ON FINSLER METRICS OF CONSTANT S-CURVATURE

XIAOHUAN MO AND XIAOYANG WANG

ABSTRACT. In this paper, we study Finsler metrics of constant S-curvature. First we produce infinitely many Randers metrics with non-zero (constant) S-curvature which have vanishing H-curvature. They are counterexamples to Theorem 1.2 in [20]. Then we show that the existence of (α, β)-metrics with arbitrary constant S-curvature in each dimension which is not Randers type by extending Li-Shen' construction.

1. Introduction

The S-curvature is one of most important non-Riemannian quantities in Finsler geometry [15]. It vanishes on a Riemannian manifold. So we call it non-Riemannian quantity.

In fact, all Berwald manifolds have zero S-curvature [15]. Locally Minkowski manifolds and Riemannian manifolds are all Berwald manifolds.

An n-dimensional Finsler metric $F$ on a manifold $M$ is said to have constant S-curvature if $S(x, y) = (n + 1)cF(x, y)$ for some constant $c$. For example, the following Finsler metric $F$ on the unit ball has constant S-curvature $S = \pm \frac{1}{2}(n + 1)\bar{F}$,

$$F = \sqrt{|y|^2 - \frac{\langle x, y \rangle^2}{1 - |x|^2}} \pm \frac{\langle x, y \rangle}{1 - |x|^2} \pm \frac{\langle a, y \rangle}{1 + \langle a, x \rangle}, \quad y \in T_x\mathbb{R}^n,$$

where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$ [3, 16]. Randers metric of constant flag curvature (or R-quadratic [8]) is of constant S-curvature [2]. Recently, S. Ohta shows that a Randers space $(M, F)$ admits a measure $m$ with $S \equiv 0$ if and only if $\beta$ is a Killing form of constant length [13]. Shen and Mo-Yu established some global rigidity theorems for Finsler manifolds with constant S-curvature [18, 11].

The aim of this paper is to study a special class of Finsler metrics $-(\alpha, \beta)$-metrics of constant S-curvature. Finsler metrics in the form $F := \alpha\phi(\frac{\beta}{2})$ are called $(\alpha, \beta)$-metrics (for definition, see Section 2). In particular, when
\( \phi(s) = 1 + s, \) \( F = \alpha + \beta \) is called a Randers metrics [14]. We first produce infinitely many Randers metrics with non-zero (constant) \( S \)-curvature which have vanishing \( H \)-curvature (see Theorem 5.1). They are counterexamples to Theorem 1.2 in [20]. Note that \( H \)-curvature is another interesting non-Riemannian quantity and it is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. Meanwhile Theorem 5.1 means that there exists a Randers metric of any constant \( S \)-curvature in each dimension. After noting this interesting fact, we investigate the existence of non-Randers \((\alpha, \beta)\)-metrics with arbitrary constant \( S \)-curvature. By extending Li-Shen’ construction [7] we prove the following:

**Theorem 1.1.** For arbitrary real number \( k \) and arbitrary natural number \( n \), there exists an \((\alpha, \beta)\)-metric \( F \) defined on an open subset in \( \mathbb{R}^n \) which is not Randers type such that \( F \) has constant \( S \)-curvature \( k \).

The above theorem tells us that Finsler metrics of constant \( S \)-curvature form a rich class of Finsler metrics. For interesting results of \( H \)-curvature, we refer the reader to [9, 12, 19].

### 2. Preliminaries

A Finsler metric is a Riemannian metric without quadratic restriction. Precisely, a function \( F(x, y) \) on \( TM \) is called a Finsler metric on a manifold \( M \) if it has the following properties:

- (a) \( F(x, y) \) is \( C^\infty \) on \( TM \setminus \{0\} \);
- (b) \( F_x(y) := F(x, y) \) is a Minkowski norm on \( T_x M \) for any \( x \in M \).

Define the (mean) distortion \( \tau : SM \to \mathbb{R} \) by [15]

\[
\tau(x, [y]) := \ln \frac{\sqrt{\det (g_{ij}(y))}}{\sigma(x)},
\]

where \( SM \) is the projective sphere bundle of \( M \), obtained from \( TM \) by identifying non-zero vectors which differ from each other by a positive multiplicative factor and

\[
\sigma(x) = \frac{\text{Vol}(B^n)}{\text{Vol}\{y^i \in \mathbb{R}^n | F(x, y^i \frac{\partial}{\partial x^i}) < 1\}},
\]

where \( B^n \) denotes the unit ball in \( \mathbb{R}^n \) and \( \text{Vol} \) denotes the Euclidean measure on \( \mathbb{R}^n \). To measure the rate of changes of the distortion along geodesics, we define

\[
S(x, y) := \frac{d}{dt} \tau(\dot{c}(t))|_{t=0},
\]

where \( \dot{c}(t) \) is the geodesic with \( \dot{c}(0) = y \). We call the scalar function \( S \) the \( S \)-curvature. \( S \) is said to be isotropic if there is a scalar function \( c(x) \) on \( M \) such that

\[
S(x, y) = (n + 1)c(x)F(x, y).
\]

In particular, \( S \) is said to be of constant \( c \) if \( c = \) constant.
In [10], the authors constructed many new examples of Finsler manifolds of isotropic S-curvature.

Let \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) be a Riemannian metric and \( \beta = b_i(x)y^i \) be a 1-form on a manifold \( M \). Consider the following function

\[
F := \alpha \phi(s), \quad s = \frac{\beta}{\alpha},
\]

where \( \phi = \phi(s) \) is a positive \( C^\infty \) function on \([-r, r]\) satisfying

\[
\phi(s) - s\phi'(s) > 0, \quad \phi''(s) > 0, \quad |s| \leq r.
\]

Then \( F \) is a Finsler metric if \( \| \beta_x \|_\alpha \leq r \) for any \( x \in M \) [17]. A Finsler metric in the form (2.1) is called an \((\alpha, \beta)\)-metric.

Let \( r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}), \quad r_i := r_{ij}b^j, \quad s_i := s_{ij}b^j, \) where \( b_{ij} \) denote covariant derivative of \( \beta \) with respect to \( \alpha \).

For a positive \( C^\infty \) function \( \phi = \phi(s) \) on \([-r, r]\) and a number \( b \in [0, r] \), let

\[
\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'',
\]

where

\[
\Delta := 1 + sQ + (b^2 - s^2)Q', \quad Q := \frac{\phi'}{\phi - s\phi'}.
\]

Recently, Cheng-Shen proved the following [4]:

**Theorem 2.1.** Let \( F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha}, \) be an \((\alpha, \beta)\)-metric on a manifold and \( b := \| \beta_x \|_\alpha \). Suppose that \( \phi \) is not Randers type. Then \( F \) is of isotropic S-curvature if and only if one of the following holds

(i) \( \beta \) satisfies

\[
r_j + s_j = 0
\]

and \( \phi = \phi(s) \) satisfies

\[
\Phi = 0.
\]

In this case, \( S = 0 \).

(ii) \( \beta \) satisfies

\[
r_{ij} = \epsilon(b^2a_{ij} - b_ib_j), \quad s_j = 0,
\]

where \( \epsilon = \epsilon(x) \) is a scalar function, and \( \phi = \phi(s) \) satisfies

\[
\Phi = -2(n + 1)k\frac{\phi\Delta^2}{b^2 - s^2},
\]

where \( k \) is a constant. In this case, \( S = (n + 1)cF \) with \( c = k\epsilon \).

(iii) \( \beta \) satisfies

\[
r_{ij} = 0, \quad s_j = 0.
\]

In this case, \( S = 0 \), regardless of the choice of a particular \( \phi \).
By using Theorem 2.1, we will show that the existence of \((\alpha, \beta)\)-metrics with arbitrary constant \(S\)-curvature in each dimension which is not Randers type in Section 4. We are going to simplify the equation (2.5) in the next section.

The \(H\)-curvature \(H_y = H_{ij} dx^i \otimes dx^j\) is defined by \(H_{ij} = E_{ikl} y^k\) where “\(|\)” denote the covariant horizontal derivatives and \(E_{ij}\) denote the mean Berwald curvature of \(F\) [9, 12].

3. Third order nonlinear ODE

In this section we are going to give the normal type of (2.5).

Lemma 3.1. Let \(\psi := \phi - s\phi'\). Then we have

\[
(3.1) \quad 1 + sQ = \frac{\phi}{\psi}.
\]

\[
(3.2) \quad Q = \frac{\phi'}{\psi}.
\]

\[
(3.3) \quad Q' = \frac{\phi \phi''}{\psi^2}.
\]

\[
(3.4) \quad Q'' = \frac{1}{\psi^3} \left[ (\phi' \phi'' + \phi \phi''') \psi + 2s\phi \phi''^2 \right],
\]

where \(Q\) is given in (2.3).

Proof. (3.2) is obvious. By using (3.2) we obtain

\[
1 + sQ = 1 + s\frac{\phi'}{\psi} = \frac{1}{\psi} (\psi + s\phi') = \frac{\phi}{\psi}.
\]

This gives (3.1). From the definition of \(\psi\), one get \(\psi' = -s\phi''\). Together with (3.2) we get

\[
Q' = \frac{\phi'' \psi - \phi' \psi'}{\psi^2} = \frac{\phi'' (\phi - s\phi') - \phi' (-s\phi'')}{\psi^2} = \frac{\phi \phi''}{\psi^2}
\]

which implies (3.3). By a similar calculation, we get

\[
Q'' = \frac{(\phi' \phi'' + \phi \phi''') \psi^2 - 2\phi \phi'' \psi \psi'}{\psi^4} = \frac{1}{\psi^3} \left[ (\phi' \phi'' + \phi \phi''') \psi + 2s\phi \phi''^2 \right].
\]

Lemma 3.2. We have the following

\[
(3.5) \quad \Delta = \phi \frac{\psi + (b^2 - s^2)\phi''}{\psi^2},
\]

\[
(3.6) \quad Q - sQ' = \frac{\phi \phi' - s(\phi'^2 + \phi \phi'')}{\psi^2},
\]

\[
(3.7) \quad n\Delta + 1 + sQ = \frac{\phi}{\psi} \left( n + 1 + n\phi' b^2 - s^2 \right).
\]
Proof. By using (2.3), (3.1) and (3.3) we have
\[ \Delta = 1 + sQ + (b^2 - s^2)Q' = \frac{\phi'}{\psi} + (b^2 - s^2)\frac{\phi''}{\psi'^2} = \phi \cdot \frac{\psi + (b^2 - s^2)\phi''}{\psi^2}. \]
From (3.2), (3.3) and the definition of \( \psi \), we get
\[ Q - sQ' = \frac{\phi'}{\psi} - s\frac{\phi''}{\psi'^2} = \frac{\phi'(\phi - s\phi') - s\phi\phi''}{\psi'^2} = \frac{\phi\phi' - s(\phi'^2 + \phi\phi'')}{\psi'^2}. \]
Finally, we have
\[ n\Delta + 1 + sQ = n\phi \frac{\psi}{\psi^2} + (b^2 - s^2)\phi'' + \frac{\phi}{\psi} = \frac{\phi}{\psi} \left[ n\psi + (b^2 - s^2)\phi'' + 1 \right] = \frac{\phi}{\psi} \left( n + 1 + n\phi' + (b^2 - s^2) \right) \]
from (3.1) and (3.5).

\[ \text{Lemma 3.3.} \quad \text{Equation (2.2) can be rewritten as follows:} \]
\[ \Phi = -\frac{\phi}{\psi^2} \left[ \phi\phi' - s(\phi'^2 + \phi\phi'') \right] \left[ (n + 1)\psi + n\phi''(b^2 - s^2) \right] \]
\[ -\frac{\phi}{\psi^2}(b^2 - s^2) \left[ (\phi'\phi'' + \phi\phi''')\psi + 2s\phi(\phi'')^2 \right]. \]

Proof. Substituting (3.6), (3.7), (3.1) and (3.4) into (2.2) we have (3.8).

\[ \text{Lemma 3.4.} \quad \text{Equation (2.5) is equivalent to} \]
\[ 2k(n + 1)\phi^3 \left[ (b^2 - s^2)\phi'' + 2\psi\phi'' + \frac{\psi^2}{b^2 - s^2} \right] \]
\[ = \left[ \phi\phi' - s(\phi'^2 + \phi\phi'') \right] \left[ (n + 1)\psi + n\phi''(b^2 - s^2) \right] \]
\[ + (b^2 - s^2) \left[ (\phi'\phi'' + \phi\phi''')\psi + 2s\phi(\phi'')^2 \right]. \]

Proof. Plugging (3.5) and (3.8) into (2.5) yields (3.9).

From Lemma 3.4 we immediately obtain the following:

\[ \text{Lemma 3.5.} \quad \text{Equation (3.9) is equivalent to the following normal ODE:} \]
\[ \phi''' = 2k(n + 1)\phi \left[ \frac{\phi''^2}{\psi} + \frac{2\phi''}{b^2 - s^2} + \frac{\psi}{(b^2 - s^2)^2} \right] - \frac{\phi\phi''}{\psi} - 2s\phi'^2 \]
\[ \text{Remark. It is easy to see that } \phi = k_1\sqrt{1 + k_2 s^2} + k_3 s \text{ (it corresponds the Randers metrics) are not the solution of (3.10).} \]
4. Proof of Theorem 1.1

Now we are going to construct Riemannian metric $\alpha$ and 1-form $\beta$ satisfy (2.4) with $\epsilon = \text{constant}$. If, at a point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ and in the direction $y = (y^1, \ldots, y^n) \in T_x \mathbb{R}^n$, Riemannian metric $\alpha = \alpha(x, y)$ and one form $\beta = \beta(x, y)$ are given by

$$\alpha := \sqrt{(y^1)^2 + e^{2x^1}(y^2)^2 + \cdots + (y^n)^2}, \quad \beta := y^1.$$  

Then

$$a_{ij} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{2x^1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2x^1} \end{pmatrix},$$

$$b_1 = 1, \ b_2 = \cdots = b_n = 0,$

where $\alpha^2 = a_{ij} y^i y^j$ and $\beta = b_i y^i$. It follows that

$$a^{ij} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{-2x^1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-2x^1} \end{pmatrix}.$$

By using (4.3) and (4.4), we obtain

$$b = \sqrt{a^{ij} b_i b_j} = \sqrt{a^{11} b_1^2} = 1.$$

From (4.3) we have

$$\frac{\partial b_i}{\partial x^j} = 0.$$

It follows that the covariant derivatives of $\beta$ with respect to $\alpha$ are given by

$$b_{ilj} = \frac{\partial b_i}{\partial x^j} - b_k \Gamma^k_{lj} = -b_k \Gamma^i_{lj} = b_{lij}.$$

Together with (4.3) we get

$$r_{ij} = \frac{1}{2}(b_{ilj} + b_{lij}) = b_{lij} = -b_k \Gamma^k_{lj} = -\Gamma^l_{lj}$$

and $s_{ij} = \frac{1}{2}(b_{lij} - b_{lij}) = 0$. By (4.2) and (4.4), the Christoffel symbols of $\alpha$ are given by

$$\Gamma^k_{lj} = \frac{1}{2} a^{kl} \left( \frac{\partial a_{lj}}{\partial x^i} + \frac{\partial a_{li}}{\partial x^j} - \frac{\partial a_{ij}}{\partial x^l} \right) = \frac{1}{2} a^{kk} \left( \frac{\partial a_{ik}}{\partial x^j} + \frac{\partial a_{jk}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^k} \right)$$

$$= \begin{cases} -e^{2x^1} & \text{if } i = j \neq k = 1, \\ 1 & \text{if } i = k \neq j = 1, j = k \neq i = 1, \\ 0 & \text{others}. \end{cases}$$
Together with (4.2), (4.3), (4.5) and (4.6) we get
\[ r_{ij} = b^2 a_{ij} - b^2 b_j = \begin{cases} e^{2x^1} & \text{if } i = j = 2, \ldots, n, \\ 0 & \text{others}. \end{cases} \]

Hence \( \alpha \) and \( \beta \) satisfy
\[ r_{ij} = \epsilon(b^2 a_{ij} - b^2 b_j), \quad s_j = 0 \]
with \( \epsilon = b = 1 \).

Remark. When \( n = 3 \), our construction have been studied by Li-Shen [7].

Now we are going to show the existence of regular solution of (2.5) for arbitrary \( k \in \mathbb{R} \) when \( \alpha \) and \( \beta \) are given by (4.1).

Let \( k \) be an arbitrary constant. We consider the solution of (2.5). By Lemma 3.4 and Lemma 3.5, (2.5) is equivalent to 3-order nonlinear ODE (3.10). Put
\[ \phi_0 := \phi, \quad \phi_1 := \phi', \quad \phi_2 := \phi''. \]

One can express (3.10) in the following form
\[ \phi'_0 := \phi_1, \quad \phi'_1 := \phi_2, \]
\[ \phi'_2 = 2k(n+1)\phi_0 \left[ \frac{\phi_2^2}{\phi_0 - s\phi_1} + \frac{2\phi_2 + \phi_0 - s\phi_1}{1 - s^2} \right] - \frac{\phi_1\phi_2}{\phi_0} - \frac{2s\phi_2^2}{\phi_0 - s\phi_1} \]
\[ := f(s, \phi_0, \phi_1, \phi_2). \]

Let \( \Omega := (-1, 1) \times \left[ \frac{N}{2}, \frac{3N}{2} \right] \times [0, 2\epsilon] \times [0, 2\tau] \) where \( N > 4\epsilon \). Then
\[ \phi_0 - s\phi_1 \geq \frac{N}{2} - 2\epsilon > 0, \quad \phi_2 \geq 0 \]
for \( (s, \phi_0, \phi_1, \phi_2) \in \Omega \). Consider the following 3th order system
\[ y' = F(s, y), \]
where
\[ \phi_0, \quad \phi_1, \quad \phi_2 \]
\[ (s, \phi_0, \phi_1, \phi_2) \]
\[ F(s, \phi_0, \phi_1, \phi_2) = \left( \frac{\phi_1}{\phi_2}, \frac{\phi_1^2}{\phi_2}, f(s, \phi_0, \phi_1, \phi_2) \right). \]

From (4.8) and (4.11), we can expand \( F(s, \phi_0, \phi_1, \phi_2) \) into convergence power series of \( s, \phi_0 - N, \phi_1 - \epsilon \) and \( \phi_2 - \tau \). By using the Cauchy theorem, there exists an analytic solution \( y^*(s) \), defined uniquely in \( \Omega \) which satisfies \( y^*(0) = (N, \epsilon, \tau) \) (cf. [6]). Put
\[ y^*(s) = \left( \begin{array}{c} \phi_0^*(s) \\ \phi_1^*(s) \\ \phi_2^*(s) \end{array} \right). \]
Then \( \phi^*(s) := \phi_0^*(s) \) is an analytic solution of (2.5), which is defined in \((-1,1)\) and satisfies \( \phi^*(0) = N \). Note that
\[
\phi^{**}(s) = \phi_1^*(s), \quad \phi^{***}(s) = \phi_2^*(s)
\]
we get \((s, \phi^*(s), \phi^{**}(s), \phi^{***}(s)) \in \Omega\). Together with (4.11) we obtain
\[
\phi^*(s) - s\phi^{**}(s) > 0, \quad \phi^{***}(s) \geq 0, \quad |s| < 1.
\]
It follows that \( \alpha\phi^*(\beta/\alpha) \) is an \((\alpha, \beta)\)-metric where \( \alpha \) and \( \beta \) is defined in (4.1).

5. Counterexamples to Tang’s Theorem 1.2

In this section, we are going to manufacture Randers metrics with non-zero \( S \)-curvature which have zero \( H \)-curvature. For a Finsler manifold \((M, F)\), the flag curvature is a function \( K(P, y) \) of tangent planes \( P \subset T_xM \) and directions \( y \in P \). \( F \) is said to be of scalar curvature if the flag curvature \( K(P, y) = K(x, y) \) is independent of flags \( P \) associated with any fixed flagpole \( y \) [5]. In particular, \( F \) is said to be of constant flag curvature if the flag curvature \( K(P, y) = \text{constant} \) [16]. By a basic result of Arbar-Zadeh [1, 12] for a Finsler metric of scalar flag curvature, the flag curvature is constant on the manifold if and only if \( H = 0 \).

**Theorem 5.1.** Let \( h = |y| \) be the Euclidean metric on \( \mathbb{R}^n \), and \( V \) be a vector field on \( \mathbb{R}^n \) given by
\[
V_x := -2cx + xQ + b,
\]
where \( c \) is a constant, \( Q \) is a skew-symmetric matrix and \( b \) is a constant vector with \( |b| < 1 \). Then Finsler metric \( F \) is determined by
\[
F(x, y) = h(x, y - F(x, y)V_x)
\]
is a Randers metric which has the following non-Riemannian curvature properties:

(a) vanishing \( H \)-curvature
\[
H = 0.
\]

(b) constant \( S \)-curvature
\[
S(x, y) = (n + 1)cF(x, y).
\]

**Proof.** (b) is an immediate conclusion of Theorem 7.3.8 in [5]. On the other hand, Chern-Shen’ result tells us \( F \) has constant flag curvature. Combining this with Arbar-Zadeh’ result yields (a). \( \square \)

Let us take a look at the special case when \( c \neq 0 \), \( F \) is a Randers metric with non-zero \( S \)-curvature which have zero \( H \)-curvature. Thus \( F \) is a counterexample to Theorem 1.2 in [20].
ON FINSLER METRICS OF CONSTANT $S$-CURVATURE

References


XIAOHUAN MO
KEY LABORATORY OF PURE AND APPLIED MATHEMATICS
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING, 100871, P. R. CHINA
E-mail address: moxh@pku.edu.cn
XIAOHUAN MO AND XIAOYANG WANG

XIAOYANG WANG
School of Mathematical Sciences
Beijing Institute of Technology
Beijing 100081, P. R. China
E-mail address: wxy314159@126.com