A CLASS OF ARITHMETIC FUNCTIONS ON PSL₂(ℤ)

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Abstract. In [3] and [2], Atanassov introduced the two arithmetic functions

\[ I(n) = \prod_{p^\alpha \mid n} p^{1/\alpha} \] and \[ R(n) = \prod_{p^\alpha \mid n} p^{\alpha - 1} \]

called the irrational factor and the restrictive factor, respectively. Alkan, Ledoan, Panaitopol, and the authors explore properties of these arithmetic functions in [1], [7], [8] and [9]. In the present paper, we generalize these functions to a larger class of elements of PSL₂(ℤ), and explore some of the properties of these maps.

1. Introduction

In [3] and [2], Atanassov introduced the two arithmetic functions

\[ I(n) = \prod_{p^\alpha \mid n} p^{1/\alpha} \] and \[ R(n) = \prod_{p^\alpha \mid n} p^{\alpha - 1} \]

called the irrational factor and the strong restrictive factor, respectively. These functions are multiplicative, and satisfy the inequality

\[ I(n)R(n)^2 \geq n, \]

with equality if and only if \( n \) is squarefree. In [8], Panaitopol showed that

\[ \sum_{n=1}^{\infty} \frac{1}{I(n)R(n)\phi(n)} < e^2, \]

and proved that the function

\[ G(n) = \prod_{\nu=1}^{n} I(\nu)^{1/n} \]

satisfies the inequalities

\[ e^{-7/2}n < G(n) < n. \]
In [1], Alkan, Ledoan and one of the authors describe a precise asymptotic for the function $G(n)$, and establish further results showing that the function $I(n)$ is very regular on average.

In [7], asymptotic formulas are established for certain weighted real moments of the restrictive factor $R(n)$. In [9], the authors establish asymptotic formulas for weighted combinations $I(n)^\alpha R(n)^\beta$.

In the present paper we consider for a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\text{PSL}_2(\mathbb{Z})$ the fractional linear transformation $Az$ given by

$$Az = \frac{az + b}{cz + d}.$$ 

For each positive integer $n$, define

$$f_A(n) = \prod_{p^\alpha || n} p^{\frac{an + b}{cn + d}}.$$ 

As an example, the function $I(n)$ is equal to $f_A_0(n)$ for

$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

We shall consider weighted averages of the functions $f_A(n)$. Let

$$M_A(x) = \frac{1}{x} \sum_{n \leq x} \left(1 - \frac{n}{x}\right) f_A(n).$$

Consider the subset $\mathcal{A} \subset \text{PSL}_2(\mathbb{Z})$ given by

$$\mathcal{A} = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) : \det A = -1, a, b, d \geq 0, c \geq 1 \right\}.$$ 

Define for each positive rational number $r$

$$E_r = \{ A \in \mathcal{A} : M_A(x) \asymp x^r \text{ as } x \to \infty \}.$$ 

Note that if $r_1 \neq r_2$, then $E_{r_1} \cap E_{r_2} = \emptyset$. We will prove that each $E_r$ with $r > 0$ consists of exactly one element.

For each matrix $A$ in $\mathcal{A}$ we define the associated series $(A_n)_{n \in \mathbb{N}}$ by

$$A_n = A_n = \frac{an + b}{cn + d}.$$ 

As we shall see, the associated series plays an important role in our computations. Clearly, if $A \in \mathcal{A}$, then $A_n$ is monotone decreasing and has the finite limit $A_\infty := a/c$.

We have the following result.
Theorem 1.1. Given $A \in A$, if $A_1 > 0$, then there are positive real-valued constants $K_A$ and $c$ such that

$$M_A(x) = K_A x^{A_1} + O_A \left( x^{A_1 - 1/2} \exp\left\{ -c (\log x)^{3/5} (\log \log x)^{-1/5} \right\} \right).$$

We remark that under the Riemann hypothesis, for a restricted class of matrices one has an asymptotic formula for the error term in Theorem 1.1 of the form

$$M_A(x) - K_A x^{A_1} \sim \tilde{K}_A x^{1/2} (A_2 - 1)$$

for a real-valued constant $\tilde{K}_A$. This naturally leads one to consider the maps $\psi_j : A \to \mathbb{Q}^+$ for $j = 1, 2$ given by

$$\psi_j(A) = A_j.$$

Since, as mentioned above, each $E_r$ consists of exactly one element, it follows that there is a well-defined map $s : \mathbb{Q}^+ \to \mathbb{Q}^+$ given by

$$s(r) = \psi_2 \circ \psi_1^{-1}(r).$$

The map $s(r)$ tells us how accurately the main term $K_A x^{A_1}$ approximates $M_A(x)$ in (1), in the sense that it gives the exact order of magnitude of the error $M_A(x) - K_A x^{A_1}$.

Although it can be shown that this map is nowhere continuous, one can obtain asymptotic formulas for the average value of $s(r)$, with $r$ in various ranges. For example, define the height function for each rational $r = p/q$ with $q \geq 1$ and $(p,q) = 1$ by

$$h(r) := \max\{|p|, |q|\}.$$

We have the following result.

Theorem 1.2. For any $\delta > 0$,

$$\sum_{r \in \mathbb{Q}^+ \cap [0,1], h(r) \leq X} s(r) = \frac{3}{2\pi^2} X^2 + O_X(X^{11/6 + \delta}).$$

2. Asymptotics of the average

Consider the Dirichlet series

$$F_A(s) = \sum_{n=1}^{\infty} \frac{f_A(n)}{n^s}$$

We will take advantage of the meromorphic continuation of $F_A(s)$ in the case where $\det A = -1$.

Proof of Theorem 1.1. We prove the result with

$$K_A = \frac{1}{(1 + A_1)(2 + A_1)\zeta(2)} T_A(1 + A_1).$$
If det \( A = -1 \), then \( p^{A_\alpha} \leq p^{A_1} \) for all \( \alpha \geq 1 \), so \( f_A(n) \leq n^{A_1} \), hence \( F_A(s) \) converges in the half plane \( \Re s = \sigma > 1 + A_1 \). Moreover, \( F_A(s) \) has an Euler product in that region. Write

\[
F_A(s) = \frac{\zeta(s - A_1)}{\zeta(2s - 2A_1)} \prod_p (1 + g_p(s)),
\]

where

\[
g_p(s) = \left( 1 + \frac{p^{A_1}}{p^s} \right)^{-1} \sum_{k=2}^{\infty} \frac{p^{A_k}}{p^{ks}}.
\]

Note that if det(\( A \)) = \( -1 \), then \( A_1 - A_2 = \frac{1}{(c+d)(2c+d)} \leq \frac{1}{2} \). Take \( \epsilon > 0 \). For \( \sigma \geq A_1 + \epsilon \) we have

\[
\left( 1 + \frac{p^{A_1}}{p^s} \right)^{-1} \ll \epsilon.
\]

Also, for \( \sigma \geq \frac{1}{2}(1 + A_2 + \epsilon) \) we have

\[
\sum_{k=2}^{\infty} \frac{p^{A_k}}{p^{ks}} \ll \epsilon \frac{1}{p^{1+\epsilon}}.
\]

Thus for \( \sigma \geq \max\{A_1 + \epsilon, \frac{1}{2}(1 + A_2 + \epsilon)\} \) the sum \( \sum_p |g_p(s)| \) converges, hence

\[
T_A(s) = \prod_p (1 + g_p(s))
\]

is analytic for \( \sigma > \sigma_0 = \max\{A_1, \frac{1}{2}(1 + A_2)\} \), so \( F_A(s) \) is meromorphic there, with a poles at \( s = 1 + A_1 \).

To continue, we utilize a variant of Perron’s formula and write

\[
\sum_{n \leq x} \left( 1 - \frac{n}{x} \right) f_A(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s - A_1)}{\zeta(2s - 2A_1)} T_A(s) \frac{x^s}{s(s+1)} \, ds,
\]

where \( 1 + A_1 < c \leq 5/4 + A_1 \).

We apply the zero-free region for \( \zeta(s) \) due to Korobov [6] and Vinogradov [12] (see Chapters 2 and 5 of the reference by Walfisz [13] for an alternative treatment)

\[
\sigma \geq 1 - c_0 (\log t)^{-2/3} (\log \log t)^{-1/3}
\]

for \( t \geq t_0 \), in which

\[
\frac{1}{|\zeta(s)|} \ll (\log t)^{2/3}(\log \log t)^{1/3}.
\]

Fix \( 0 < U < T \leq x \), let \( \nu = 1/2 + A_1 \) and

\[
\eta = \nu - c_0 (\log U)^{-2/3}(\log \log U)^{-1/3}.
\]
Deform the path of integration into the union of the line segments
\[
\begin{aligned}
\gamma_1, \gamma_9 : s = c + it & \quad \text{if } |t| \geq T \\
\gamma_2, \gamma_8 : s = \sigma + iT & \quad \text{if } \nu \leq \sigma \leq c \\
\gamma_3, \gamma_7 : s = \nu + it & \quad \text{if } U \leq |t| \leq T \\
\gamma_4, \gamma_6 : s = \nu + iT & \quad \text{if } \eta \leq \sigma \leq \nu \\
\gamma_5 : s = \eta + it & \quad \text{if } |t| \leq U.
\end{aligned}
\]

The integrand is analytic on and within this modified contour, hence by Cauchy’s theorem
\[
xMA(x) = \frac{1}{(1 + A_1)(2 + A_1)\zeta(2)} T \sum_{k=1}^{9} J_k,
\]
with the main terms coming from the residue at the simple pole at \( s = 1 + A_1 \).

In order to estimate the integral along our modified contour we will make use of the bounds
\[
|\zeta(s + it)| = \begin{cases} O(t^{1-\sigma}/2), & \text{if } 0 \leq \sigma \leq 1 \text{ and } |t| \geq 1 \\ O(\log t), & \text{if } 1 \leq \sigma \leq 2 \\ O(1), & \text{if } \sigma \geq 2 \end{cases}
\]
(see [11], §3.11 and §5.1).

On the line segments on which \( s = c + it, |t| \geq T \), we have that \( \zeta(s - A_1) \ll \log t \) and \( 1/\zeta(2s - 2A_1) \ll \log t \), so
\[
|J_1|, |J_9| \ll \int_T^\infty (\log t)^2 (c + it)(c + 1 + it) dt \ll \frac{x^c}{\log T}.
\]

On the line segments on which \( s = \sigma + iT, \nu \leq \sigma \leq c \), we have that
\[
1/\zeta(2s - 2A_1) \ll \log T, \quad \zeta(s - A_1) \ll T^{1-\sigma+A_1/2} \quad \text{for } \nu \leq \sigma \leq 1 + A_1, \quad \text{and} \quad \zeta(s - A_1) \ll \log T \text{ for } 1 + A_1 \leq \sigma \leq c.
\]
So
\[
|J_2|, |J_8| \ll \int_\nu^{1+A_1} T^{1/2(1-\sigma+A_1)} \log T \frac{x^\sigma}{T^2} d\sigma + \int_{1+A_1}^c (\log T)^{3/2} \frac{x^\sigma}{T^2} d\sigma \ll T^{1/2(1+A_1)} \log T \max \left\{ \left( \frac{x}{\sqrt{T}} \right)^\nu, \left( \frac{x}{\sqrt{T}} \right)^{1+A_1} \right\} + (\log T)^2 x^c.
\]

On the line segments on which \( s = \nu + it, U \leq |t| \leq T \), we have that \( \zeta(s - A_1) \ll t^{1-\nu+A_1/2} \) and \( 1/\zeta(2s - 2A_1) \ll \log t \), so
\[
|J_4|, |J_7| \ll \int_U^T (\log t)t^{1/2(1-\nu+A_1)} \frac{x^\nu}{(\nu + it)(\nu + 1 + it)} dt \ll \frac{\log T}{U^{3/4}} x^\nu.
\]
On the line segments on which \( s = \sigma + iU, \eta \leq \sigma \leq \nu, \) we have that \( \zeta(s - A_1) \ll U^{(1 - \sigma + A_1)/2} \) and \( 1/\zeta(2s - 2A_1) \ll \log U, \) so

\[
|J_4|, |J_6| \ll \int_\eta^{\nu} (\log U)U^{\frac{1}{2}(1 - \sigma + A_1)}x^\sigma \sqrt{d\sigma} \ll U^{\frac{1}{2}(1 + A_1)^{-2}} \log U \max \left\{ \left( \frac{x}{\sqrt{U}} \right)^\nu, \left( \frac{x}{\eta} \right)^\eta \right\}.
\]

On the line segment on which \( s = \eta + it, \) \(|t| \leq U, \) we have that \( \zeta(s - A_1) \ll (|t| + 1)^{(1 - \eta + A_1)/2} \) and \( 1/\zeta(2s - 2A_1) \ll \log(|t| + 1), \) so

\[
|J_5| \ll \int_{-U}^{U} (|t| + 1)^{1 - \eta + A_1} \log(|t| + 1) x^\eta \frac{dt}{|\eta + it||\eta + 1 + it|} \ll x^\eta \int_{-U}^{U} (|t| + 1)^{\frac{1}{2}(1 - \eta + A_1)^{-2}} \log(|t| + 1) \, dt.
\]

Since \( \frac{1}{2}(1 - \eta + A_1) - 2 \leq -\frac{3}{2} \) for \( U \) sufficiently large, the above integral converges, hence \( |J_5| \ll x^\eta. \)

We collect all estimates, and take \( T = x^2 \) and \( U = \exp\{c_2(\log x)^{3/5}(\log \log x)^{-1/5}\} \)

to obtain the desired result. \( \square \)

One could instead factor

\[
\left( 1 + \frac{p^{A_1}}{p^s} + \frac{p^{A_2}}{p^{2s}} + \frac{p^{A_3}}{p^{3s}} + \cdots \right) = \left( 1 + \frac{p^{A_1}}{p^s} \right) \left( 1 + \frac{p^{A_2}}{p^{2s}} \right) (1 + g_p(s))
\]

with

\[
g_p(s) = \left( 1 + \frac{p^{A_1}}{p^s} \right)^{-1} \left( 1 + \frac{p^{A_2}}{p^{2s}} \right)^{-1} \left( -\frac{p^{A_1 + A_2}}{p^{3s}} + \sum_{k=3}^{\infty} \frac{p^{A_k}}{p^{ks}} \right)
\]

so that

\[
F_A(s) = \frac{\zeta(s - A_1)}{\zeta(2s - 2A_1)} \frac{\zeta(2s - A_2)}{\zeta(4s - 2A_2)} \prod_p (1 + g_p(s)).
\]

Under the Riemann hypothesis, we get a second order term of the form \( \tilde{K}_{A_1, A_2} \) in the asymptotic formula for \( F_A(s) \) provided that \( \frac{1}{4} + A_1 < \frac{1}{2}(1 + A_2). \) That is, provided that

\[
a + b < \frac{c + d}{2} - \frac{1}{2c + d}.
\]

This occurs for matrices \( A \) in \( \mathcal{A} \) with restrictions on \( c \) and \( d. \) One can see that \( A_1 \) will lie in the interval \((0, 1/2)\).
3. Mapping through PSL₂(𝐙)

We now return to the two maps ψ₁ and ψ₂ defined in (2).

**Lemma 3.1.** The map ψ₁ is bijective.

**Proof.** For $\frac{p}{q} \in \mathbb{Q}_+$, consider the set of matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathcal{A}$ such that $\psi_1(A) = \frac{p}{q}$. We note that any such quadruple $(a, b, c, d)$ is constrained by $c \geq 0, d \geq 0$,

(4) \quad ad - bc = -1

and

(5) \quad c + d = \frac{q}{p}(a + b)

(Note that (4) implies that $p$ cannot be zero). By (5) we have

$$c = \frac{q}{p}(a + b) - d.$$

Inserting this into (4) gives us

$$ad - b(a + b)\frac{q}{p} + bd = -1$$

so

$$(a + b)(pd - qb) = -p.$$

Write $a + b = \pm n$ for some positive integer $n \mid p$. By (5) we have $c + d = \frac{\pm n}{p} \in \mathbb{Z}$ so $p \mid n$, hence $p = n$.

There are two cases: If $a + b = -p$, then $c + d = -q$. This contradicts the assumptions that $q \geq 1$ and $c$ and $d$ are non-negative. On the other hand, if $a + b = p$, then $c + d = q$, so (4) gives us

$$a(q - c) - bc = -1$$

so

(6) \quad pc = 1 + aq.

So $c$ is uniquely determined by $cp \equiv 1 \pmod{q}$ and $1 \leq c < q$. Then $d$ is uniquely determined by $d = q - c$, and $a$ and $b$ by $a = \frac{1 - pc}{q}$ and $b = p - a$. □

In the case where $p/q \in (0, 1]$, we identify $p/q$ as an element of $\mathcal{F}_Q$, the Farey fractions of order $Q$, with $Q \geq q$. If we consider the “minimal” set of Farey fractions $\mathcal{F}_q$ containing $p/q$, then elementary properties of Farey fractions (see for example Chapter 3 of [5]) give that the adjacent Farey fractions $p'/q' < p/q < p''/q''$ satisfy $q' = \bar{p}$, $p' = \bar{q}$, $p'' = p - \bar{q}$ and $q'' = q - \bar{p}$. Here $\bar{p}$ is the
unique integer $1 \leq \bar{p} < q$ satisfying $p\bar{p} \equiv 1 \pmod{q}$ and $\bar{q}$ is the unique integer $1 \leq \bar{q} < p$ satisfying $q\bar{q} \equiv 1 \pmod{p}$. We can write

$$\psi_1(p/q) = \left( \frac{q}{\bar{p}}, \frac{p - \bar{q}}{\bar{p}} \right).$$

That is, the matrix $\psi_1(p/q)$ is comprised of the “parent” Farey fractions in $F_{q-1}$.

Additionally, we can write the function $s(p/q)$ from (3) uniquely as

$$(7) \quad s(p/q) = \frac{\bar{p}p - 1 + pq}{q(\bar{p} + q)}.$$  

To prove Theorem 1.2, we will use the following result (see Lemma 2.3 of [4]).

**Lemma 3.2.** Assume that $q \geq 1$ and $h$ are two given integers, $\mathcal{I}$ and $\mathcal{J}$ are intervals of length less than $q$, and $f : \mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R}$ is a $C^1$ function. Then for any integer $T \geq 1$ and any $\delta > 0$

$$\sum_{a \in \mathcal{I}, b \in \mathcal{J}} f(a, b) = \frac{\phi(q)}{q^2} \int_{[1/q, 1]^2} f(x, y) dxdy + \mathcal{E},$$

with

$$\mathcal{E} \ll T^2\|f\|_\infty q^{1/2+\delta} \text{gcd}(h, q)^{1/2} + T\|\nabla f\|_\infty q^{3/2+\delta} \text{gcd}(h, q)^{1/2} + \frac{||\nabla f||_\infty ||\mathcal{I}||\mathcal{J}|}{T},$$

where $||f||_\infty$ and $||\nabla f||_\infty$ denote the sup-norm of $f$ and respectively $|\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}|$ on $\mathcal{I} \times \mathcal{J}$.

**Proof of Theorem 1.2.** Let $Q = \lfloor X \rfloor$. Since $r \in F_Q$ we have

$$\sum_{r \in Q \cap [0, 1]} s(r) = \sum_{1 \leq p < q \leq Q, (p, q) = 1} s(p/q).$$

We use (7) and Lemma 3.2 with $T = \frac{q^2}{4}$ to get that the right-hand sum is equal to

$$\sum_{1 \leq q \leq Q} \sum_{1 \leq p < q \leq Q, pp \equiv 1 \pmod{q}} \frac{\bar{p}p - 1 + pq}{q(\bar{p} + q)} = \sum_{1 \leq q \leq Q} \frac{\phi(q)}{q^2} \int_{[1/q, 1]^2} \frac{vu - 1 + uq}{q(v + q)} dvdu + \mathcal{E}$$

$$= \sum_{1 \leq q \leq Q} \phi(q) \int_{[1/q, 1]^2} \frac{xy - \frac{1}{2} + x}{y + 1} dxdy + \mathcal{E}.$$
where $E \ll \delta q^{5/6 + \delta}$. The integral is equal to
\[
\frac{1}{2} \left( 1 - \frac{1}{q^2} \right) \left( 1 - \frac{1}{q} \right) - \frac{q - 1}{q^2} \left( \log 2 - \log \left( 1 + \frac{1}{q} \right) \right) = \frac{1}{2} + O \left( \frac{1}{q} \right)
\]
so
\[
\sum_{r \in \mathbb{Q}^+ \cap [0,1]} h(r) = \frac{1}{2} \sum_{1 \leq q \leq Q} \phi(q) + O \left( \sum_{1 \leq q \leq Q} \frac{\phi(q)}{q} \right) + O \left( \sum_{1 \leq q \leq Q} q^{5/6 + \delta} \right).
\]
One can use the methods of Section 2 to estimate the sums over $\phi(q)$, or use partial summation along with standard estimates (see for example [13] or Chapter 18 of [5]). This gives the main term of our theorem; the first error term above is $O(X)$, and the second is $O(X^{11/6 + \delta})$. □

References

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