MATLIS INJECTIVE MODULES

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Abstract. In this paper, Matlis injective modules are introduced and studied. It is shown that every \( R \)-module has a (special) Matlis injective preenvelope over any ring \( R \) and every right \( R \)-module has a Matlis injective envelope when \( R \) is a right Noetherian ring. Moreover, it is shown that every right \( R \)-module has an \( F^{\perp} \)-envelope when \( R \) is a right Noetherian ring and \( F \) is a class of injective right \( R \)-modules.

1. Introduction

Throughout this paper, \( R \) will denote an associative ring with identity and all modules will be unitary right \( R \)-modules.

The motivation of this paper is from [4], where the notion of Whitehead modules was studied. Recall that an \( R \)-module \( M \) is called a Whitehead module or \( W \)-module if \( \text{Ext}^1_R(M, R) = 0 \). We introduce the notion of Matlis injective modules as a dual notion of Whitehead modules in some sense. An \( R \)-module \( M \) is called Matlis injective if \( \text{Ext}^1_R(E(R), M) = 0 \), where \( E(R) \) denotes the injective envelope of \( R \). Let \( R \) be an integral domain and \( Q \) its field of quotients, an \( R \)-module \( C \) is called Matlis cotorsion or weakly cotorsion if \( \text{Ext}^1_R(Q, C) = 0 \).

Then, it is easy to see that the notion of Matlis injective \( R \)-modules coincides with the notion of Matlis cotorsion \( R \)-modules when \( R \) is an integral domain.

Following [7], an \( R \)-module \( M \) is called copure injective if \( \text{Ext}^1_R(E, M) = 0 \) for any injective \( R \)-module \( E \). Clearly, every copure injective \( R \)-module is Matlis injective, but it is easy to see that the converse is not true in general. Thus Matlis injective \( R \)-modules can be seen as a generalization of copure injective \( R \)-modules.

Let \( C \) be a class of \( R \)-modules. Enochs defined a \( C \)-(pre)cover (\( C \)-(pre)envelope) of an \( R \)-module in [6]. Therefore, it is natural to study the existence of Matlis injective (pre)covers and Matlis injective (pre)envelopes. Obviously, the class of Matlis injective \( R \)-modules is closed under direct summands, but we show that it is not closed under direct sums in general. So there exist a ring...
R and an R-module M such that M doesn’t have a Matlis injective precover. Then, we are only interested in the existence of Matlis injective (pre)-envelopes in this paper. Let \( \mathcal{F} \) be a class of R-modules, we denote by \( \mathcal{F}^{-1} \) the class of R-modules N such that \( \text{Ext}_R^1(F,N) = 0 \) for every \( F \in \mathcal{F} \). In [5, Theorem 10], Eklof and Trlifaj proved that if there is a set \( \mathcal{S} \) of R-modules such that \( \mathcal{F}^{-1} = \mathcal{S}^{-1} \), then every R-module has an \( \mathcal{F}^{-1} \)-preenvelope. Using this result, we show that every R-module has a Matlis injective preenvelope. If R is a right Noetherian ring, we show that every R-module has an \( \mathcal{F}^{\perp 1} \)-envelope, where \( \mathcal{F} \) is any subclass of the class of injective R-modules. As a byproduct, we show that every R-module has a Matlis injective envelope when R is a right Noetherian ring.

2. Preliminaries

In this section we briefly recall some definitions and results required in this paper.

For a ring \( R \), \( \text{Mod-} R \) will denote the category of all right R-modules and \( \text{pd}(M) \) will denote the projective dimension of \( M \). For an R-module \( M \), we denote by \( E(M) \) the injective envelope of \( M \). We frequently identify \( M \) with its image in \( E(M) \) and think of \( M \) as a submodule of \( E(M) \).

Let \( C \subseteq \text{Mod-} C \). Define

\[
\begin{align*}
\mathcal{C}^{\perp 1} &= \{ X \in \text{Mod-} R \mid \text{Ext}_R^1(C,X) = 0 \text{ for all } C \in \mathcal{C} \}, \\
\mathcal{C}^{\perp 1} &= \{ X \in \text{Mod-} R \mid \text{Ext}_R^1(X,C) = 0 \text{ for all } C \in \mathcal{C} \}.
\end{align*}
\]

\( \text{Add}(C) =\{ X \in \text{Mod-} R \mid X \text{ is a direct summand of } \bigoplus_{i \in I} C_i, \text{ where } I \text{ is a set and where for any } i \in I, C_i \text{ is isomorphic to an element of } \mathcal{C} \}. \)

For \( \mathcal{C} = \{ C \} \), we write \( \mathcal{C}^{\perp 1} \), \( \mathcal{C}^{\perp 1} \) and \( \text{Add}(C) \) in place of \( \{ C \}^{\perp 1} \), \( \{ C \}^{\perp 1} \) and \( \text{Add} \{ C \} \), respectively.

Let \( M \in \text{Mod-} R \). A homomorphism \( f \in \text{Hom}_R(M,C) \) with \( C \in \mathcal{C} \) is called a \( \mathcal{C} \)-preenvelope of \( M \) provided that the abelian group homomorphism \( \text{Hom}_R(f,C') : \text{Hom}_R(C,C') \to \text{Hom}_R(M,C') \) is surjective for each \( C' \in \mathcal{C} \). The \( \mathcal{C} \)-preenvelope \( f \) is called a \( \mathcal{C} \)-envelope of \( M \) provided that \( f = gf \) implies \( g \) is an automorphism for each \( g \in \text{End}_R(C) \). Moreover, a \( \mathcal{C} \)-preenvelope \( f : M \to C \) of \( M \) is called special provided that \( f \) is injective and \( \text{Coker} \ f \in \mathcal{C}^{\perp 1} \). \( \mathcal{C} \)-envelopes may not exist in general, but if they exist, they are unique up to isomorphism. If \( \mathcal{C} \) is the class of injective modules, then we get the usual injective envelopes.

\( \mathcal{C} \)-precovers and \( \mathcal{C} \)-covers are defined dually. These generalize the projective covers introduced by Bass in the 1960’s.

A pair \( (A, B) \) of R-module classes is called a cotorsion theory (or cotorsion pair) provided that \( A^{\perp 1} = B \) and \( A = B^{\perp 1} \). An R-module M is called cotorsion if \( \text{Ext}_R^1(F,M) = 0 \) for any flat R-module F. Let \( \mathcal{F} \) be the class of flat R-modules and \( \mathcal{C} \) be the class of cotorsion R-modules, it is known that \( (\mathcal{F}, \mathcal{C}) \) is a cotorsion theory.

For any class \( \mathcal{F} \) of R-modules. The following theorem, due to Eklof and
Trlifaj, says that every $R$-module has a special $F^{⊥1}$-preenvelope if there is a set $S$ of $R$-modules such that $S^{⊥1} = F^{⊥1}$. Before stating the result, we need more notions:

A sequence of modules $A = (A_α | α ≤ μ)$ is called a continuous chain of modules provided that $A_0 = 0$, $A_α ⊆ A_{α+1}$ for all $α < μ$ and $A_α = \bigcup_{β < α} A_β$ for all limit ordinals $α ≤ μ$.

Let $M$ be a module and $C$ a class of modules. Then $M$ is called $C$-filtered provided that there are an ordinal $κ$ and a continuous chain, $\langle M_α | α ≤ κ \rangle$, consisting of submodules of $M$ such that $M = M_κ$, and such that each of the modules $M_{α+1}/M_α (α < κ)$ is isomorphic to an element of $C$. The chain $\langle M_α | α ≤ κ \rangle$ is called a $C$-filtration of $M$.

**Theorem 2.1** ([10], Theorem 3.2.1, p. 117). Let $S$ be a set of $R$-modules and $M$ an $R$-module. Then there is a short exact sequence $0 → M → P → N → 0$, where $P ∈ S^{⊥1}$ and $N$ is $S$-filtered. In particular, $M → P$ is a special $S^{⊥1}$-preenvelope of $M$.

The following theorem from [10] gives a criterion to judge when an $R$-module $M$ has a $C^{⊥1}$-envelope.

**Theorem 2.2** ([10], Theorem 2.3.2, p. 107). Let $R$ be a ring and $M$ be an $R$-module. Let $C$ be a class of $R$-modules closed under extensions and direct limits. Assume that $M$ has a special $C^{⊥1}$-preenvelope $ν$ with $\text{Coker } ν ∈ C$. Then $M$ has a $C^{⊥1}$-envelope.

A short exact sequence $0 → A → B → C → 0$ of $R$-modules is called pure if the induced sequence $0 → \text{Hom}_R(F, A) → \text{Hom}_R(F, B) → \text{Hom}_R(F, C) → 0$ of abelian groups is exact for every finitely presented $R$-module $F$. A submodule $A$ of an $R$-module $B$ is called a pure submodule of $B$ if the canonical exact sequence $0 → A → B → B/A → 0$ is pure. An $R$-module $M$ is called pure injective if the sequence $0 → \text{Hom}_R(C, M) → \text{Hom}_R(B, M) → \text{Hom}_R(A, M) → 0$ is exact for every pure exact sequence $0 → A → B → C → 0$ of $R$-modules.

Let $M$ be an $R$-module. $M$ is said to be $Σ$-pure injective if for every index set $I$ the direct sum $M^{(I)}$ is pure injective. $M$ is said to be $Σ$-self orthogonal if $\text{Ext}_R^1(M, M^{(I)}) = 0$ for every index set $I$.

The following property of $Σ$-pure injective modules will be used in this paper.

**Proposition 2.3** ([9], Corollary 1.42, p. 30). Every pure submodule of a $Σ$-pure injective module $B$ is a direct summand of $B$.

For unexplained terminology and notation, we refer the reader to [1, 3, 8, 10, 13].

### 3. Properties of Matlis injective modules

We start with the following definition.
Definition 3.1. Let $R$ be a ring and $M$ an $R$-module. $M$ is said to be Matlis injective if $\text{Ext}^1_R(E(R), M) = 0$. An $R$-module $N$ is said to be Matlis projective if $\text{Ext}^1_R(E(R), C) = 0$ implies $\text{Ext}^1_R(N, C) = 0$ for any $R$-module $C$. $R$ is said to be a right Matlis ring if $E(R)$ is flat and $\text{pd}(E(R)) \leq 1$.

In what follows, we denote by $\mathcal{MI}$ ($\mathcal{MP}$) the class of Matlis injective (projective) $R$-modules. For $\mathcal{C} = \mathcal{MI}$, $\mathcal{C}$-(pre)envelopes will simply be called Matlis injective (pre)envelopes.

Proposition 3.2. Let $R$ be a ring. Then $\mathcal{MI}$ is closed under extensions, direct products and direct summands; $\mathcal{MI} = \text{Mod}_R$ if and only if $E(R)$ is projective.

Proof. It is easy to see that the assertion holds by definition. □

Corollary 3.3. Let $R$ be an integral domain. Then every $R$-module is Matlis injective if and only if $R$ is a field.

Proof. “$\Leftarrow$” is trivial. “$\Rightarrow$”. By Proposition 3.2, $E(R)$ is projective, then there exists a non-zero homomorphism $f \in \text{Hom}_R(E(R), R)$. So $f(E(R))$ is a non-zero divisible submodule of $R$. Let $r$ be any non-zero element from $R$. We choose a non-zero element $x \in f(E(R))$. Since $rx$ is non-zero and $f(E(R))$ is divisible, there is an element $y \in f(E(R))$ with $(rx)y = x$, and so $(ry - 1)x = 0$. But $R$ is an integral domain, then $ry - 1 = 0$, i.e., $ry = 1$. Hence $R$ is a field. □

Remark 3.4. Recall that a commutative domain $R$ is called almost perfect provided that $R/I$ is a perfect ring for each ideal $0 \neq I \neq R$. We will show that $\mathcal{MI}$ is not closed under direct sums if $R$ is an almost perfect domain but not a field. If $R$ is an almost perfect domain, then $\mathcal{MI}$ coincides with the class of cotorsion $R$-modules by [10, Theorem 4.4.16, p. 172]. But the class of cotorsion $R$-modules is closed under direct sums if and only if $R$ is a perfect ring by [11, Theorem 19]. Note that $E(R)$ is flat when $R$ is a commutative domain, and so $R$ is a perfect ring if and only if $R$ is a field by Corollary 3.3. Hence $\mathcal{MI}$ is not closed under direct sums when $R$ is an almost perfect domain but not a field. Then we will show that there exist a ring $R$ and an $R$-module $M$ such that $M$ doesn’t have a Matlis injective precover. For example, let $R$ be an almost perfect domain but not a field, then there exists a family $\{M_i\}_{i \in I}$ of Matlis injective $R$-modules such that $\bigoplus_{i \in I} M_i$ is not Matlis injective. But since $\mathcal{MI}$ is closed under direct summands by Proposition 3.2, it is easy to check that $\bigoplus_{i \in I} M_i$ doesn’t have a Matlis injective precover.

Lemma 3.5. Let $R$ be a ring. Then every cotorsion $R$-module is Matlis injective if and only if $E(R)$ is flat.

Proof. “$\iff$” is clear. “$\Rightarrow$”. Let $C$ be any cotorsion $R$-module. By hypothesis, we have $\text{Ext}^1_R(E(R), C) = 0$.

Hence $E(R)$ is flat by the fact that $(\mathcal{F}, \mathcal{E})$ is a cotorsion theory. □
Proposition 3.6. Let \( R \) be a ring. Then \( MI = \mathcal{C} \) if and only if \( E(R) \) is flat and every Matlis injective \( R \)-module is cotorsion.

Proof. “\( \iff \)” holds by assumption and Lemma 3.5.

“\( \Rightarrow \)” By assumption, we have \( M \) is cotorsion if and only if it is Matlis injective. Then the assertion holds by Lemma 3.5.

Proposition 3.7. Let \( R \) be a ring. Then the following are equivalent.

1. Every quotient module of any Matlis injective \( R \)-module is Matlis injective.
2. Every quotient module of any injective \( R \)-module is Matlis injective.
3. The projective dimension of \( E(R) \) is at most 1.

Proof. (1) \( \iff \) (2) is trivial.

(2) \( \Rightarrow \) (3). Let \( K \) be any \( R \)-module. It is enough to show that \( \text{Ext}^2_R(E(R), K) = 0 \). Let us consider the exact sequence \( 0 \to K \to E(K) \to E(K)/K \to 0 \). We then have the exact sequence \( \text{Ext}^1_R(E(R), E(K)/K) \to \text{Ext}^2_R(E(R), K) \to \text{Ext}^2_R(E(R), E(K)) = 0 \). Note that \( \text{Ext}^1_R(E(R), E(K)/K) = 0 \) by (2), we get \( \text{Ext}^2_R(E(R), K) = 0 \).

(3) \( \Rightarrow \) (1). Let \( M \) be a Matlis injective \( R \)-module and \( N \) a submodule of \( M \). Let us consider the exact sequence \( 0 \to N \to M \to M/N \to 0 \). Applying the functor \( \text{Hom}_R(E(R), -) \) to the above exact sequence, we get the exact sequence \( 0 = \text{Ext}^1_R(E(R), M) \to \text{Ext}^1_R(E(R), M/N) \to \text{Ext}^2_R(E(R), N) \). Note that \( \text{Ext}^1_R(E(R), N) = 0 \) by (3), so \( \text{Ext}^1_R(E(R), M/N) = 0 \) and (1) follows.

Remark 3.8. If \( E(R) \) is flat, then the condition that every quotient module of any cotorsion \( R \)-module is Matlis injective is also equivalent to the conditions of Proposition 3.7.

Lemma 3.9. Let \( R \) be a ring. Then \( (\mathcal{MP}, MI) \) is a cotorsion theory.

Proof. Straightforward.

Theorem 3.10. Let \( R \) be a ring. Then the following are equivalent.

1. \( R \) is a right Matlis ring.
2. Every quotient module of any Matlis injective \( R \)-module is Matlis injective and every cotorsion \( R \)-module is Matlis injective.
3. Every quotient module of any injective \( R \)-module is Matlis injective and every cotorsion \( R \)-module is Matlis injective.
4. Every Matlis projective \( R \)-module is flat and its projective dimension is at most 1.

Proof. (1) \( \iff \) (2) \( \iff \) (3) hold by Lemma 3.5 and Proposition 3.7.

(1) \( \Rightarrow \) (4). By Lemma 3.9 and [10, Corollary 3.2.4, p. 119], every Matlis projective \( R \)-module is a direct summand of some \( \{E(R), R\} \)-filtered \( R \)-module. Note that every \( \{E(R), R\} \)-filtered \( R \)-module is flat and its projective dimension is at most \( \text{pd}(E(R)) \) by (1) and [10, Lemma 3.1.2, p. 113]. So every Matlis projective \( R \)-module is flat and its projective dimension is at most 1.
we have the exact sequence 0 = \text{Ext}^1 \rightarrow \text{Ext}^2 \rightarrow \text{Ext}^3 \rightarrow \cdots 

\square

by assumption.

Let us consider the exact sequence 0 = \text{Ext}^1 \rightarrow \text{Ext}^2 \rightarrow \text{Ext}^3 \rightarrow \cdots

Proof. Let us consider the exact sequence 0 = \text{Ext}^1 \rightarrow \text{Ext}^2 \rightarrow \text{Ext}^3 \rightarrow \cdots

\square

Hence (1) holds.

\square

Proposition 3.11. Let R be a ring. If pd(\text{Ext}(R)) \leq 1, then tight submodules of Matlis projective R-modules are also Matlis projective.

Proof. Let us consider the exact sequence 0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0, where M is Matlis projective and N is a tight submodule of M. Then pd(M) \leq 1 by hypothesis and the proof of Theorem 3.10. For any Matlis injective R-module C, we have the induced exact sequence

\text{Ext}^1_R(M, C) \rightarrow \text{Ext}^2_R(N, C) \rightarrow \text{Ext}^2_R(M/N, C).

The two ends vanish, since M is Matlis projective and pd(M/N) \leq pd(M) \leq 1. So the middle term is 0, and hence the assertion holds.

\square

Proposition 3.12. Let R be a ring. Then the following are equivalent.

(1) C \in \mathcal{M}\mathcal{Z} whenever 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 is an exact sequence of R-modules such that A, B \in \mathcal{M}\mathcal{Z}.

(2) E(M)/M is Matlis injective when M is Matlis injective.

(3) For any R-module M, Ext^1_R(\text{Ext}(R), M) = 0 implies Ext^2_R(\text{Ext}(R), M) = 0.

Proof. (1) \implies (2) is trivial.

(2) \implies (3). Let M be an R-module such that Ext^1_R(\text{Ext}(R), M) = 0, i.e., M is Matlis injective. Then E(M)/M is Matlis injective by (2). Applying the functor \text{Hom}_R(\text{Ext}(R), -) to the exact sequence 0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0, we have the exact sequence 0 = Ext^1_R(\text{Ext}(R), E(M)/M) \rightarrow Ext^2_R(\text{Ext}(R), M) \rightarrow Ext^2_R(\text{Ext}(R), E(M)) = 0. So Ext^2_R(\text{Ext}(R), M) = 0.

(3) \implies (1). Let 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 be an exact sequence of R-modules such that A, B \in \mathcal{M}\mathcal{Z}. Applying the functor \text{Hom}_R(\text{Ext}(R), -) to the above sequence, we have the exact sequence 0 = Ext^1_R(\text{Ext}(R), B) \rightarrow Ext^1_R(\text{Ext}(R), C) \rightarrow Ext^2_R(\text{Ext}(R), A) = 0 by (3). So Ext^2_R(\text{Ext}(R), C) = 0, i.e., C is Matlis injective. Hence (1) holds.

\square

Proposition 3.13. Let R be a commutative Artinian ring. Then \mathcal{M}\mathcal{Z} is closed under direct sums, pure submodules and direct limits. Moreover, \mathcal{M}\mathcal{Z} is a definable class, i.e., it is closed under pure submodules, direct products and direct limits.

Proof. By hypothesis, E(R) is finitely presented by [12, Theorem 3.64, p. 90]. Then \mathcal{M}\mathcal{Z} is closed under direct sums by the isomorphism

\bigoplus \text{Ext}^1_R(F, M_\alpha) \cong \text{Ext}^1_R(F, \bigoplus M_\alpha)
for any finitely presented \( R \)-module \( F \) and any family \( \{ M_\alpha \} \) of \( R \)-modules. Suppose that \( A \) is a pure submodule of a Matlis projective \( R \)-module \( B \). Then we have the exact sequences
\[
0 \rightarrow \text{Hom}_R(E(R), A) \rightarrow \text{Hom}_R(E(R), B) \rightarrow \text{Hom}_R(E(R), B/A) \rightarrow 0 \text{ and } \text{Hom}_R(E(R), B) \rightarrow \text{Hom}_R(E(R), B/A) \rightarrow \text{Ext}_R^1(E(R), A) \rightarrow \text{Ext}_R^1(E(R), B) = 0.
\]
Hence \( \text{Ext}_R^1(E(R), A) = 0 \), i.e., \( A \) is Matlis injective. So \( MZ \) is closed under pure submodules. That \( MZ \) is closed under direct limits follows from the isomorphism \( \text{Ext}_R^1(F, \varinjlim M_\alpha) \cong \varinjlim \text{Ext}_R^1(F, M_\alpha) \) for any finitely presented \( R \)-module \( F \) and any family \( \{ M_\alpha \} \) of \( R \)-modules since \( R \) is a commutative Artinian ring. So \( MZ \) is definable by Proposition 3.2.

**Proposition 3.14.** Let \( R \) be a commutative Artinian ring and \( S \subset R \) be a multiplicative set. If \( M \) is a Matlis injective \( R \)-module, then \( S^{-1}M \) is a Matlis injective \( S^{-1}R \)-module.

**Proof.** By assumption, \( E(R) \) is finitely generated by [12, Theorem 3.64, p. 90] and \( R \) is a Noetherian ring. So,
\[
\text{Ext}_{S^{-1}R}(S^{-1}E_R(R), S^{-1}M) \cong S^{-1}\text{Ext}_R^1(E_R(R), M)
\]
by [8, Theorem 3.2.5, p. 76]. But \( S^{-1}E_R(R) \cong E_{S^{-1}R}(S^{-1}R) \) by [8, Theorem 3.3.3, p. 84]. Thus \( S^{-1}M \) is a Matlis injective \( S^{-1}R \)-module when \( M \) is a Matlis injective \( R \)-module.

**Proposition 3.15.** Let \( R \) be a commutative Noetherian ring and \( S \subset R \) be a multiplicative set. If \( M \) is a Matlis projective \( R \)-module, then \( S^{-1}M \) is a Matlis projective \( S^{-1}R \)-module.

**Proof.** Note that \( S^{-1}E_R(R) \cong E_{S^{-1}R}(S^{-1}R) \) by [8, Theorem 3.3.3, p. 84] and by hypothesis. Then, every Matlis projective \( S^{-1}R \)-module is a direct summand of some \( \{ S^{-1}E_R(R), S^{-1}R \} \)-filtered \( S^{-1}R \)-module by [10, Corollary 3.2.4, p. 119]. Since \( M \) is a Matlis projective \( R \)-module, \( M \) is a direct summand of some \( \{ E(R), R \} \)-filtered \( R \)-module by [10, Corollary 3.2.4, p. 119]. Let \( N \) be an \( \{ E(R), R \} \)-filtered \( R \)-module and the chain \( \{ N_\alpha \mid \alpha \leq \kappa \} \) be a \( \{ E(R), R \} \)-filtration of \( N \). Then \( S^{-1}N \) is an \( \{ S^{-1}E_R(R), S^{-1}R \} \)-filtered \( S^{-1}R \)-module and the chain \( \{ S^{-1}N_\alpha \mid \alpha \leq \kappa \} \) is a \( \{ S^{-1}E_R(R), S^{-1}R \} \)-filtration of \( S^{-1}N \) by [8, Theorem 1.5.7, p. 33, and Proposition 2.2.4, p. 44] and by definition. So \( S^{-1}M \) is a Matlis projective \( S^{-1}R \)-module and the assertion holds.

4. The existence of Matlis injective (pre)envelopes

According to Theorem 2.1, we immediately have the following proposition.

**Proposition 4.1.** Let \( R \) be a ring. Then every \( R \)-module has a special Matlis injective preenvelope.

The following lemmas are needed to prove the main result of this paper.
**Lemma 4.2.** Let $R$ be a ring and $M$ an $R$-module. If $M$ is $\Sigma$-pure injective and $\Sigma$-self orthogonal, then $\text{Add}(M)$ is closed under extensions and direct limits.

**Proof.** Let $0 \to A \to B \to C \to 0$ be an exact sequence of $R$-modules such that both $A$ and $C$ are in $\text{Add}(M)$. Without loss of generality, we may assume that both $A$ and $C$ are direct summands of $M^{(I)}$ for an index set $I$. Since $M$ is $\Sigma$-self orthogonal, we have $\text{Ext}^1_{\text{R}}(C, A) = 0$. Then the exact sequence $0 \to A \to B \to C \to 0$ splits, and so $B \cong A \oplus C$. Obviously, $A \oplus C \in \text{Add}(M)$.

Therefore, $B \in \text{Add}(M)$. So $\text{Add}(M)$ is closed under extensions. We claim that any $R$-module $N$ from $\text{Add}(M)$ is $\Sigma$-pure injective. It is clear that $N$ is a direct sum of indecomposable injective $R$-modules. Then every $R$-module is the direct sum of copies of $\text{Add}(M)$ where $I$ is a directed set. Then there exists a short exact sequence $0 \to K \to \bigoplus_{i \in I} M_i \to \lim_{\to} M_i \to 0$ with $K$ a pure submodule of $\bigoplus_{i \in I} M_i$. But $\bigoplus_{i \in I} M_i$ is $\Sigma$-pure injective, then the exact sequence $0 \to K \to \bigoplus_{i \in I} M_i \to \lim_{\to} M_i \to 0$ splits by Proposition 2.3. So $\lim_{\to} M_i$ is isomorphic to a direct summand of $\bigoplus_{i \in I} M_i$, i.e., $\lim_{\to} M_i \in \text{Add}(M)$.

Hence $\text{Add}(M)$ is closed under direct limits. $\square$

**Lemma 4.3.** Let $R$ be a ring and $M$ an $R$-module. Assume that $M$ is $\Sigma$-pure injective and $\Sigma$-self orthogonal. Then every $R$-module $N$ has an $M^{1-}$-envelope.

**Proof.** Obviously, $M^{1-} = (\text{Add}(M))^{1-}$. Thus it is equivalent to show that every $R$-module $N$ has an $(\text{Add}(M))^{1-}$-envelope. By Theorem 2.1, $N$ has a special $(\text{Add}(M))^{1-}$-envelope $f$ with $\text{Coker} f$ is $\{M\}$-filtered. Note that every $\{M\}$-filtered $R$-module is in $\text{Add}(M)$ by Lemma 4.2 and transfinite induction. So $N$ has an $(\text{Add}(M))^{1-}$-envelope by Lemma 4.2 and Theorem 2.2. $\square$

We are now in a position to prove the following

**Theorem 4.4.** Let $R$ be a right Noetherian ring and $F$ a class of injective $R$-modules. Then every $R$-module $M$ has an $F^{1-}$-envelope; in particular, every $R$-module $M$ has a Matlis injective envelope.

**Proof.** If $R$ is right Noetherian, then every injective $R$-module is the direct sum of indecomposable injective $R$-modules. Each such module is the injective envelope of a cyclic $R$-module. Hence, we can find a representative set of such modules. So there is a family $\{E_i\}_{i \in I}$ of indecomposable injective $R$-modules such that every injective $R$-module is the direct sum of copies of $E_i$.

Let $S = \{E_i \mid E_i \text{ is isomorphic to a direct summand of an element of } F \}$. It is easy to see that $(\bigoplus_{E_i \in S} E_i)^{1-} = F^{1-}$. Note that $\bigoplus_{E_i \in S} E_i$ is $\Sigma$-pure injective and $\Sigma$-self orthogonal by the fact that the class of right injective $R$-modules is closed under direct sums when $R$ is right Noetherian. So the assertion holds by Lemma 4.3. $\square$

We end this paper with the following remark.
Remark 4.5. If \( R \) is a commutative Artinian ring, then every \( R \)-module has a Matlis injective cover by Proposition 3.13 and [2, Corollary 2.6 and Proposition 4.3(3)].

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References


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