sigma_3(4L+2) = 11\sigma_3(2L+1) - \sigma_3(2L+1) = 2\sigma_3(2L+1).
\]

Comparing (23), (22), (26) with (24), we have
\[
\wp''(\frac{\tau}{2}, \tau) = 32\pi^4 \left\{ \sum_{L=1}^{\infty} \sigma_3(2L - 1)q^{2L-1} + \sum_{K=1}^{\infty} [\sigma_3(2K) - \sigma_3(K)]q^{2K} \right\}
\]
\[
(27)
\]
From (2), (3) and (27) it is easy to see that
\[
\wp''(\frac{\tau}{2}, \tau) = 32\pi^4 q\left(\frac{q^2; q^2}{(q)^8}\right)_{\infty}
\]
\[
= 32\pi^4 \left(\sum_{n=1}^{\infty} \sigma_3(n)q^n - \sum_{n=1}^{\infty} \sigma_3(n)q^{2n}\right).
\]

Summarizing the above results, obtained from (23) to (28) are as follow.
Theorem 3.1. Let \( \varphi''(\frac{1}{2}, \tau) := \pi^4 \sum_{n=0}^\infty d_n q^n \) and let \( 2n = 2^l Q \) be an integer with \( Q \) odd.

(a) \( \varphi''(\frac{1}{2}, \tau) = 32 \pi^4 q^{\frac{(\tau^2 - \tau^2)^{16}}{9q^0}} \).
(b) \( d_0 = 0 \).
(c) \( d_{2n-1} = 32 \sigma_3(2n - 1) \).
(d) \( d_{2l}Q = 32 \cdot 8^l \sigma_3(Q) \).

And let us evaluate \( \varphi''(1, \tau) = 2(e_2 - e_1)(e_2 - e_3) \)

\[
\begin{align*}
&= 2 \left[ \frac{2\pi^2}{3} (1 + 24S_2) + \frac{\pi^2}{3} (1 + 24S_1 + 24S_2) \right] \\
&\times \left[ \frac{2\pi^2}{3} (1 + 24S_2) + \frac{\pi^2}{3} (1 - 24S_1 + 24S_2) \right] \\
&= 2\pi^4 \left[ -64 \left( \sum_{M=1}^\infty \sigma_{1,1}(2M - 1; 2)q^{2M-1} \right) \left( \sum_{N=1}^\infty \sigma_{1,1}(2N - 1; 2)q^{2N-1} \right) \\
&+ \left( 1 + 24 \sum_{L=1}^\infty \sigma_{1,1}(2L; 2)q^{2L} \right) \left( 1 + 24 \sum_{K=1}^\infty \sigma_{1,1}(2K; 2)q^{2K} \right) \right] \\
&= 2\pi^4 + 96\pi^4 \sum_{L=1}^\infty \sigma_{1,1}(2L; 2)q^{2L} \\
&\quad - 128\pi^4 \sum_{M,N=1}^\infty \sigma_{1,1}(2M - 1; 2)\sigma_{1,1}(2N - 1; 2)q^{2(M+N-1)} \\
&\quad + 1152\pi^4 \sum_{K,L=1}^\infty \sigma_{1,1}(L; 2)\sigma_{1,1}(K; 2)q^{2(L+K)}.
\end{align*}
\]

Similarly, we can use the Glaisher’s proof (22) for the term

\[
\sum_{M,N=1}^\infty \sigma_{1,1}(2M - 1; 2)\sigma_{1,1}(2N - 1; 2)
\]

and (23) with \( M = N + 1 \) for the term

\[
\sum_{K,L=1}^\infty \sigma_{1,1}(L; 2)\sigma_{1,1}(K; 2).
\]

At last, we get

\[
\varphi''(\frac{1}{2}, \tau) = 2\pi^4 + \pi^4 \sum_{N=1}^\infty [544\sigma_3(N) - 64\sigma_3(2N)]q^{2N}.
\]

Theorem 3.2. Let \( \varphi''(\frac{1}{2}, \tau) := \pi^4 \sum_{n=0}^\infty e_n q^n \) and let \( 2n = 2^l Q \) be an integer with \( Q \) odd.
Let \( Q \) with \( \mathcal{P} \).

Corollary 3.4. Let \( N \) be any non-negative integer.

(a) If \( N \equiv 2 \) (mod 4), then \( e_N < 0 \) and \( e_N \equiv 0 \) (mod 32).
Corollary 3.5. (a) If $N = 2k - 1$, then $d_N = -f_N$.

(b) If $N \equiv 0 \pmod{4}$, then $e_N > 0$ and $e_N \equiv 0 \pmod{32}$, where $N > 1$.

Proof. In (29), if $N = 2k - 1$ with $k \in \mathbb{N}$, then the coefficients of $q^{2(2k-1)}$ becomes like this:

$$e_{4k-2} = 544\sigma_3(2k-1) - 64\sigma_3(2(2k-1))$$

$$= 544\sigma_3(2k-1) - 64\sigma_3(2)\sigma_3(2k-1)$$

$$= -32\sigma_3(2k-1).$$

So, $e_{4k-2}$ always has a negative sign and $e_{4k-2} \equiv 0 \pmod{32}$.

But for $N = 2k$, the coefficients of $q^{2(2k)}$ is $544\sigma_3(2k) - 64\sigma_3(2(2k))$. Let $k = 2^r\sigma Q$ with $r_1 \geq 0$ and $Q$ be odd. Then,

$$544\sigma_3(2k) - 64\sigma_3(2(2k)) = 544\sigma_3(2^{r_1+1}Q) - 64\sigma_3(2^{r_1+2}Q)$$

$$= [544\sigma_3(2^{r_1+1}) - 64\sigma_3(2^{r_1+2})]\sigma_3(Q)$$

$$= 32[-2 \cdot 8^{r_1+2} + 15(8^{r_1+1} + 8^{r_1} + \cdots + 1)]\sigma_3(Q)$$

$$= 32 \cdot \frac{8^{r_1+2} - 15}{7}\sigma_3(Q).$$

Since $r_1 \geq 0$, so $544\sigma_3(2k) - 64\sigma_3(2(2k)) > 0$. And $544\sigma_3(2k) - 64\sigma_3(2(2k)) = 32[-2 \cdot 8^{r_1+2} + 15(8^{r_1+1} + 8^{r_1} + \cdots + 1)]\sigma_3(Q)$ shows that $e_{4k} \equiv 0 \pmod{32}$.

Now, let us investigate the relation of coefficients of $\varphi''(\frac{1}{2}, \tau)$ and $\varphi''(\frac{2^k-1}{2}, \tau)$ in (27) and (30), respectively.

Theorem 3.6. Let $(\varphi(\frac{1}{2}) - \varphi(\frac{x}{2}))^2 := \pi^4 \sum_{n=0}^\infty \alpha_n q^n$, $(\varphi(\frac{1}{4}) - \varphi(\frac{x+1}{4}))^2 := \pi^4 \sum_{n=0}^\infty \beta_n q^n$ and $(\varphi(\frac{2^k-1}{2}) - \varphi(\frac{x}{2}))^2 := \pi^4 \sum_{n=0}^\infty \gamma_n q^n$. Then we get the following.

(a)

$$\alpha_n = \begin{cases} 1 & n = 0 \\ 16\sigma_3(n) & n \text{ odd} \\ 256\sigma_3(\frac{n}{2}) - 16\sigma_3(n) & n \text{ even.} \end{cases}$$

(b)

$$\beta_n = \begin{cases} 1 & n = 0 \\ -16\sigma_3(n) & n \text{ odd} \\ 256\sigma_3(\frac{n}{2}) - 16\sigma_3(n) & n \text{ even.} \end{cases}$$
\[
\gamma_n = \begin{cases} 
32\sigma_3(n) - 32\sigma_3(\frac{n}{2}) & \text{n even} \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** (a) It follows from (11) that

\[
(\psi(1) - \psi(\frac{r}{2}))^2 = \pi^2 \left( 1 + 8 \sum_{N \text{ odd}} \sigma_1(N;2)q^N + 24 \sum_{N \geq 2 \text{ even}} \sigma_1(N;2)q^N \right)^2
\]

\[
= \pi^4 \left( 1 + 8 \sum_{n=1}^{\infty} \sigma_1(2n - 1;2)q^{2n-1} + 24 \sum_{k=1}^{\infty} \sigma_1(2k;2)q^{2k} \right)
\]

\[
N \quad \frac{dN}{N} \quad \frac{\sigma_3(N)}{N} \quad \frac{\sigma_1(N)}{N} \\
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\times \left( 1 + 8 \sum_{m=1}^{\infty} \sigma_{1,1}(2m - 1; 2)q^{2m-1} + 24 \sum_{l=1}^{\infty} \sigma_{1,1}(2l; 2)q^{2l} \right)
\]
\[
= \pi^4 \left[ 1 + 16 \sum_{n=1}^{\infty} \sigma_{1,1}(2n - 1; 2)q^{2n-1} + 48 \sum_{k=1}^{\infty} \sigma_{1,1}(2k; 2)q^{2k} \\
+ 64 \sum_{n,m=1}^{\infty} \sigma_{1,1}(2n - 1; 2)\sigma_{1,1}(2m - 1; 2)q^{2(n+m-1)} \\
+ 384 \sum_{k,m=1}^{\infty} \sigma_{1,1}(2k; 2)\sigma_{1,1}(2m - 1; 2)q^{2(k+m)-1} \\
+ 576 \sum_{k,l=1}^{\infty} \sigma_{1,1}(2k; 2)\sigma_{1,1}(2l; 2)q^{2(k+l)} \right].
\]

It follows from (22) that
\[
\sum_{n,m=1}^{\infty} \sigma_{1,1}(2n - 1; 2)\sigma_{1,1}(2m - 1; 2)q^{2(n+m-1)} \\
= \sum_{M=1}^{\infty} \sum_{k=1}^{M} \sigma_{1,1}(2k - 1; 2)\sigma_{1,1}(2(M - k) + 1; 2)q^{2M} \\
= \sum_{M=1}^{\infty} \frac{1}{8}[\sigma_3(2M) - \sigma_3(M)]q^{2M},
\]
from (25) that
\[
\sum_{k,m=1}^{\infty} \sigma_{1,1}(2k; 2)\sigma_{1,1}(2m - 1; 2)q^{2(k+m)-1} \\
= \sum_{M=1}^{\infty} \sum_{n=1}^{M} \sigma_{1,1}(2n; 2)\sigma_{1,1}(2(M - n) + 1; 2)q^{2M+1} \\
= \sum_{M=1}^{\infty} \frac{1}{24}[\sigma_3(2M + 1) - \sigma_3(2M + 1)]q^{2M+1},
\]
and from (23) that
\[
\sum_{k,l=1}^{\infty} \sigma_{1,1}(2k; 2)\sigma_{1,1}(2l; 2)q^{2(k+l)} \\
= \sum_{k,l=1}^{\infty} \sigma_{1,1}(k; 2)\sigma_{1,1}(l; 2)q^{2(k+l)} \\
= \sum_{M=1}^{\infty} \sum_{n=1}^{M} \sigma_{1,1}(n; 2)\sigma_{1,1}(M + 1 - n; 2)q^{2(M+1)}
\]
CONGRUENCES OF $\wp(x)$ AND $\wp'(x) = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$-FUNCTIONS ON DIVISORS

$$\frac{1}{24} \sum_{M=1}^{\infty} [11\sigma_3(M+1) - \sigma_3(2M+1) - 2\sigma_{1,1}(M+1; 2)]q^{2(M+1)},$$

and thus we can obtain the desired result.

(b) In a similar manner like (a).

(c) Using (13) and (22), we find, upon direct calculation, that

$$\left( \wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{\tau}{2}\right) \right)^2 = \left( \pi^2 \sum_{N \text{ odd}} 16\sigma_{1,1}(N; 2)q^N \right)^2$$

$$= 256\pi^4 \sum_{n,m=1}^{\infty} \sigma_{1,1}(2n-1; 2)\sigma_{1,1}(2m-1; 2)q^{2(n+m-1)}. \square$$

Thus, let us make a table about $\frac{1}{16}\alpha_N$, $\frac{1}{16}\beta_N$ and $\frac{1}{16}\gamma_N$ for $0 \leq N \leq 51$.

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Corollary 3.7. Let $N$ be any non-negative integer. Then
(a) \(d_{2N-1} = 2\alpha_{2N-1}\).

(b) \(f_{2N} = \gamma_{2N}\).

Hahn (see [11]) proved a congruence for the function \(\nu(n)\) which is defined by

\[
(31) \quad \sum_{n=0}^{\infty} \nu(n)q^n := \prod_{n=1}^{\infty} (1 - q^{2n})^8(1 + q^n)^8.
\]

Thus \(\nu(n)\) is the number of partitions of \(n\) into 16 colors, 8 appear at most once (say \(S_1\)), and 8 are even and appear at most once (say \(S_2\)), weighted by the parity of colors from the set \(S_2\).

**Proposition 3.8** ([11]). If \(\nu(n)\) is defined by (31), then

\[
\nu(n-1) \equiv \tilde{\sigma}_3(n) \pmod{3}.
\]

**Remark 3.9.** By (27) we see that

\[
\psi''(\frac{T}{2}, \tau) = 32\pi^4 \left\{ \sum_{L=1}^{\infty} \sigma_3(2L-1)q^{2L-1} + \sum_{K=1}^{\infty} [\sigma_3(2K) - \sigma_3(K)]q^{2K} \right\}
\]

and

\[
\nu(n-1) = \begin{cases} 
\sigma_3(n) & \text{if } n \text{ odd}, \\
\sigma_3(n) - \sigma_3(\frac{n}{2}) & \text{otherwise}.
\end{cases}
\]

If \(n\) is odd, then \(\nu(n-1) = \sigma_3(n) = \tilde{\sigma}_3(n)\). If \(2n\) is even, then

\[
\sigma_3(2n) - \sigma_3(n) - \tilde{\sigma}_3(2n) = \sigma_3(2n) - \sigma_3(n) - \sigma_3(2n) + 2^4\sigma_3(n) = 15\sigma_3(n).
\]

Therefore, \(\sigma_3(2n) - \sigma_3(n) = \nu(2n-1) \equiv \tilde{\sigma}_3(2n) \pmod{3}\).

We can reprove Proposition 3.8.

We prove a congruence for the function \(\mu(n)\) [11, 6.3] which is defined by

\[
(32) \quad \sum_{n=0}^{\infty} \mu(n)q^n := \prod_{n=1}^{\infty} (1 - q^n)^8(1 - q^{2n})^8.
\]

It follows that

\[
\mu(n) = \mu_e(n) - \mu_o(n),
\]

where \(\mu_e(n)\) and \(\mu_o(n)\) are the number of 16-colored partitions into an even (respectively, odd) number of distinct parts, where all the parts of the latter eight colors are even.

Next, we can also retrieve \(\mu(3n-1) \equiv 0 \pmod{3}\) shown in [11].

**Theorem 3.10.** Let \(\sum_{n=0}^{\infty} \mu(n)q^n := \prod_{n=1}^{\infty} (1 - q^n)^8(1 - q^{2n})^8\) in (32). Then

(a) \(\mu(2n-1) \equiv \sigma_1(2n-1) \pmod{6}\).

(b) \(\mu(2n) \equiv (2n+1)[\sigma_1(2n) + \sigma_1(n)] \pmod{6}\).
Proof. By (27) and Theorem 3.6(a), we can obtain:

\[ \psi''(\frac{a}{2}) \times (e_2 - e_1)^2 \]

\[ = 32\pi^4 \left\{ \sum_{L=1}^{\infty} \sigma_3(2L - 1)q^{2L-1} + \sum_{K=1}^{\infty} [\sigma_3(2K) - \sigma_3(K)]q^{2K} \right\} \]

\[ \times \pi^4 \left\{ 1 + 16 \sum_{M=1}^{\infty} \sigma_3(2M - 1)q^{2M-1} + 16 \sum_{M=1}^{\infty} [16\sigma_3(M) - \sigma_3(2M)]q^{2M} \right\} \]

\[ = 32\pi^8 \left\{ \sum_{L=1}^{\infty} \sigma_3(2L - 1)q^{2L-1} + 240 \sum_{K,M=1}^{\infty} \sigma_3(2K - 1)\sigma_3(M)q^{2(L+M)-1} \right. \]

\[ + 16 \sum_{L,M=1}^{\infty} \sigma_3(2L - 1)\sigma_3(2M - 1)q^{2(L+M-1)} + \sum_{K=1}^{\infty} [\sigma_3(2K) - \sigma_3(K)]q^{2K} \]

\[ + \sum_{K,M=1}^{\infty} [272\sigma_3(2K)\sigma_3(M) - 16\sigma_3(2K)\sigma_3(2M) - 256\sigma_3(K)\sigma_3(M)]q^{2(M+K)} \right\} \].

(a) Let us pay attention to \( q^{N-1} \) in (33):

\[ \sum_{L=1}^{\infty} \sigma_3(2L - 1)q^{2L-1} + 240 \sum_{K,M=1}^{\infty} \sigma_3(2K - 1)\sigma_3(M)q^{2(L+M)-1} \]

\[ \equiv \sum_{L=1}^{\infty} \sigma_3(2L - 1)q^{2L-1} \pmod{6}. \]

By the definition of \( \mu(n) \), it means that \( \mu(2n - 1) \equiv \sigma_1(2n - 1) \pmod{6} \).

(b) Now, let us consider \( q^{2N} \) in (33). Since \( \sigma_3(M) \equiv \sigma_1(M) \pmod{6} \), the term with \( q^{2N} \) can be changed like this;

\[ S := 16 \sum_{L,M=1}^{\infty} \sigma_3(2L - 1)\sigma_3(2M - 1)q^{2(L+M-1)} + \sum_{K=1}^{\infty} [\sigma_3(2K) - \sigma_3(K)]q^{2K} \]

\[ + \sum_{K,M=1}^{\infty} [272\sigma_3(2K)\sigma_3(M) - 16\sigma_3(2K)\sigma_3(2M) - 256\sigma_3(K)\sigma_3(M)]q^{2(M+K)} \]

\[ \equiv 16 \sum_{L,M=1}^{\infty} \sigma_1(2L - 1)\sigma_1(2M - 1)q^{2(L+M-1)} + \sum_{K=1}^{\infty} [\sigma_1(2K) - \sigma_1(K)]q^{2K} \]

\[ + \sum_{K,M=1}^{\infty} [272\sigma_1(2K)\sigma_1(M) - 16\sigma_1(2K)\sigma_1(2M) - 256\sigma_1(K)\sigma_1(M)]q^{2(M+K)} \pmod{6}. \]
Then, by (22)
\[16 \sum_{L,M=1}^{\infty} \sigma_1(2L-1)\sigma_1(2M-1)q^{2(L+M-1)} = \sum_{K=1}^{\infty} 2[\sigma_3(2K) - \sigma_3(K)]q^{2K},\]
and by [12, (4.4)],
\[272 \sum_{K,M=1}^{\infty} \sigma_1(2K)\sigma_1(M)q^{2(M+K)}
= 272 \sum_{K=1}^{\infty} \sum_{l=1}^{K} \sigma_1(l)\sigma_1(2(K-l+1))q^{2(K+1)}
= \frac{34}{3} \sum_{K=1}^{\infty} [2\sigma_3(2(K+1)) - (6K+5)\sigma_1(2(K+1))
+ 8\sigma_3(K+1) - (12K+11)\sigma_1(K+1)]q^{2(K+1)}.
\]
Also by [12, (3.10)]
\[-256 \sum_{K,M=1}^{\infty} \sigma_1(K)\sigma_1(M)q^{2(M+K)}
= -256 \sum_{K=1}^{\infty} \sum_{l=1}^{K} \sigma_1(l)\sigma_1(K+1-l)q^{2(K+1)}
= \sum_{K=1}^{\infty} -\frac{64}{3}[5\sigma_3(K+1) - (6K+5)\sigma_1(K+1)]q^{2(K+1)}.
\]
Lastly, we get
\[-16 \sum_{K,M=1}^{\infty} \sigma_1(2K)\sigma_1(2M)q^{2(M+K)}
= -16 \sum_{K=1}^{\infty} \sum_{n=1}^{K} \sigma_1(2n)\sigma_1(2(K-n+1))q^{2(K+1)}.
\]
Since
\[\sum_{n=1}^{K} \sigma_1(2n)\sigma_1(2(K-n+1))
= \sum_{n=1}^{2K+1} \sigma_1(n)\sigma_1(2(K+1) - n) - \sum_{n=1}^{K+1} \sigma_1(2n-1)\sigma_1(2(K+1) - 2n + 1)
= \frac{1}{12}[5\sigma_3(2(K+1)) - (12K+11)\sigma_1(2(K+1)]
- \frac{1}{8}[\sigma_3(2(K+1)) - \sigma_3(K+1)],\]
we have
\[ -16 \sum_{K,M=1}^{\infty} \sigma_1(2K)\sigma_1(2M)q^{2(M+K)} \]
\[ = \sum_{K=1}^{\infty} \left\{ \frac{-4}{3}[5\sigma_3(2(K+1)) - 12K\sigma_1(2(K+1)) - 11\sigma_1(2(K+1))] \right. \]
\[ + 2\sigma_3(2(K+1)) - 2\sigma_3(K+1) \right\} q^{2(K+1)}. \]

Again applying \( \sigma_3(M) \equiv \sigma_1(M) \pmod{6} \) to the results of the above calculations, we can ultimately get (35) like this:
\[ S \equiv \sum_{K=1}^{\infty} \left[-21\sigma_1(2(K+1)) - 52K\sigma_1(2(K+1)) \right. \]
\[ - 39\sigma_1(K+1) + 8K\sigma_1(K+1)\] \( q^{2(K+1)} \pmod{6} \).

Then, we claim that
\[ \mu(2n) \equiv (2n+1)[\sigma_1(2n) + \sigma_1(n)] \pmod{6} \]
for \( n = K + 1 \).

\[ \square \]

**Corollary 3.11.** If \( \mu(n) \) is defined by (32), then
\[ \mu(3n - 1) \equiv 0 \pmod{6}. \]

**Proof.** Let us consider the odd and even cases in Theorem 3.10 for \( 3n - 1 \).
From (34), \( \mu(6n + 5) \equiv \sigma_3(6n + 5) \equiv \sigma_1(6n + 5) \pmod{6} \). Let \( 6n + 5 = p_1^{r_1}p_2^{r_2} \cdots q_r^{r_r} \cdot 2^{s_1} \sigma_1 \cdot 2^{s_2} \cdots 2^{s_s} \), with distinct primes \( p_1 \equiv p_2 \equiv \cdots \equiv p_r \equiv 1 \pmod{6} \) and \( q_1 \equiv q_2 \equiv \cdots \equiv q_r \equiv -1 \pmod{6} \). Because of \( 6n + 5 \), we have \( f_1 + f_2 + \cdots + f_r \equiv 1 \pmod{2} \).

Without loss of generality, suppose that \( f_1 \equiv 1 \pmod{2} \). Then,
\[ \sigma_1(6n + 5) = \sigma_1(p_1)^{r_1}\sigma_1(p_2)^{r_2} \cdots \sigma_1(q_r)^{r_r}\sigma_1(2^{s_1})\sigma_1(2^{s_2}) \cdots \sigma_1(2^{s_s}) \]
\[ \equiv 0 \pmod{6}, \]

since \( 1 + q_1 + q_1^2 + \cdots + q_1^{f_1} \equiv 0 \pmod{6} \). Thus, \( \mu(6n + 5) \equiv 0 \pmod{6} \).

On the other hand, from (36) we evaluate that \( \mu(6n + 2) \equiv (6n + 3)[\sigma_1(2(3n+1)) + \sigma_1(3n+1)] \pmod{6} \).
Let \( 3n + 1 = 2^rQ \) with \( r \geq 0 \) and odd \( Q \).
Then,
\[ \sigma_1(2(3n+1)) + \sigma_1(3n+1) = \sigma_1(2^{r+1}Q) + \sigma_1(2^rQ) = 2(3 \cdot 2^r - 1)\sigma_1(Q). \]
So,
\[ \mu(6n + 2) \equiv 6(2n + 1)(3 \cdot 2^r - 1)\sigma_1(Q) \equiv 0 \pmod{6}. \]
Remark 3.12. $\mu(3n - 1) \equiv 0 \pmod{6}$ shown by us induces that $\mu(3n - 1) \equiv 0 \pmod{3}$ which is also the Hahn’s result in [11, Theorem 6.1].

References

[10] , , On the square of the series in which the coefficients are the sums of the divisors of the exponents, Mess. Math. 14 (1884).
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