SEMI-SLANT SUBMERSIONS

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ABSTRACT. We introduce semi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds as a generalization of slant submersions, semi-invariant submersions, anti-invariant submersions, etc. We obtain characterizations, investigate the integrability of distributions and the geometry of foliations, etc. We also find a condition for such submersions to be harmonic. Moreover, we give lots of examples.

1. Introduction

Let $F$ be a $C^\infty$-submersion from a semi-Riemannian manifold $(M, g_M)$ onto a semi-Riemannian manifold $(N, g_N)$. Then according to the conditions on the map $F : (M, g_M) \rightarrow (N, g_N)$, we have the following submersions:

Semi-Riemannian submersion and Lorentzian submersion [7], Riemannian submersion ([8], [14]), slant submersion ([5], [17]), almost Hermitian submersion [20], contact-complex submersion [9], quaternionic submersion [10], almost h-slant submersion and h-slant submersion [15], anti-invariant submersion [19], semi-invariant submersion [18], h-semi-invariant submersion [16], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([3], [21]), Kaluza-Klein theory ([2], [11]), Supergravity and superstring theories ([12], [13]), etc. Let $(M, g_M)$ and $(N, g_N)$ be Riemannian manifolds and $F : M \rightarrow N$ a $C^\infty$-submersion. The map $F$ is said to be Riemannian submersion if the differential $F_*$ preserves the lengths of horizontal vectors [10]. Let $(M, g_M, J)$ and $(M_1, g_{M_1}, J_1)$ be almost Hermitian manifolds. A Riemannian submersion $F : (M, g_M, J) \rightarrow (N, g_N)$ is called a slant submersion if the angle $\theta(X)$ between $JX$ and the space $\ker(F_*)_p$ is constant for any nonzero $X \in T_pM$ and $p \in M$ [17]. We call $\theta(X)$ a slant angle. A Riemannian submersion $F : (M, g_M, J) \rightarrow (N, g_N)$ is called an anti-invariant submersion if $JX \in \Gamma((\ker F_*)_p)$ for $X \in \Gamma(\ker F_*)$ [19]. A Riemannian submersion $F : (M, g_M, J) \rightarrow (M_1, g_{M_1}, J_1)$ is called an almost Hermitian submersion if $F$ is an almost complex map, i.e., $F_* \circ J = J_1 \circ F_*$ [20]. A Riemannian
submersion $F : (M,g_M,J) \mapsto (N,g_N)$ is called a semi-invariant submersion if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1, \quad J(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_* [17]$. Let $(M,g_M)$ and $(N,g_N)$ be Riemannian manifolds and $F : (M,g_M) \mapsto (N,g_N)$ a smooth map. The second fundamental form of $F$ is given by

$$(\nabla F_*)(X,Y) := \nabla_X F Y - F_*(\nabla_X Y) \quad \text{for } X,Y \in \Gamma(TM),$$

where $\nabla^F$ is the pullback connection and we denote conveniently by $\nabla$ the Levi-Civita connections of the metrics $g_M$ and $g_N [4]$. Recall that $F$ is said to be harmonic if $\text{trace}(\nabla F_*) = 0$ and $F$ is called a totally geodesic map if $(\nabla F_*)(X,Y) = 0$ for $X,Y \in \Gamma(TM) [4]$. The paper is organized as follows. In Section 2 we give the definition of the semi-slant submersion and obtain some interesting properties on them. In Section 3 we construct some examples for the semi-slant submersion.

2. Semi-slant submersions

**Definition 2.1.** Let $(M,g_M,J)$ be an almost Hermitian manifold and $(N,g_N)$ a Riemannian manifold. A Riemannian submersion $F : (M,g_M,J) \mapsto (N,g_N)$ is called a semi-slant submersion if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between $JX$ and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$. We call the angle $\theta$ a semi-slant angle.

**Remark 2.2.** As we know, a semi-slant submersion is the generalized version of a slant submersion. There are some similarities and differences between them. For the condition for such submersions to be harmonic, a semi-slant submersion has much more nice form than a slant submersion. But for the one for such submersions to be totally geodesic, two cases have the same condition. With the tensor $\omega$ to be parallel, we obtain some results on the slant submersions. For the semi-slant submersions with totally umbilical fibers, we have some results for the mean curvature vector field.

Let $F : (M,g_M,J) \mapsto (N,g_N)$ be a semi-slant submersion. Then there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between $JX$ and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$. Then for $X \in \Gamma(\ker F_*)$, we have

$$X = PX + QX,$$
where \( P_X \in \Gamma(D_1) \) and \( Q_X \in \Gamma(D_2) \).

For \( X \in \Gamma(\ker F) \), we get
\[
JX = \phi X + \omega X,
\]
where \( \phi X \in \Gamma(\ker F) \) and \( \omega X \in \Gamma((\ker F)^\perp) \).

For \( Z \in \Gamma((\ker F)^\perp) \), we obtain
\[
JZ = BZ + CZ,
\]
where \( BZ \in \Gamma(\ker F) \) and \( CZ \in \Gamma((\ker F)^\perp) \).

For \( U \in \Gamma(TM) \), we have
\[
U = VU + HU,
\]
where \( VU \in \Gamma(\ker F) \) and \( HU \in \Gamma((\ker F)^\perp) \).

Then
\[
(\ker F)^\perp = \omega D_2 \oplus \mu,
\]
where \( \mu \) is the orthogonal complement of \( \omega D_2 \) in \( (\ker F)^\perp \) and is invariant under \( J \). Furthermore,
\[
\phi D_1 = D_1, \quad \omega D_1 = 0, \quad \phi D_2 \subset D_2, \quad B((\ker F)^\perp) = D_2
\]
\[
\phi^2 + B\omega = -id, \quad C^2 + \omega B = -id, \quad \omega \phi + C\omega = 0, \quad BC + \phi B = 0.
\]

Define the tensors \( T \) and \( A \) by
\[
A_EF = \mathcal{H}\nabla_{HE}VF + \mathcal{V}\nabla_{HE}HF, \quad T_EF = \mathcal{H}\nabla_{VE}VF + \mathcal{V}\nabla_{VE}HF
\]
for vector fields \( E, F \) on \( M \), where \( \nabla \) is the Levi-Civita connection of \( g_M \).

Define
\[
(\nabla_X \phi)Y := \nabla_X \phi Y - \phi \nabla_X Y
\]
and
\[
(\nabla_X \omega)Y := \mathcal{H}\nabla_X \omega Y - \omega \nabla_X Y
\]
for \( X, Y \in \Gamma(\ker F) \), where \( \nabla_X Y := \mathcal{V}\nabla_X Y \). Then we easily have:

**Lemma 2.3.** Let \((M, g_M, J)\) be a Kähler manifold and \((N, g_N)\) a Riemannian manifold. Let \( F : (M, g_M, J) \to (N, g_N) \) be a semi-slant submersion. Then we get:

(a) \( \nabla_X \phi Y + T_X \omega Y = \phi \nabla_X Y + BT_X Y, \)
\[
T_X \phi Y + \mathcal{H}\nabla_X \omega Y = \omega \nabla_X Y + CT_X Y
\]
for \( X, Y \in \Gamma(\ker F) \).

(b) \( \mathcal{V}\nabla_Z BW + A_Z CW = \phi A_Z W + B\mathcal{H}\nabla_Z W, \)
\[
A_Z BW + \mathcal{H}\nabla_Z CW = \omega A_Z W + C\mathcal{H}\nabla_Z W
\]
for \( Z, W \in \Gamma((\ker F)^\perp) \).

(c) \( \nabla_X BZ + T_X CZ = \phi T_X Z + B\mathcal{H}\nabla_X Z, \)
\[
T_X BZ + \mathcal{H}\nabla_X CZ = \omega T_X Z + C\mathcal{H}\nabla_X Z
\]
for $X \in \Gamma(\ker F_1)$ and $Z \in \Gamma((\ker F_1)^\perp)$.

**Theorem 2.4.** Let $F$ be a semi-slant submersion from an almost Hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the complex distribution $\mathcal{D}_1$ is integrable if and only if we have

$$\omega(\hat{\nabla}_X Y - \hat{\nabla}_Y X) = C(T_X X - T_X Y) \quad \text{for } X, Y \in \Gamma(\mathcal{D}_1).$$

**Proof.** For $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma((\ker F_1)^\perp)$, since $[X, Y] \in \Gamma(\ker F_1)$, we obtain

$$g_M(J[X, Y], Z) = g_M(J(\nabla_X Y - \nabla_Y X), Z)$$

$$= g_M(\omega \hat{\nabla}_X Y + \omega \hat{\nabla}_Y X + BT_X Y + CT_X Y - \phi \hat{\nabla}_Y X - \omega \hat{\nabla}_Y X - B T_X Y - CT_X X, Z)$$

$$= g_M(\omega \hat{\nabla}_X Y + CT_X Y - \omega \hat{\nabla}_Y X - CT_Y X, Z).$$

Therefore, we have the result. \hfill \Box

Similarly, we get:

**Theorem 2.5.** Let $F$ be a semi-slant submersion from an almost Hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the slant distribution $\mathcal{D}_2$ is integrable if and only if we obtain

$$P(\phi \hat{\nabla}_X Y - \hat{\nabla}_Y X)^\ast + B(T_X Y - T_Y X) = 0 \quad \text{for } X, Y \in \Gamma(\mathcal{D}_2).$$

**Lemma 2.6.** Let $(M, g_M, J)$ be a Kähler manifold and $(N, g_N)$ a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a semi-slant submersion. Then the slant distribution $\mathcal{D}_2$ is integrable if and only if we obtain

$$P(\hat{\nabla}_X \phi Y - \hat{\nabla}_Y \phi X + \nabla_X \omega Y - \nabla_Y \omega X) = 0 \quad \text{for } X, Y \in \Gamma(\mathcal{D}_2).$$

**Proof.** For $X, Y \in \Gamma(\mathcal{D}_2)$ and $Z \in \Gamma(\mathcal{D}_1)$, since $[X, Y] \in \Gamma(\ker F_1)$, we have

$$g_M(J[X, Y], Z) = g_M(\nabla_X J Y - \nabla_Y J X, Z)$$

$$= g_M(\phi \hat{\nabla}_X Y + T_X \phi Y + T_X \omega Y + \mathcal{H} \nabla_X \omega Y - \hat{\nabla}_Y \phi X - T_Y \phi X - T_Y \omega X - \mathcal{H} \nabla_Y \omega X, Z)$$

$$= g_M(\hat{\nabla}_X \phi Y + T_X \phi Y - \hat{\nabla}_Y \phi X - T_Y \omega X, Z).$$

Therefore, the result follows. \hfill \Box

In a similar way, we have:

**Lemma 2.7.** Let $(M, g_M, J)$ be a Kähler manifold and $(N, g_N)$ a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a semi-slant submersion. Then the complex distribution $\mathcal{D}_1$ is integrable if and only if we get

$$Q(\hat{\nabla}_X \phi Y - \hat{\nabla}_Y \phi X ) = 0 \quad \text{and } T_X \phi Y = T_Y \phi X \quad \text{for } X, Y \in \Gamma(\mathcal{D}_1).$$
Define an endomorphism $\tilde{F}$ of $\ker F_*$ by
$$\tilde{F} := JP + \phi Q,$$
where $(\nabla_X \tilde{F})Y := \hat{\nabla}_X \tilde{F}Y - \tilde{F} \hat{\nabla}_X Y$ for $X, Y \in \Gamma(\ker F_*)$. Then it is not difficult to get.

**Lemma 2.8.** Let $F$ be a semi-slant submersion from a Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then we have
$$(\nabla_X \tilde{F})Y = \phi(\hat{\nabla}_X PY - \hat{\nabla}_X Y) + BT_X PY + \hat{\nabla}_X \phi QY$$ for $X, Y \in \Gamma(\ker F_*)$.

**Proposition 2.9.** Let $F$ be a semi-slant submersion from an almost Hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then we obtain
$$\phi^2 X = -\cos^2 \theta X$$ for $X \in \Gamma(D_2)$.

**Proof.** Since
$$\cos \theta = \frac{g_M(JX, \phi X)}{|JX| \cdot |\phi X|} = -\frac{g_M(X, \phi^2 X)}{|X| \cdot |\phi X|}$$ and $\cos \theta = \frac{|\phi X|}{|JX|}$,
we have
$$\cos^2 \theta = -\frac{g_M(X, \phi^2 X)}{|X|^2}$$ for $X \in \Gamma(D_2)$.
Hence,
$$\phi^2 X = -\cos^2 \theta X$$ for $X \in \Gamma(D_2)$. □

**Remark 2.10.** In particular, we easily see that the converse of Proposition 2.9 is also true.

Assume that the semi-slant angle $\theta$ is not equal to $\frac{\pi}{2}$ and define an endomorphism $\tilde{J}$ of $\ker F_*$ by
$$\tilde{J} := JP + \frac{1}{\cos \theta} \phi Q.$$
Then,
$$\tilde{J}^2 = -id$$ on $\ker F_*$.

**Remark 2.11.** Let $F$ be a semi-slant submersion from an almost Hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Assume that $\dim M = 2m$, $\dim N = n$, and $\theta \in [0, \frac{\pi}{2})$. From (1), we have
$$\dim(\ker(F_*))_p = 2k$$ and $\dim((\ker(F_*))_p)^\perp = 2m - 2k$ for $p \in M$,
where $k$ is a non-negative integer.

Therefore, $n$ must be even.

**Theorem 2.12.** Let $F$ be a semi-slant submersion from an almost Hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$ with the semi-slant angle $\theta \in [0, \frac{\pi}{2})$. Then $N$ is an even-dimensional manifold.
Proposition 2.13. Let $F$ be a semi-slant submersion from a Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the distribution $\ker F_*$ defines a totally geodesic foliation if and only if
\[
\omega(\hat{\nabla}_X \phi Y + T_X \omega Y) + C(T_X \phi Y + H \nabla_X \omega Y) = 0 \quad \text{for } X, Y \in \Gamma(\ker F_*).
\]
Proof. For $X, Y \in \Gamma(\ker F_*),
\nabla_X Y = -J \nabla_X J Y
= -J(\hat{\nabla}_X \phi Y + T_X \phi Y + T_X \omega Y + H \nabla_X \omega Y)
= -(\phi \hat{\nabla}_X \phi Y + \omega \hat{\nabla}_X \phi Y + B T_X \phi Y + C T_X \phi Y + \phi T_X \omega Y + \omega T_X \omega Y
+ B H \nabla_X \omega Y + C H \nabla_X \omega Y).
\]
Thus,
\[
\nabla_X Y \in \Gamma(\ker F_*) \Leftrightarrow \omega(\hat{\nabla}_X \phi Y + T_X \omega Y) + C(T_X \phi Y + H \nabla_X \omega Y) = 0. \quad \Box
\]

Proposition 2.14. Let $F$ be a semi-slant submersion from a Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation if and only if
\[
\phi (V \nabla_X BY + A_X CY) + B(A_X BY + H \nabla_X CY) = 0 \quad \text{for } X, Y \in \Gamma((\ker F_*)^\perp).
\]

Proposition 2.15. Let $F$ be a semi-slant submersion from a Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the distribution $D_1$ defines a totally geodesic foliation if and only if
\[
Q(\phi \hat{\nabla}_X \phi Y + B T_X \phi Y) = 0 \quad \text{and} \quad \omega \hat{\nabla}_X \phi Y + C T_X \phi Y = 0
\]
for $X, Y \in \Gamma(D_1)$.
Proof. For $X, Y \in \Gamma(D_1), we get
\nabla_X Y = -J \nabla_X J Y
= -J(\hat{\nabla}_X \phi Y + T_X \phi Y)
= -(\phi \hat{\nabla}_X \phi Y + \omega \hat{\nabla}_X \phi Y + B T_X \phi Y + C T_X \phi Y).
\]
Hence,
\[
\nabla_X Y \in \Gamma(D_1) \Leftrightarrow Q(\phi \hat{\nabla}_X \phi Y + B T_X \phi Y) = 0 \quad \text{and} \quad \omega \hat{\nabla}_X \phi Y + C T_X \phi Y = 0. \quad \Box
\]

In a similar way, we obtain:

Proposition 2.16. Let $F$ be a semi-slant submersion from a Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the distribution $D_2$ defines a totally geodesic foliation if and only if
\[
P(\phi (\hat{\nabla}_X \phi Y + T_X \omega Y) + B(T_X \phi Y + H \nabla_X \omega Y)) = 0,
\]
\[
\omega(\hat{\nabla}_X \phi Y + T_X \omega Y) + C(T_X \phi Y + H \nabla_X \omega Y) = 0
\]
Theorem 2.17. Let $F$ be a semi-slant submersion from a Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then $F$ is a totally geodesic map if and only if
\[
\omega(\tilde{\nabla}_X \phi Y + T_X \omega Y) + C(T_X \phi Y + H\nabla_X \omega Y) = 0,
\]
\[
\omega(\tilde{\nabla}_X B Z + T_X C Z) + C(T_X B Z + H\nabla_X C Z) = 0
\]
for $X, Y \in \Gamma(\ker F^*)$.

Proof. Since $F$ is a Riemannian submersion, we have
\[
(\nabla F_*)(Z_1, Z_2) = 0 \quad \text{for} \quad Z_1, Z_2 \in \Gamma((\ker F^*)^\perp).
\]
For $X, Y \in \Gamma(\ker F_*)$, we obtain
\[
(\nabla F_*)(X, Y) = -F_*(\nabla_X Y)
\]
\[
= F_*(J\nabla_X (\phi Y + \omega Y))
\]
\[
= F_*(\phi \tilde{\nabla}_X \phi Y + \omega \tilde{\nabla}_X \phi Y + B T_X \phi Y + C T_X \phi Y + \phi T_X \omega Y
\]
\[
+ \omega T_X \omega Y + B H\nabla_X \omega Y + C H\nabla_X \omega Y).
\]
Thus,
\[
(\nabla F_*)(X, Y) = 0 \iff \omega(\tilde{\nabla}_X \phi Y + T_X \omega Y) + C(T_X \phi Y + H\nabla_X \omega Y) = 0.
\]
For $X \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$, we get
\[
(\nabla F_*)(X, Z) = -F_*(\nabla_X Z)
\]
\[
= F_*(J\nabla_X (B Z + C Z))
\]
\[
= F_*(\phi \tilde{\nabla}_X B Z + \omega \tilde{\nabla}_X B Z + B T_X B Z + C T_X B Z + \phi T_X C Z
\]
\[
+ \omega T_X C Z + B H\nabla_X C Z + C H\nabla_X C Z).
\]
Hence,
\[
(\nabla F_*)(X, Z) = 0 \iff \omega(\tilde{\nabla}_X B Z + T_X C Z) + C(T_X B Z + H\nabla_X C Z) = 0.
\]
Since $(\nabla F_*)(X, Z) = (\nabla F_*)(Z, X)$, we get the result. \qed

Let $F$ be a semi-slant submersion from a Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Assume that $D_1$ is integrable. Choose a local orthonormal frame $\{v_1, \ldots, v_l\}$ of $D_2$ and a local orthonormal frame $\{e_1, \ldots, e_k\}$ of $D_1$ such that $e_{2i} = J e_{2i-1}$ for $1 \leq i \leq k$. Since
\[
F_*(\nabla_{J e_{2i-1}} J e_{2i-1}) = -F_*(\nabla_{e_{2i-1}} e_{2i-1})
\]
for $1 \leq i \leq k$, we have
\[
\text{trace}(\nabla F_*) = 0 \iff \sum_{j=1}^{l} F_*(\nabla_{v_j} v_j) = 0.
\]
Theorem 2.18. Let $F$ be a semi-slant submersion from a Kähler manifold $(M,g_M,J)$ onto a Riemannian manifold $(N,g_N)$ such that $D_1$ is integrable. Then $F$ is a harmonic map if and only if

$$\text{trace}(\nabla F) = 0 \quad \text{on } D_2.$$ 

Let $F : (M,g_M) \mapsto (N,g_N)$ be a Riemannian submersion. The map $F$ is called a Riemannian submersion with totally umbilical fibers if

$$(2) \quad T_X Y = g_M(X,Y)H \quad \text{for } X,Y \in \Gamma(\ker F_*),$$

where $H$ is the mean curvature vector field of the fiber.

In a similar way with Lemma 4.2 of [18], we obtain:

Lemma 2.19. Let $F$ be a semi-slant submersion with totally umbilical fibers from a Kähler manifold $(M,g_M,J)$ onto a Riemannian manifold $(N,g_N)$. Then we have

$$H \in \Gamma(\omega D_2).$$

Proof. For $X,Y \in \Gamma(D_1)$ and $W \in \Gamma(\mu)$, we get

$$T_X Y + \nabla_X JY = \nabla_X Y = B T_X Y + C T_X Y + \phi \nabla_X Y + \omega \nabla_X Y$$

so that

$$g_M(T_X Y, W) = g_M(C T_X Y, W).$$

By (2), with a simple calculation we obtain

$$g_M(X, JY)g_M(H, W) = -g_M(X, Y)g_M(H, JW).$$

Interchanging the role of $X$ and $Y$, we get

$$g_M(Y, JX)g_M(H, W) = -g_M(Y, X)g_M(H, JW)$$

so that combining the above two equations, we have

$$g_M(X, Y)g_M(H, JW) = 0$$

which means $H \in \Gamma(\omega D_2)$, since $J\mu = \mu$. Therefore, we obtain the result. □

Remark 2.20. Let $F$ be a semi-slant submersion from a Kähler manifold $(M,g_M,J)$ onto a Riemannian manifold $(N,g_N)$. Then there is a distribution $D_1 \subset \ker F_*$ such that

$$\ker F_* = D_1 \oplus D_2, \quad J(D_1) = D_1,$$

and the angle $\theta = \theta(X)$ between $JX$ and the space $(D_2)_q$ is constant for nonzero $X \in (D_2)_q$ and $q \in M$, where $D_2$ is the orthogonal complement of $D_1$ in $\ker F_*$. Furthermore,

$$\phi D_2 \subset D_2, \quad \omega D_2 \subset (\ker F_*)^\perp, \quad (\ker F_*)^\perp = \omega D_2 \oplus \mu,$$

where $\mu$ is the orthogonal complement of $\omega D_2$ in $(\ker F_*)^\perp$ and is invariant under $J$. As we know, the holomorphic sectional curvatures determine the Riemannian curvature tensor in a Kähler manifold.
Given a plane $P$ being invariant by $J$ in $T_pM$, $p \in M$, there is an orthonormal basis $\{X, JX\}$ of $P$. Denote by $K(P)$, $K_*(P)$, and $\tilde{K}(P)$ the sectional curvatures of the plane $P$ in $M$, $N$, and the fiber $F^{-1}(F(p))$, respectively, where $K_*(P)$ denotes the sectional curvature of the plane $P_* = \langle F, X, F_JX \rangle$ in $N$. Let $K(X \wedge Y)$ be the sectional curvature of the plane spanned by the tangent vectors $X, Y \in T_pM$, $p \in M$. Using both Corollary 1 of [14, p. 465] and (1.27) of [7, p. 12], we obtain the following:

1. If $P \subset (D_1)_p$, then with some computations we have
   \[ K(P) = \tilde{K}(P) + |\nabla X|^2 - |\nabla JX|^2 - g_M(\nabla X, J[\nabla X, X]). \]
2. If $P \subset (D_2 \oplus \omega D_2)_p$ with $X \in (D_2)_p$, then we get
   \[ K(P) = \cos^2 \theta \cdot K(X \wedge \phi X) + 2g_M((\nabla_\phi X)(X, X)) \]
   \[ - (\nabla X)(\phi X, X), \omega X) + \sin^2 \theta \cdot K(X \wedge \omega X). \]
3. If $P \subset (\mu)_p$, then we obtain
   \[ K(P) = K_*(P) - 3|\nabla J\nabla X|^2. \]

3. Examples

Example 3.1. Let $F$ be a slant submersion from an almost Hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$ [17]. Then the map $F$ is a semi-slant submersion with $D_2 = \ker F_*$. 

Example 3.2. Let $F$ be a semi-invariant submersion from an almost Hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$ [18]. Then the map $F$ is a semi-slant submersion with the semi-slant angle $\theta = \frac{\pi}{2}$. 

Example 3.3. Let $F$ be an almost $h$-slant submersion from a hyperkähler manifold $(M, g_M, I, J, K)$ onto a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost $h$-slant basis [15]. Then the map $F : (M, g_M, R) \mapsto (N, g_N)$ is a semi-slant submersion with $D_2 = \ker F_*$ for $R \in \{I, J, K\}$. 

Example 3.4. Let $F$ be an almost $h$-semi-invariant submersion from a hyperkähler manifold $(M, g_M, I, J, K)$ onto a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost $h$-semi-invariant basis [16]. Then the map $F : (M, g_M, R) \mapsto (N, g_N)$ is a semi-slant submersion with the semi-slant angle $\theta = \frac{\pi}{2}$ for $R \in \{I, J, K\}$. 

Example 3.5. Define a map $F : \mathbb{R}^6 \mapsto \mathbb{R}^2$ by
   \[ F(x_1, x_2, \ldots, x_6) = (x_3 \sin \alpha - x_5 \cos \alpha, x_6), \]
where $\alpha \in (0, \frac{\pi}{2})$. Then the map $F$ is a semi-slant submersion such that

\[ D_1 = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) \] and \[ D_2 = \left( \frac{\partial}{\partial x_4}, \cos \alpha \frac{\partial}{\partial x_3} + \sin \alpha \frac{\partial}{\partial x_5} \right) \]
with the semi-slant angle $\theta = \alpha$. 

Example 3.6. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^2$ by

$$F(x_1, x_2, \ldots, x_8) = \left( \frac{x_5 - x_8}{\sqrt{2}}, x_6 \right).$$

Then the map $F$ is a semi-slant submersion such that

$$D_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle$$

and

$$D_2 = \left\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle$$

with the semi-slant angle $\theta = \frac{\pi}{4}$.

Example 3.7. Define a map $F : \mathbb{R}^{10} \mapsto \mathbb{R}^5$ by

$$F(x_1, x_2, \ldots, x_{10}) = \left( x_2, x_1, \frac{x_5 + x_6}{\sqrt{2}}, \frac{x_7 + x_9}{\sqrt{2}}, x_8 + x_{10} \right).$$

Then the map $F$ is a semi-slant submersion such that

$$D_1 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}} \right\rangle$$

and

$$D_2 = \left\langle -\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \right\rangle$$

with the semi-slant angle $\theta = \frac{\pi}{2}$.

Example 3.8. Define a map $F : \mathbb{R}^{10} \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \ldots, x_{10}) = \left( \frac{x_3 - x_5}{\sqrt{2}}, x_6, \frac{x_7 - x_9}{\sqrt{2}}, x_8 \right).$$

Then the map $F$ is a semi-slant submersion such that

$$D_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle$$

and

$$D_2 = \left\langle \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}} \right\rangle$$

with the semi-slant angle $\theta = \frac{\pi}{7}$.

Example 3.9. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \ldots, x_8) = (x_1, x_2, x_3 \cos \alpha - x_5 \sin \alpha, x_4 \sin \beta - x_6 \cos \beta),$$

where $\alpha$ and $\beta$ are constant. Then the map $F$ is a semi-slant submersion such that

$$D_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle$$

and

$$D_2 = \left\langle \sin \alpha \frac{\partial}{\partial x_3} + \cos \alpha \frac{\partial}{\partial x_5}, \cos \beta \frac{\partial}{\partial x_4} + \sin \beta \frac{\partial}{\partial x_6} \right\rangle$$

with the semi-slant angle $\theta$ with $\cos \theta = | \sin(\alpha + \beta) |$.

Example 3.10. Let $G$ be a slant submersion from an almost Hermitian manifold $(M_1, g_{M_1}, J_1)$ onto a Riemannian manifold $(N, g_N)$ with the slant angle $\theta$ and $(M_2, g_{M_2}, J_2)$ an almost Hermitian manifold. Denote by $(M, g, J)$ the warped product of $(M_1, g_{M_1}, J_1)$ and $(M_2, g_{M_2}, J_2)$ by a positive function $f$ on $M_1$ [7], where $J = J_1 \times J_2$. Define a map $F : (M, g, J) \mapsto (N, g_N)$ by

$$F(x, y) = G(x)$$

for $x \in M_1$ and $y \in M_2$.

Then the map $F$ is a semi-slant submersion such that

$$D_1 = TM_2 \text{ and } D_2 = \ker G_*.$$
with the semi-slant angle $\theta$.

References


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