INTERPOLATIONS FOR HÖLDER’S INEQUALITY

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Abstract. Kwon and Bae gave an interpolation for a continuous form of Hölder’s inequality for a real-valued bounded measurable function on a product of measure spaces. It is given some generalizations of their result.

1. Introduction

For real numbers $A$ and $B$ satisfying $A \leq B$, a continuous and increasing function $h$ on the open interval $(0,1)$ is said to be an interpolation for $A \leq B$ if

$$\lim_{t \to 0} h(t) = A \quad \text{and} \quad \lim_{t \to 1} h(t) = B.$$ 

Let $(X, \mu)$ be a measure space with $\mu(X) = 1$ and $f \in L^1(\mu)$ satisfy $f > 0$ on $X$. By [4, p. 71],

$$h(t) := \left( \int_X f^t \, d\mu \right)^{1/t}, \quad 0 < t \leq 1,$$

is an interpolation for the inequality

$$\exp \int_X \log f \, d\mu \leq \int_X f \, d\mu.$$ 

Let $X_{ij}$ and $p_j$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, be positive numbers satisfying

$$\sum_{j=1}^n \frac{1}{p_j} = 1.$$ 

The inequality

$$(1.1) \quad \sum_{i=1}^m \prod_{j=1}^n X_{ij} \leq \prod_{j=1}^n \left( \sum_{i=1}^m X_{ij}^{p_j} \right)^{1/p_j}.$$
is well known as the classical Hölder inequality [1]. In [5], Yang gave an interpolation for Hölder’s inequality as follows.

**Theorem A.** A function

\[ h(t) := \prod_{k=1}^{n} \left( \sum_{i=1}^{m} \left( \prod_{j=1}^{n} X_{ij} \right)^{1-t} X_{ik}^{tp_k} \right)^{\frac{1}{p_k}}, \quad 0 \leq t \leq 1, \]

is an interpolation for Hölder’s inequality given in (1.1), and \( h \) is continuous on \([0, 1]\).

Let \((X, \mu)\) and \((Y, \nu)\) be \(\sigma\)-finite measure spaces with positive measures \(\mu, \nu\) and \(\mu(X) = 1\). Let \((X \times Y, \mu \times \nu)\) be their product measure space. In [2], Kwon gave a continuous form of Hölder’s inequality as follows.

**Theorem B.** Let \(\mu(X) = 1\) and \(f \in L^1(\mu \times \nu)\) satisfy \(f(x, y) > 0\) on \(X \times Y\). Then

\[
\int_Y \exp \left( \int_X \log f(x, y) \, d\mu(x) \right) \, d\nu(y) \leq \exp \left( \int_X \log \left( \int_Y f(x, y) \, d\nu(y) \right) \, d\mu(x) \right).
\]

Equality holds in (1.2) as a nonzero valued if and only if \(f(x, y) = g(x)h(y)\) for \(\mu \times \nu\)-a.e. on \(X \times Y\) with \(-\infty < \int_X \log g \, d\mu\).

It may happen that \(\log f \notin L^1(\mu \times \nu)\); in that case, \(\int_X \log f(x, y) \, d\mu(x)\) exists in the extended sense. Since \(f \in L^1(\mu \times \nu)\),

\[-\infty \leq \int_X \log f(x, y) \, d\mu(x) < \infty\]

for \(\nu\)-a.e. on \(Y\). We consider that \(\exp(-\infty) = 0\) and \(\log 0 = -\infty\).

Let \(\mu(X) = 1\) and \(f \in L^1(\mu \times \nu)\) satisfy \(f > 0\) on \(X \times Y\). Then

\[-\infty \leq \int_X \log f(z, y) \, d\mu(z) \leq \int_X f(z, y) \, d\mu(z) < \infty.\]

For \(0 \leq t < 1\), we define

\[
g(t, x, y) := f^t(x, y) \exp \left( (1-t) \int_X \log f(z, y) \, d\mu(z) \right)
\]

and

\[
h(t) := \exp \left( \int_X \log \left( \int_Y g(t, x, y) \, d\nu(y) \right) \, d\mu(x) \right).
\]

When \(f = f_n\), we write \(g_n(t, x, y)\) and \(h_n(t)\). In [3], Kwon and Bae gave an interpolation for the inequality given in (1.2) as follows.
Theorem C. Let \( \mu(X) = 1 \), \( 0 < \nu(Y) < \infty \) and \( f \) be a measurable function on \( X \times Y \) satisfying \( 0 < \delta \leq f(x, y) \leq M < \infty \) for some positive numbers \( \delta \) and \( M \). In this case, we may define \( h \) on \([0,1]\) and \( h \) is an interpolation for the inequality given in (1.2), and \( h \) is a continuous and convex function on \([0,1]\).

In this paper, we shall study Theorem C and relax the assumption \( "0 < \delta \leq f(x, y) \leq M < \infty" \).

2. Generalizations of Kwon and Bae’s result

Let \( \mu(X) = 1 \) and \( f \in L^1(\mu) \) satisfy \( f \geq 0 \) a.e. on \( X \). Then Jensen’s inequality [4, Theorem 3.3] says that

\[
\exp \int_X \log f \, d\mu \leq \int_X f \, d\mu.
\]

Lemma 2.1. For \( a, b > 0 \) and \( 0 \leq t \leq 1 \), we have \( a^t b^{1-t} \leq a + b \).

Proof. We have

\[
a^t b^{1-t} = \left( \frac{a}{b} \right)^t b \leq \max\{a, b\} \leq a + b. \tag*{□}
\]

Lemma 2.2. Let \( \mu(X) = 1 \) and \( f \in L^1(\mu \times \nu) \) satisfy \( f > 0 \) on \( X \times Y \). Then for \( 0 \leq t < 1 \), we have the following.

(i) \( 0 \leq g(t, x, y) = f^t(x, y) \exp \left( (1 - t) \int_X \log f(z, y) \, d\mu(z) \right) \)

\[
\leq f^t(x, y) \exp \left( (1 - t) \log \int_X f(z, y) \, d\mu(z) \right) \text{ by Jensen’s inequality}
\]

\[
= f^t(x, y) \left( \int_X f(z, y) \, d\mu(z) \right)^{1-t}
\]

\[
\leq f(x, y) + \int_X f(z, y) \, d\mu(z) \text{ by Lemma 2.1}
\]

\[
\in L^1(\mu \times \nu).
\]

The second inequality follows from Jensen’s inequality.

(ii) By (i), we have

\[
\log \int_Y g(t, x, y) \, d\nu(y) \leq \int_Y g(t, x, y) \, d\nu(y)
\]

\[
\leq \int_Y f(x, y) \, d\nu(y) + \int_{X \times Y} f \, d\mu \times \nu \in L^1(\mu). \tag*{□}
\]
Theorem 2.3. Let \( \mu(X) = \nu(Y) = 1 \) and \( f \in L^1(\mu \times \nu) \) satisfy \( \delta \leq f \) on \( X \times Y \) for some positive number \( \delta \). Then \( h \) is an interpolation for the inequality given in (1.2). Moreover \( h(1) \) is well defined and \( h \) is a continuous and convex function on [0, 1].

Proof. We may assume that \( 0 < \delta < 1 \). For each positive integer \( n \), let
\[
f_n(x, y) := \min\{f(x, y), n\}.
\]
Then \( 0 < \delta \leq f_n \leq n \) and \( f_n \uparrow f \) as \( n \to \infty \) on \( X \times Y \). By Theorem C, \( h_n \) is an interpolation for the inequality
\[
\int_Y \exp \left( \int_X \log f_n \, d\mu \right) \, d\nu \leq \exp \left[ \int_X \log \left( \int_Y f_n \, d\nu \right) \, d\mu \right]
\]
and, \( h_n \) is a continuous and convex function on [0, 1]. Letting \( n \to \infty \), by the Lebesgue monotone convergence theorem (we write LMCT in short) we have
\[
\int_Y \exp \left( \int_X \log f_n \, d\mu \right) \, d\nu \to \int_Y \exp \left( \int_X \log f \, d\mu \right) \, d\nu,
\]
\[
\exp \left[ \int_X \log \left( \int_Y f_n \, d\nu \right) \, d\mu \right] \to \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right]
\]
and
\[
h_n(t) \to h(t), \quad n \to \infty
\]
for \( 0 \leq t \leq 1 \). Since \( h_n \) is a convex and increasing function, \( h \) is also a convex and increasing function on [0, 1]. By [4, Theorem 3.2], \( h \) is a continuous function on [0, 1].

Since \( \mu(X) = 1 \), we have
\[
h(0) = \exp \left[ \int_X \log \left( \int_Y \exp \left( \int_X \log f(z, y) \, d\mu(z) \right) \, d\nu(y) \right) \, d\mu(x) \right]
\]
\[
= \int_Y \exp \left( \int_X \log f(z, y) \, d\mu(z) \right) \, d\nu(y).
\]
Since \( \delta \leq f \in L^1(\mu \times \nu) \), we have
\[
-\infty < \log \delta \leq \int_X \log f(z, y) \, d\mu(z) \leq \int_X f(z, y) \, d\mu(z) < \infty
\]
for \( \nu \)-a.e. on \( Y \). Hence
\[
h(1) = \exp \left[ \int_X \log \left( \int_Y f(x, y) \, d\nu(y) \right) \, d\mu(x) \right]
\]
and
\[
\int_Y \exp \left( \int_X \log f \, d\mu \right) \, d\nu = h(0) \leq h(t) \leq \lim_{t \to 1} h(t) \leq h(1)
\]
\[
= \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right].
\]
To finish the proof, it is sufficient to show that
\[ (2.1) \quad \lim_{t \to 1} h(t) = \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right]. \]

Since
\[ 0 \leq g(t, x, y) = f'(x, y) \exp \left( (1 - t) \int_X \log f(z, y) \, d\mu(z) \right) \to f(x, y) \]
as \( t \to 1 \), by Lemma 2.2(i) and the Lebesgue dominated convergence theorem (we write LDCT in short) we have
\[ \int_Y g(t, x, y) \, d\nu(y) \to \int_Y f(x, y) \, d\nu(y), \]

so
\[ \log \int_Y g(t, x, y) \, d\nu(y) \to \log \int_Y f(x, y) \, d\nu(y) \]
as \( t \to 1 \) for \( \mu \)-a.e. on \( X \). Since \( \delta = \delta^t \delta^{1-t} \leq g(t, x, y) \),
\[-\infty < \log \delta \leq \log \int_Y g(t, x, y) \, \nu(y). \]

Hence by Lemma 2.2(ii) and LDCT, we get (2.1).

**Theorem 2.4.** Let \( \mu(X) = \nu(Y) = 1 \) and \( f \in L^1(\mu \times \nu) \) satisfy \( f > 0 \) on \( X \times Y \). Then \( h \) is a continuous, convex and increasing function on \([0, 1)\), and
\[ \int_Y \exp \left( \int_X \log f \, d\mu \right) \, d\nu = h(0) \leq h(t) \leq \lim_{t \to 1} h(t) \leq \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right]. \]

**Proof.** For each positive integer \( N \), let
\[ f_N(x, y) = \max \left\{ f(x, y), \frac{1}{N} \right\}. \]

Then \( 0 < 1/N \leq f_N \in L^1(\mu \times \nu) \) and \( f_N \downarrow f \) as \( N \to \infty \) on \( X \times Y \). By Theorem 2.3, \( h_N \) is an interpolation for the inequality
\[ \int_Y \exp \left( \int_X \log f_N \, d\mu \right) \, d\nu \leq \exp \left[ \int_X \log \left( \int_Y f_N \, d\nu \right) \, d\mu \right], \]

and \( h_N \) is a continuous and convex function on \([0, 1] \). Since \( \log f_N \downarrow \log f \) and \( \log f_N \leq \log f_1 \leq f_1 \in L^1(\mu \times \nu) \), by LMCT we have
\[ \exp \left( \int_X \log f_N \, d\mu \right) \downarrow \exp \left( \int_X \log f \, d\mu \right). \]

By Jensen’s inequality,
\[ \exp \left( \int_X \log f_N(x, y) \, d\mu(x) \right) \leq \int_X f_N(x, y) \, d\mu(x) \leq \int_X f_1(x, y) \, d\mu(x) \in L^1(\nu). \]
Hence by LMCT, we have
\[
\int_Y \exp \left( \int_X \log f_N \, d\mu \right) \, d\nu \to \int_Y \exp \left( \int_X \log f \, d\mu \right) \, d\nu.
\]
Since
\[
\log \left( \int_Y f_N \, d\nu \right) \downarrow \log \left( \int_Y f \, d\nu \right)
\]
and
\[
\log \int_Y f_N \, d\nu \leq \int_Y f_N \, d\nu \leq \int_Y f_1 \, d\nu \in L^1(\mu),
\]
by LMCT we have
\[
\int_X \log \left( \int_Y f_N \, d\nu \right) \, d\mu \to \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu,
\]
so
\[
h_N(1) = \exp \left[ \int_X \log \left( \int_Y f_N \, d\nu \right) \, d\mu \right] \to \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right].
\]
Let fix \( t \) with \( 0 \leq t < 1 \). By (1.3), we have
\[
0 < g_N(t, x, y) \downarrow g(t, x, y), \quad N \to \infty.
\]
By Lemma 2.2(i) and LMCT, we have
\[
\log \int_Y g_N(t, x, y) \, d\nu(y) \downarrow \log \int_Y g(t, x, y) \, d\nu(y).
\]
By Lemma 2.2(ii),
\[
\log \int_Y g_N(t, x, y) \, d\nu(y) \leq \int_Y f_N(x, y) \, d\nu(y) + \int_{X \times Y} f_N \, d\mu \times \nu
\]
\[
\leq \int_Y f_1(x, y) \, d\nu(y) + \int_{X \times Y} f_1 \, d\mu \times \nu
\]
\[
\in L^1(\mu).
\]
By (1.4) and LMCT, we have \( h_N(t) \to h(t) \) as \( N \to \infty \) for \( 0 \leq t < 1 \). Then \( h \) is a convex and increasing function on \([0,1]\), so \( h \) is continuous on \([0,1]\). We note that
\[
h(0) = \int_Y \exp \left( \int_X \log f \, d\mu \right) \, d\nu.
\]
Since
\[
h_N(t) \leq h_N(1) = \exp \left[ \int_X \log \left( \int_Y f_N \, d\nu \right) \, d\mu \right],
\]
we have
\[
h(t) \leq \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right], \quad 0 \leq t < 1.
\]
Therefore we get our assertion. \( \square \)
Suppose that $0 < \nu(Y) < \infty$. We have

$$\int_Y \exp \left( \int_X \log f \, d\mu \right) \frac{d\nu}{\nu(Y)} = \frac{1}{\nu(Y)} \int_Y \exp \left( \int_X \log f \, d\mu \right) d\nu,$$

and

$$\exp \left[ \int_X \log \left( \int_Y f \, \frac{d\nu}{\nu(Y)} \right) d\mu \right] = \frac{1}{\nu(Y)} \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) d\mu \right].$$

By Theorem 2.4, we have the following.

**Corollary 2.5.** Let $\mu(X) = 1$, $0 < \nu(Y) < \infty$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. Then $h$ is a continuous, convex and increasing function on $[0,1)$, and

$$\int_Y \exp \left( \int_X \log f \, d\mu \right) d\nu = h(0) \leq h(t) \leq \lim_{t \to 1} h(t) \leq \exp \left( \int_X \log \left( \int_Y f \, d\nu \right) d\mu \right).$$

Next, we shall study the case of $\nu(Y) = \infty$.

**Theorem 2.6.** Let $\mu(X) = 1$, $0 < \nu(Y) \leq \infty$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. Then $h$ is a continuous, convex and increasing function on $[0,1)$, and

$$\int_Y \exp \left( \int_X \log f \, d\mu \right) d\nu = h(0) \leq h(t) \leq \lim_{t \to 1} h(t) \leq \exp \left( \int_X \log \left( \int_Y f \, d\nu \right) d\mu \right).$$

**Proof.** By Corollary 2.5, it is sufficient to show the case of $\nu(Y) = \infty$.

First, suppose that

$$\int_Y \exp \left( \int_X \log f \, d\mu \right) d\nu = 0.$$

Hence

$$\int_X \log f(z, y) \, d\mu(z) = -\infty$$

$\nu$-a.e. on $Y$. So by (1.3), $g(t, x, y) = 0$ for $\mu \times \nu$-a.e. on $X \times Y$ and $0 \leq t < 1$.

Thus by (1.4), we get $h(t) = 0$ for $0 \leq t < 1$ and the assertion.

Next, suppose that

$$0 < \int_Y \exp \left( \int_X \log f \, d\mu \right) d\nu.$$

By Jensen’s inequality,

$$0 < \int_Y \exp \left( \int_X \log f \, d\mu \right) d\nu \leq \int_{X \times Y} f \, d\mu \times \nu < \infty.$$
Since $\nu$ is a $\sigma$-finite measure, there is a sequence of measurable subsets $\{Y_n\}_n$ of $Y$ such that $Y_n \subset Y_{n+1}$, $\nu(Y_n) < \infty$ for every $n \geq 1$, $Y = \bigcup_{n=1}^{\infty} Y_n$ and by (2.2),

$$(2.3) \quad 0 < \int_{Y_1} \exp \left( \int_X \log f \, d\mu \right) \, d\nu.$$ 

Let

$$h_n(t) := \exp \left[ \int_X \log \left( \int_{Y_n} g(t, x, y) \, d\nu(y) \right) \, d\mu(x) \right], \quad 0 \leq t < 1.$$ 

By Corollary 2.5, $h_n$ is a continuous, convex and increasing function on $[0, 1)$, and

$$(2.4) \quad \int_{Y_n} \exp \left( \int_X \log f \, d\mu \right) \, d\nu = h_n(0) \leq h_n(t) \leq \lim_{t \to 1} h_n(t) \leq \exp \left[ \int_X \log \left( \int_{Y_n} f \, d\nu \right) \, d\mu \right] < \infty.$$ 

For each $y \in Y$, we have

$$0 \leq \chi_{Y_n}(y) \exp \left( \int_X \log f(z, y) \, d\mu(z) \right) \leq \exp \left( \int_X \log f(z, y) \, d\mu(z) \right).$$

By LMCT,

$$\int_{Y_n} \exp \left( \int_X \log f \, d\mu \right) \, d\nu \uparrow \int_Y \exp \left( \int_X \log f \, d\mu \right) \, d\nu.$$ 

By (2.3) and (2.4),

$$\log \int_{Y_1} f \, d\nu \in L^1(\mu).$$

We have

$$\log \int_Y f \, d\nu \leq \log \int_{Y_n} f \, d\nu \uparrow \log \int_Y f \, d\nu$$

for $\mu$-a.e. on $X$. By LMCT, we have

$$\exp \left[ \int_X \log \left( \int_{Y_n} f \, d\nu \right) \, d\mu \right] \uparrow \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right].$$

In the similar way, we obtain

$$h_n(t) \uparrow h(t), \quad 0 \leq t < 1.$$ 

By (2.4), $h$ is a continuous, convex and increasing function on $[0, 1)$, and

$$\int_Y \exp \left( \int_X \log f \, d\mu \right) \, d\nu = h(0) \leq h(t) \leq \lim_{t \to 1} h(t) \leq \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right].$$

$\square$
3. Interpolations for Hölder’s inequality

Let $\mu(X) = 1$, $0 < \nu(Y) \leq \infty$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. If
$$\exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right] = 0,$$
then by Theorem 2.6, trivially $h$ is an interpolation for (1.2). Since
$$\log \int_Y f \, d\nu \leq \int_Y f \, d\nu \in L^1(\mu),$$
if $\log \int_Y f \, d\nu \notin L^1(\mu)$, then
$$\exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right] = 0,$$
so we assume that
$$\log \int_Y f \, d\nu \in L^1(\mu).$$

Let
$$Y_\infty := \{ y \in Y : \int_X \log f(z,y) \, d\mu(z) \neq -\infty \}.$$
Suppose that $0 < \nu(Y \setminus Y_\infty)$. Then for $0 \leq t < 1$, we have
$$\int_Y \exp \left( \int_X \log f \, d\mu \right) \, d\nu = \int_{Y_\infty} \exp \left( \int_X \log f \, d\mu \right) \, d\nu,$$
$$\int_Y f^t \exp \left( (1-t) \int_X \log f \, d\mu \right) \, d\nu = \int_{Y_\infty} f^t \exp \left( (1-t) \int_X \log f \, d\mu \right) \, d\nu,$$
$$h(t) = \exp \left[ \int_X \log \left( \int_{Y_\infty} g(t,x,y) \, d\nu \right) \, d\mu \right]$$
and by (3.1),
$$\exp \left[ \int_X \log \left( \int_{Y_\infty} f \, d\nu \right) \, d\mu \right] < \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right].$$
By Theorem 2.6, we have the following.

**Corollary 3.1.** Let $\mu(X) = 1$, $0 < \nu(Y) \leq \infty$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. If (3.1) holds and $0 < \nu(Y \setminus Y_\infty)$, then
$$\int_Y \exp \left( \int_X \log f \, d\mu \right) \, d\nu = h(0) \leq h(t) \leq \lim_{t \to 1} h(t)$$
$$\leq \exp \left[ \int_X \log \left( \int_{Y_\infty} f \, d\nu \right) \, d\mu \right] < \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right].$$
So $h$ is not an interpolation for the inequality given in (1.2).

Hence for the study of interpolations, we may assume that
$$\nu(Y \setminus Y_\infty) = 0.$$
Lemma 3.2. Let $\mu(X) = 1$, $0 < \nu(Y) \leq \infty$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. If (3.2) holds and there is a function $F(x) \in L^1(\mu)$ such that
\[
F(x) \leq \log \int_Y g(t, x, y) \, d\nu(y)
\]
for every $0 \leq t < 1$, then $h$ is an interpolation for Hölder’s inequality given in (1.2), and $h$ is a convex function on $[0, 1)$.

Proof. By Theorem 2.6, it is sufficient to show that
\[
\lim_{t \to 1} h(t) = \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right].
\]
By (3.2),
\[
-\infty < \int_X \log f(z, y) \, d\mu(z) < \infty
\]
for $\nu$-a.e. on $Y$. Hence by (1.3),
\[
0 \leq g(t, x, y) \to f(x, y), \quad t \to 1
\]
for $\mu \times \nu$-a.e. on $X \times Y$. By Lemma 2.2(i) and LDCT, we have
\[
\log \int_Y g(t, x, y) \, d\nu(y) \to \log \int_Y f(x, y) \, d\nu(y), \quad t \to 1.
\]
By Lemma 2.2(ii), the assumption and LDCT, we get
\[
h(t) = \exp \left[ \int_X \log \left( \int_Y g(t, x, y) \, d\nu(y) \right) \, d\mu(x) \right]
\to \exp \left[ \int_X \log \left( \int_Y f \, d\nu \right) \, d\mu \right], \quad t \to 1.
\]
\[
\square
\]

Theorem 3.3. Let $\mu(X) = 1$ and $0 < \nu(Y) \leq \infty$. Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be real valued functions in $L^1(\mu)$ and $\psi_1, \psi_2, \ldots, \psi_n$ be real valued measurable functions on $Y$. Let
\[
f(x, y) = \exp \sum_{j=1}^n \varphi_j(x) \psi_j(y).
\]
Suppose that $f \in L^1(\mu \times \nu)$. Then $h$ is an interpolation for Hölder’s inequality given in (1.2), and $h$ is a convex function on $[0, 1)$.

Proof. We have
\[
\int_X \log f(z, y) \, d\mu(z) = \sum_{j=1}^n \psi_j(y) \int_X \varphi_j(z) \, d\mu(z).
\]
Hence (3.2) holds and by (1.3),
\[
g(t, x, y) = \exp \left( t \sum_{j=1}^n \varphi_j(x) \psi_j(y) + (1 - t) \sum_{j=1}^n \psi_j(y) \int_X \varphi_j \, d\mu \right)
\]
\[
= \exp \left( \sum_{j=1}^{n} \psi_j(y) \left( t \varphi_j(x) + (1 - t) \int_{X} \varphi_j \, d\mu \right) \right)
\geq \exp \left( - \sum_{j=1}^{n} |\psi_j(y)||\varphi_j(x)| + \int_{X} |\varphi_j| \, d\mu \right).
\]

Hence for \(0 < \lambda < \infty\), we have
\[
\int_{Y} g(t, x, y) \, d\nu(y)
\geq \int_{Y} \exp \left( - \sum_{j=1}^{n} |\psi_j(y)||\varphi_j(x)| + \int_{X} |\varphi_j| \, d\mu \right) \, d\nu(y)
\geq \int_{\bigcap_{j=1}^{n} \{|\psi_j| \leq \lambda\}} \exp \left( - \sum_{j=1}^{n} |\psi_j(y)||\varphi_j(x)| + \int_{X} |\varphi_j| \, d\mu \right) \, d\nu(y)
\geq \nu \left( \bigcap_{j=1}^{n} \{|\psi_j| \leq \lambda\} \right) \times \exp \left( - \lambda \sum_{j=1}^{n} (|\varphi_j(x)| + \int_{X} |\varphi_j| \, d\mu) \right).
\]

Since \(f \in L^1(\mu \times \nu)\) and \(\varphi_j \in L^1(\mu)\) for \(1 \leq j \leq n\), by Lemma 2.2(ii) we have
\[
\nu \left( \bigcap_{j=1}^{n} \{|\psi_j| \leq \lambda\} \right) < \infty.
\]

Since \(\psi_1, \psi_2, \ldots, \psi_n\) are real valued measurable functions on \(Y\),
\[
0 < \nu \left( \bigcap_{j=1}^{n} \{|\psi_j| \leq \lambda\} \right)
\]
for some \(0 < \lambda < \infty\). Therefore
\[
\log \int_{Y} g(t, x, y) \, d\nu(y)
\geq \log \nu \left( \bigcap_{j=1}^{n} \{|\psi_j| \leq \lambda\} \right) - \lambda \sum_{j=1}^{n} (|\varphi_j(x)| + \int_{X} |\varphi_j| \, d\mu)
\in L^1(\mu) \quad \text{by the assumption.}
\]

By Lemma 3.2, we get the assertion. \(\square\)

**Corollary 3.4.** Let \(\mu(X) = 1\), \(0 < \nu(Y) \leq \infty\) and \(f \in L^1(\mu \times \nu)\) satisfy \(f > 0\) on \(X \times Y\). Suppose that there are real valued functions \(\varphi_1, \varphi_2, \ldots, \varphi_n \in L^1(\mu)\) and real valued measurable functions \(\psi_1, \psi_2, \ldots, \psi_n\) on \(Y\) such that
\[
\sum_{j=1}^{n} \varphi_j(x) \psi_j(y) \leq \log f(x, y).
\]

Then \(h\) is an interpolation for Hölder’s inequality given in (1.2), and \(h\) is a convex function on \([0, 1]\).


Proof. We have
\[ \exp \sum_{j=1}^{n} \varphi_j(x) \psi_j(y) \leq f(x, y) \]
and
\[ \sum_{j=1}^{n} \psi_j(y) \int_X \varphi_j(z) d\mu(z) \leq \int_X \log f(z, y) d\mu(z). \]
Hence
\[ \log \int_Y \exp \left( \sum_{j=1}^{n} \psi_j(y) \left( t \varphi_j(x) + (1-t) \int_X \varphi_j d\mu \right) \right) d\nu(y) \leq \log \int_Y g(t, x, y) d\nu(y). \]
By the proof of Theorem 3.3, there exists \( F(x) \in L^1(\mu) \) satisfying
\[ F(x) \leq \log \int_Y \exp \left( \sum_{j=1}^{n} \psi_j(y) \left( t \varphi_j(x) + (1-t) \int_X \varphi_j d\mu \right) \right) d\nu(y) \]
for every \( 0 \leq t < 1 \). By Lemma 3.2, we get the assertion. \( \square \)

Example 3.5. Let \( X = \{1, 2\} \) and \( 0 < \lambda < 1 \). Let \( \mu \) be the measure on \( X \) satisfying \( \mu(\{1\}) = \lambda \) and \( \mu(\{2\}) = 1 - \lambda \). Let \( \nu \) be a \( \sigma \)-finite positive measure on \( Y \). Let
\[ f(x, y) = e^{\chi_{\{1\}} \psi_1(y)} + \chi_{\{2\}} \psi_2(y) \]
for real valued measurable functions \( \psi_1, \psi_2 \) on \( Y \). Then we have
\[ \int_Y \exp \left( \int_X f d\mu \right) d\nu = \int_Y (e^{\psi_1})^\lambda (e^{\psi_2})^{1-\lambda} d\nu \]
and
\[ \exp \left[ \int_X \log \left( \int_Y f d\nu \right) d\mu \right] = \left( \int_Y e^{\psi_1} d\nu \right)^\lambda \left( \int_Y e^{\psi_2} d\nu \right)^{1-\lambda}. \]
We have also that \( f \in L^1(\mu \times \nu) \) if and only if \( e^{\psi_1}, e^{\psi_2} \in L^1(\nu) \). Under the condition that \( e^{\psi_1}, e^{\psi_2} \in L^1(\nu) \), for \( 0 \leq t < 1 \) by Theorem 3.3
\[ h(t) = \left( \int_Y e^{t \psi_1 + (1-t)(\lambda \psi_1 + (1-\lambda) \psi_2)} d\nu \right)^\lambda \left( \int_Y e^{t \psi_2 + (1-t)(\lambda \psi_1 + (1-\lambda) \psi_2)} d\nu \right)^{1-\lambda} \]
is an interpolation for Hölder’s inequality;
\[ \int_Y (e^{\psi_1})^\lambda (e^{\psi_2})^{1-\lambda} d\nu \leq \left( \int_Y e^{\psi_1} d\nu \right)^\lambda \left( \int_Y e^{\psi_2} d\nu \right)^{1-\lambda}. \]
In the case of \( 0 < \nu(Y) < \infty \), we have another condition on \( f \) for the interpolation.

Proposition 3.6. Let \( \mu(X) = 1, 0 < \nu(Y) < \infty \) and \( f \in L^1(\mu \times \nu) \) satisfy \( f > 0 \) on \( X \times Y \) and \( \log f \in L^1(\mu \times \nu) \). Then \( h \) is an interpolation for the inequality given in (1.2), and \( h \) is a convex function on \([0, 1)\).
Proof. By Fubini’s theorem, 
\[ \int_X \log f(z, y) \, d\mu(z) \in L^1(\nu), \]
so (3.2) holds. By Jensen’s inequality, we have
\[
\log \int_Y g(t, x, y) \, d\nu(y) \geq \int_Y \left( t \log f(x, y) + (1 - t) \int_X \log f(z, y) \, d\mu(z) \right) \frac{d\nu(y)}{\nu(Y)}
\]
\[
= t \int_Y \log f(x, y) \, d\nu(Y) + (1 - t) \int_{X \times Y} \log f \, d\frac{d\mu \times \nu}{\nu(Y)}
\]
\[
\geq - \int_Y |\log f(x, y)| \, d\nu(Y) - \int_{X \times Y} |\log f| \, d\frac{d\mu \times \nu}{\nu(Y)}
\]
\[\in L^1(\mu). \]

By Lemma 3.2, we get the assertion. \( \square \)

In the case of \( \nu(Y) = \infty \), Proposition 3.6 does not have meaning. For, there are no functions \( f \in L^1(\mu \times \nu) \) satisfying \( f > 0 \) on \( X \times Y \) and \( \log f \in L^1(\mu \times \nu) \). If \( f \in L^1(\mu \times \nu) \), then
\[ (\mu \times \nu)(\{ f \geq 1/2 \}) < \infty, \]
so
\[ (\mu \times \nu)(\{ f < 1/2 \}) = \infty. \]

Hence
\[ - \int_{\{ f < 1/2 \}} \log f \, d\mu \times \nu > (\log 2)(\mu \times \nu)(\{ f < 1/2 \}) = \infty. \]

Therefore \( \log f \notin L^1(\mu \times \nu) \).

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