RECURRENT JACOBI OPERATOR OF REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS

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Abstract. In this paper we give a non-existence theorem for Hopf hypersurfaces in the complex two-plane Grassmannian $G_2(C^{m+2})$ with recurrent normal Jacobi operator $\bar{R}_N$.

1. Introduction

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold. The Jacobi operator $\bar{R}_X$, for any tangent vector field $X$ at $x \in \bar{M}$, is defined by

$$(\bar{R}_X Y)(x) = (\bar{R}(Y, X) X)(x)$$

for any $Y \in T_x \bar{M}$. It becomes a self adjoint endomorphism of the tangent bundle $TM$ of $\bar{M}$, where $\bar{R}$ denotes the curvature tensor of $(\bar{M}, \bar{g})$. That is, the Jacobi operator satisfies $\bar{R}_X \in \text{End}(T_x \bar{M})$ and is symmetric in the sense of $\bar{g}(\bar{R}_X Y, Z) = \bar{g}(\bar{R}_X Z, Y)$ for any vector fields $Y$ and $Z$ on $\bar{M}$.

Let $M$ be a hypersurface in a Riemannian manifold $\bar{M}$. Now by putting a unit normal vector $N$ into the curvature tensor $\bar{R}$ of $\bar{M}$, the normal Jacobi operator $\bar{R}_N$ is defined by

$$\bar{R}_N X = \bar{R}(X, N) N$$

for any tangent vector field $X$ on $M$ in $\bar{M}$.

Related to the commuting problem with the shape operator for real hypersurfaces $M$ in quaternionic projective space $\mathbb{HP}^m$ or in quaternionic hyperbolic space $\mathbb{H}^m$, Berndt [1] has introduced the notion of normal Jacobi operator $\bar{R}_N \in \text{End}(T_x M)$, $x \in M$, where $\bar{R}$ denotes the curvature tensor of the ambient spaces $\mathbb{HP}^m$ or $\mathbb{H}^m$. He [1] showed that the curvature adaptedness, when

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the normal Jacobi operator commutes with the shape operator $A$, is equivalent to the fact that the distributions $\mathcal{D}$ and $\mathcal{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator $A$ of $M$, where $T_x M = \mathcal{D} \oplus \mathcal{D}^\perp$, $x \in M$. Here, $\{J_\nu \mid \nu = 1, 2, 3\}$ is a canonical local basis of quaternionic Kähler structure $J$ and $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Moreover, he gave a complete classification of curvature adapted real hypersurfaces in quaternionic projective space $\mathbb{HP}^m$ and in quaternionic hyperbolic space $\mathbb{HH}^m$, respectively.

We say that the normal Jacobi operator $\bar{R}_N$ is parallel on $M$ if the covariant derivative of the normal Jacobi operator $\bar{R}_N$ identically vanishes, that is, $\nabla_X \bar{R}_N = 0$ for any vector field $X$ on $M$.

Parallelness of the normal Jacobi operator means that the normal Jacobi operator $\bar{R}_N$ is parallel on a real hypersurface $M$ in ambient space $\mathbb{C}$. This means that the eigenspaces of the normal Jacobi operator are parallel along any curve $\gamma$ in $M$. Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be parallel along any curve $\gamma$ if they are invariant with respect to any parallel displacement along the curve $\gamma$.

The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$ has a remarkable geometric structure. It is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $\bar{J}$ not containing $J$ (See [2]). From these two structures $J$ and $\bar{J}$, we have geometric conditions naturally induced on a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$: That $[\xi] = \text{Span} \{\xi\}$ or $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. From these two conditions, Berndt and Suh [3] have proved the following:

**Theorem A.** Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{D}^\perp$ are invariant under the shape operator of $M$ if and only if

(A) $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{HP}^n$ in $G_2(\mathbb{C}^{m+2})$.

The structure vector field $\xi$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ is said to be a Reeb vector field. If the Reeb vector field $\xi$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ is invariant under the shape operator, $M$ is said to be a Hopf hypersurface. In such a case the integral curves of the Reeb vector field $\xi$ are geodesics (See [4]). Moreover, the flow generated by the integral curves of the structure vector field $\xi$ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a geodesic Reeb flow.

In paper [9], Jeong, Kim and Suh considered the notion of parallel normal Jacobi operator, that is, $\nabla_X \bar{R}_N = 0$ for any vector field $X$ on $M$ in $G_2(\mathbb{C}^{m+2})$, where $\nabla$ denotes the induced connection from the Levi-Civita connection $\nabla$ of $G_2(\mathbb{C}^{m+2})$. They proved a non-existence theorem for Hopf hypersurfaces...
in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator as follows:

**Theorem B.** There does not exist any connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator.

On the other hand, in [10] Jeong, Lee and Suh have considered a Lie parallelness of the normal Jacobi operator, that is, $\mathcal{L}_X R_N = 0$, where $\mathcal{L}_X$ denotes the Lie derivative along any direction $X$ on $M$ in $G_2(\mathbb{C}^{m+2})$, and asserted the following:

**Theorem C.** There does not exist any Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator if the integral curves of $\mathcal{D}$ and $\mathcal{D}^\perp$ components of the Reeb vector field are totally geodesics.

The purpose of this paper is to study a generalized condition weaker than parallel normal Jacobi operator for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Let $T$ be a tensor field of type (1,1) on $M$. $T$ is said to be recurrent if there exists a certain 1-form $\omega$ on $M$ such that for any vector fields $X, Y$ tangent to $M$, $(\nabla_X T)(Y) = \omega(X) T(Y)$. This notion generalizes the fact of $T$ being parallel (see [13]).

Hamada [5], [6] investigated real hypersurfaces $M$ in complex projective space $\mathbb{C}P^m$ with recurrent shape operator. This means that the eigenspaces of the shape operator are parallel along any curve $\gamma$ in $M$. That is, they are invariant with respect to parallel translation along $\gamma$. He proved that there does not exist real hypersurface in complex projective space $\mathbb{C}P^m$ with recurrent shape operator. In [7] he also proved that there does not exist any real hypersurface $M$ in complex projective space $\mathbb{C}P^m$ with recurrent Ricci tensor if the structure vector field $\xi$ is principal.

For a real hypersurface in complex projective space $\mathbb{C}P^m$, P´erez and Santos [15] introduced a new notion of $\mathcal{D}$-recurrent, which is weaker than the structure Jacobi operator being recurrent. Here, the structure Jacobi operator $R_\xi$ is said to be $\mathcal{D}$-recurrent if it satisfies

$$(\nabla_X R_\xi)(Y) = \omega(X) R_\xi(Y),$$

where $\omega$ and $\mathcal{D}$ respectively denote an 1-form on $M$ and the orthogonal complement of the Reeb vector field $\xi$ in $TM$, and any vector fields $X \in \mathcal{D}, Y \in TM$. Namely, they proved the following:

**Theorem D.** Let $M$ be a real hypersurface of $\mathbb{C}P^m$, $m \geq 3$. Then its structure Jacobi operator is $\mathcal{D}$-recurrent if and only if it is a minimal ruled real hypersurface.

Related to the shape operator, in paper due to [12], first they have applied Hamada’s results to hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and next obtained a non-existence property for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator.
Motivated by such a recurrent shape operator, and in order to make a generalization of Theorem B, in this paper we introduce a new notion of recurrent Jacobi operator; that is, the recurrent normal Jacobi operator for a real hypersurface $M$ in complex two-plane Grassmannians $G_2(C^{m+2})$. A hypersurface $M$ in $G_2(C^{m+2})$ with recurrent normal Jacobi operator is defined by

$$(\nabla_X R_N)(Y) = \omega(X)R_N(Y),$$

where $\omega$ denotes an 1-form on $M$ and any vector fields $X, Y$ tangent to $M$ (see Kobayashi and Nomizu [13] page 305). Consequently, we prove the following:

**Theorem 1.1.** There does not exist any Hopf hypersurface in complex two-plane Grassmannians $G_2(C^{m+2})$ with recurrent normal Jacobi operator.

## 2. Riemannian geometry of $G_2(C^{m+2})$

In this section we summarize basic material about $G_2(C^{m+2})$, for details refer to [2], [3] and [4].

By $G_2(C^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $C^{m+2}$. The special unitary group $G = SU(m + 2)$ acts transitively on $G_2(C^{m+2})$ with stabilizer isomorphic to $K = SU(2) \times U(m) \subset G$. The space $G_2(C^{m+2})$ can be identified with the homogeneous space $G/K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_2(C^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of $G$ and $K$, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $B$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$.

We put $o = eK$ and identify $T_oG_2(C^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since $B$ is negative definite on $\mathfrak{g}$, negative $B$ restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of $B$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_2(C^{m+2})$.

In this way $G_2(C^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $g$ such that the maximum sectional curvature of $(G_2(C^{m+2}), g)$ is eight.

When $m = 1$, $G_2(C^4)$ is isometric to the two-dimensional complex projective space $CP^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(C^4)$ and the real Grassmann manifold $G^+_2(R^6)$ of oriented two-dimensional linear subspaces of $R^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = su(m) \oplus su(2) \oplus \mathfrak{r}$, where $\mathfrak{r}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(C^{m+2})$, the center $\mathfrak{r}$ induces a Kähler structure $J$ and the $su(2)$-part a quaternionic Kähler structure $\mathfrak{j}$ on $G_2(C^{m+2})$.

If $J_1$ is any almost Hermitian structure in $\mathfrak{j}$, then $JJ_1 = J_1J$, and $JJ_1$ is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$. 


A canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ consists of three local almost Hermitian structures $J_\nu$ in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\nabla$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ three local one-forms $q_1, q_2, q_3$ such that

$$\nabla_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2}$$

for all vector fields $X$ on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\tilde{R}(X, Y) Z = g(Y, Z) X - g(X, Z) Y + g(JY, Z) JX - g(JX, Z) JY - 2g(JX, Y) JZ - \sum_{\nu=1}^{3} \left( g(J_\nu Y, Z) J_{\nu} X - g(J_\nu X, Z) J_\nu Y \right)$$

$$- 2g(J_\nu X, Y) J_\nu Z + \sum_{\nu=1}^{3} \left( g(J_\nu JY, Z) J_\nu JX - g(J_\nu JX, Z) J_\nu JY \right)$$

for any vector fields $X, Y$ and $Z$ on $G_2(\mathbb{C}^{m+2})$, where $J_1, J_2, J_3$ is any canonical local basis of $\mathfrak{J}$ [2].

3. Some fundamental formulas

In this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (See [2], [3] and [4]).

Let $M$ be a real hypersurface of $G_2(\mathbb{C}^{m+2})$. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal field of $M$ and $A$ the shape operator of $M$ with respect to $N$. The Kähler structure $J$ of $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. Furthermore, let $J_1, J_2, J_3$ be a canonical local basis of $\mathfrak{J}$. Then each $J_\nu$ induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$. Using the above expression for $\tilde{R}$, the Codazzi equation becomes

$$(\nabla_X A) Y - (\nabla_Y A) X = \eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \xi + \sum_{\nu=1}^{3} \left\{ \eta_\nu(X) \phi_\nu Y - \eta_\nu(Y) \phi_\nu X - 2g(\phi_\nu X, Y) \xi_\nu \right\}$$

$$+ \sum_{\nu=1}^{3} \left\{ \eta_\nu(\phi X) \phi_\nu Y - \eta_\nu(\phi Y) \phi_\nu X \right\}$$

$$+ \sum_{\nu=1}^{3} \left\{ \eta(X) \eta_\nu(\phi Y) - \eta(Y) \eta_\nu(\phi X) \right\} \xi_\nu.$$
The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

\[
\begin{align*}
\phi_{\nu+1} \xi_{\nu} &= -\xi_{\nu+2}, \\
\phi_{\nu} \xi_{\nu+2} &= \xi_{\nu+1}, \\
\eta_{\nu}(\phi_{\nu} X) &= \eta(\phi_{\nu} X), \\
\phi_{\nu} \phi_{\nu+1} X &= \phi_{\nu+2} X + \eta_{\nu+1}(X) \xi_{\nu}, \\
\phi_{\nu+1} \phi_{\nu} X &= -\phi_{\nu+2} X + \eta_{\nu}(X) \xi_{\nu+1}.
\end{align*}
\]

(3.1)

Now let us put

\[
(3.2) \quad JX = \phi X + \eta(X) N, \quad J_{\nu} X = \phi_{\nu} X + \eta_{\nu}(X) N
\]

for any vector field \( X \) tangent to a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \), where \( N \) denotes a normal vector field of \( M \) in \( G_2(\mathbb{C}^{m+2}) \). Then from this and the formulas (2.1) and (3.1) we have that

\[
(3.3) \quad (\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y) \xi, \quad \nabla_X \xi = \phi AX,
\]

(3.4)

\[
(\nabla_X \phi_{\nu}) Y = -q_{\nu+1}(X) \phi_{\nu+2} Y + q_{\nu+2}(X) \phi_{\nu+1} Y + \eta_{\nu}(Y) AX
\]

(3.5)

Moreover, from \( JJ_{\nu} = J_{\nu} J \), \( \nu = 1, 2, 3 \), it follows that

\[
(3.6) \quad \phi \phi_{\nu} X = \phi_{\nu} \phi X + \eta_{\nu}(X) \xi - \eta(X) \xi_{\nu}.
\]

4. Recurrent normal Jacobi operator

From (2.2) the normal Jacobi operator \( \bar{R}_N \) of \( M \) is given by

\[
\bar{R}_N(X) = \bar{R}(X, N) N
\]

\[
= X + 3\eta(X) \xi + 3 \sum_{\nu=1}^{3} \eta_{\nu}(X) \xi_{\nu} \\
- \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\xi_{\nu})(\phi_{\nu} \phi X - \eta(X) \xi_{\nu}) - \eta_{\nu}(\phi_{\nu} \phi \xi) \right\}.
\]

Now let us consider the covariant derivative of the normal Jacobi operator \( \bar{R}_N \) along the direction \( X \) (see [9]). It is given by

\[
(\nabla_X \bar{R}_N) Y = 3g(\phi AX, Y) \xi + 3\eta(Y) \phi
\]

\[
+ 3 \sum_{\nu=1}^{3} \left\{ g(\phi_{\nu} AX, Y) \xi_{\nu} + \eta_{\nu}(Y) \phi_{\nu} AX \right\}
\]

\[
- \sum_{\nu=1}^{3} \left[ 2\eta_{\nu}(\phi AX) \phi_{\nu} \xi + \eta(\phi_{\nu} \phi Y - \eta(Y) \xi_{\nu}) - g(\phi_{\nu} AX, \phi Y) \phi_{\nu} \xi
\]

\[- \eta(Y) \eta_{\nu}(AX) \phi_{\nu} \xi - \eta_{\nu}(\phi Y)(\phi_{\nu} \phi AX - g(AX, \xi) \xi_{\nu}) \right].
\]
From this, together with formulas given in Section 3, a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \) with recurrent normal Jacobi operator satisfies the following

\[
3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX + 3\sum_{\nu=1}^{3} \left\{ g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \right\} \\
- \sum_{\nu=1}^{3} \left[ 2\eta_\nu(\phi AX)(\phi_\nu \phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu AX, \phi Y)\phi_\nu \xi \right] \\
(4.1) - \eta(Y)\eta_\nu(AX)\phi_\nu \xi - \eta_\nu(\phi Y)(\phi_\nu \phi AX - g(AX, \xi)\xi_\nu) \\
= \omega(X) \left[ Y + 3\eta(Y)\xi + 3\sum_{\nu=1}^{3} \eta_\nu(Y)\xi_\nu \\
- \sum_{\nu=1}^{3} \left\{ \eta_\nu(\phi_\nu \phi Y - \eta(Y)\xi_\nu) - \eta_\nu(\phi Y)\phi_\nu \xi \right\} \right].
\]

In order to prove our Main Theorem in the introduction, we give the following.

**Lemma 4.1.** Let \( M \) be a Hopf hypersurface in \( G_2(\mathbb{C}^{m+2}) \) with recurrent normal Jacobi operator. Then the Reeb vector field \( \xi \) belongs to either the distribution \( D \) or the distribution \( D^\perp \).

**Proof.** From (4.1), we take \( X = Y = \xi \) and suppose that \( M \) is Hopf, that is, \( A\xi = \alpha \xi \) for a certain function \( \alpha \). Then this yields

\[
\alpha \sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu \xi = \omega(\xi)(\xi + \sum_{\nu=1}^{3} \eta_\nu(\xi)\xi_\nu).
\]

Taking its scalar product with \( \xi \) we get \( \omega(\xi) = 0 \). As \( \omega(\xi) = 0 \), (4.2) yields

\[
\alpha \sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu \xi = 0.
\]

From this, we consider the case that the function \( \alpha \) is non-vanishing. Now let us put \( \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \) for some unit \( X_0 \in D \) and non-vanishing functions \( \eta(X_0) \) and \( \eta(\xi_1) \).

Then (4.3) yields

\[
0 = \eta(\xi_1)\phi_1 \xi = \eta(X_0)\eta(\xi_1)\phi_1 X_0.
\]

This gives a contradiction with \( \eta(\xi_1) \neq 0 \) and \( \eta(X_0) \neq 0 \). So we get \( \eta(\xi_1) = 0 \) or \( \eta(X_0) = 0 \), which means \( \xi \in D \) or \( \xi \in D^\perp \).

When the function \( \alpha \) vanishes, we can differentiate \( A\xi = 0 \). Then by a theorem due to Berndt and Suh [4] we know that

\[
\sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu \xi = 0.
\]

This also gives \( \xi \in D \) or \( \xi \in D^\perp \). \( \square \)
5. Recurrent normal Jacobi operator with $\xi \in \mathcal{D}$

In paper [14], Lee and Suh gave a characterization of real hypersurfaces of type B in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field $\xi \in \mathcal{D}$. Now we introduce the following:

Proposition 5.1. Let $M$ be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector field $\xi$ belongs to the distribution $\mathcal{D}$, then the distribution $\mathcal{D}$ is invariant under the shape operator $A$ of $M$, that is, $g(AD, \mathcal{D}^\perp) = 0$.

Then by Proposition 5.1 and Theorem A in the introduction, we know naturally that a Hopf hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ with recurrent normal Jacobi operator and $\xi \in \mathcal{D}$ is a tube over a totally geodesic quaternionic projective space $\mathbb{HP}^n$, $m = 2n$.

Now let us check if a real hypersurface of type (B) in $G_2(\mathbb{C}^{m+2})$, that is, a tube over a totally geodesic $\mathbb{HP}^n$, satisfies the notion of recurrent normal Jacobi operator. Corresponding to such a real hypersurface of type (B), we introduce a proposition due to Berndt and Suh [3] as follows:

Proposition 5.2. Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $AD \subset \mathcal{D}$, $A\xi = \alpha \xi$, and $\xi$ is tangent to $\mathcal{D}$. Then the quaternionic dimension $m$ of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and $M$ has five distinct constant principal curvatures

$$\alpha = -2 \tan(2r), \quad \beta = 2 \cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = J\mathbb{J}\xi, \quad T_\gamma = \mathbb{J}\xi, \quad T_\lambda, \quad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{HC}\xi)^\perp, \quad JT_\lambda = T_\lambda, \quad JT_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

Now let us suppose $M$ is of type (B) with recurrent normal Jacobi operator $\bar{R}_N$. From (4.1), by putting $X = \xi_2$ and $Y = \xi$ we have

$$\omega(\xi_2)\xi - \beta \phi_2 \xi = 0.$$

Then it follows that $\omega(\xi_2) = 0$ and $\beta = 0$. Since $\beta$ is not zero, this makes a contradiction. Thus we conclude the following:

Theorem 5.1. There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with recurrent normal Jacobi operator if the Reeb vector $\xi$ belongs to the distribution $\mathcal{D}$. 

6. Recurrent normal Jacobi operator with $\xi \in \mathcal{D}^\perp$

In this section, we consider the case that $\xi \in \mathcal{D}^\perp$. Then the unit normal vector field $N$ is a singular tangent vector of $G_2(C^{m+2})$ of type $JN \in \mathfrak{J}N$. So there exists an almost Hermitian structure $J_1 \in \mathfrak{J}$ such that $JN = J_1N$. Then we have

$$\xi = \xi_1, \ \phi_1 \xi_2 = -\xi_3, \ \phi_3 \xi_2 = \xi_2, \ \phi \mathcal{D} \subset \mathcal{D}.$$ 

Then, by putting $X = \xi_\mu$ and $Y = \xi$ in (4.1), we get

$$3\phi A\xi_\mu + 5\sum_{\nu=1}^3 \eta_\nu (\phi A\xi_\mu) \xi_\nu + 3\phi_1 A\xi_\mu + \sum_{\nu=1}^3 \eta_\nu (A\xi_\mu) \phi_\nu \xi = 8 \omega(\xi_\mu) \xi.$$ 

From this, by taking its scalar product with Reeb vector field $\xi$ we get $\omega(\xi_\mu) = 0$. As $\omega(\xi_\mu) = 0$, we have

$$0 = (\nabla_{\xi_\mu} \hat{R}_N)X$$

$$= 3g(\phi A\xi_\mu, X) \xi + 3\eta(X)\phi A\xi_\mu$$

$$+ 3\sum_{\nu=1}^3 \left\{ g(\phi_\nu A\xi_\mu, X) \xi_\nu + \eta_\nu (X)\phi_\nu A\xi_\mu \right\}$$

$$- 3\sum_{\nu=1}^3 \left[ 2\eta_\nu (\phi A\xi_\mu) (\phi_\nu \phi X - \eta(X) \xi_\nu) - g(\phi_\nu A\xi_\mu, \phi X) \phi_\nu \xi \right.$$ 

$$- \eta(X)\eta_\nu (A\xi_\mu) \phi_\nu \xi - \eta_\nu (\phi X) (\phi_\nu \phi A\xi_\mu - g(A\xi_\mu, \xi_\nu) \phi_\nu \xi) \left. \right]$$

for any $X \in TM$. From this, by putting $X = \xi$ and using $\xi = \xi_1$, we have

$$0 = 3\phi A\xi_\mu + 5\sum_{\nu=1}^3 \eta_\nu (\phi A\xi_\mu) \xi_\nu + 3\phi_1 A\xi_\mu + \sum_{\nu=1}^3 \eta_\nu (A\xi_\mu) \phi_\nu \xi.$$ 

From this, taking its inner product with $X \in \mathcal{D}$ and using $g(\phi_\nu, X, \xi) = 0$, we obtain

$$0 = 3g(\phi A\xi_\mu, X) + 3g(\phi_1 A\xi_\mu, X).$$

(6.1)

On the other hand, by using (3.4) we know that

$$\phi A\xi_\mu = \nabla_{\xi_\mu} \xi_1 = \nabla_{\xi_\mu} \xi_1 = \eta_1 (\xi_\mu) \xi_2 - \eta_2 (\xi_\mu) \xi_3 + \phi_1 A\xi_\mu.$$ 

From this, taking its inner product with $X \in \mathcal{D}$, we have

$$g(\phi A\xi_\mu, X) = g(\phi_1 A\xi_\mu, X).$$

Substituting this formula into (6.1) gives

$$0 = g(\phi A\xi_\mu, X).$$

From this, let us replace $X$ by $\phi X$. Then it follows that

$$0 = g(\phi A\xi_\mu, \phi X) = -g(A\xi_\mu, \phi^2 X) = g(AX, \xi_\mu)$$

for any vector field $X \in \mathcal{D}$, $\mu = 1, 2, 3$.

Then we obtain the following:
Lemma 6.1. Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with recurrent normal Jacobi operator and $\xi \in D^\perp$. Then $g(\mathcal{AD}, D^\perp) = 0$.

From this together with Theorem A in the introduction we know that any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with recurrent normal Jacobi operator $\bar{R}_N$ and $\xi \in D^\perp$ is congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now let us check if real hypersurfaces of type (A) satisfy the condition of recurrent normal Jacobi operator. Then we recall a proposition given by Berndt and Suh [3] as follows:

Proposition 6.1. Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\xi \subset D$, $A\xi = \alpha \xi$, and $\xi$ is tangent to $D^\perp$. Let $J_1 \in J$ be the almost Hermitian structure such that $JN = J_1N$. Then $M$ has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and as corresponding eigenspaces we have

$$T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1,$$
$$T_\beta = \mathbb{C}\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3,$$
$$T_\lambda = \{X | X \perp \mathbb{H}\xi, JX = J_1X\},$$
$$T_\mu = \{X | X \perp \mathbb{H}\xi, JX = -J_1X\},$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector $\xi$ and $\mathbb{C}^\perp \xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Now let us suppose $M$ is of type (A) with recurrent normal Jacobi operator $R_N$ and $\xi \in D^\perp$. From (4.1), by putting $X = \xi_2$ and $Y = \xi$ we have

$$8\omega(\xi_2)\xi - 6\beta\xi_1 = 0.$$ 

Then it follows that $\omega(\xi_2) = 0$ and $\beta = 0$. Since $\beta$ is not zero, this gives a contradiction. Thus we conclude the following:

Theorem 6.1. There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with recurrent normal Jacobi operator if the Reeb vector $\xi$ belongs to the distribution $D^\perp$.

Accordingly, by Lemma 4.1 together with Theorems 5.1 and 6.1 we give a complete proof of our Theorem 1.1 mentioned in the introduction.
References


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