MELTING OF THE EUCLIDEAN METRIC TO NEGATIVE SCALAR CURVATURE

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Abstract. We find a $C^\infty$-continuous path of Riemannian metrics $g_t$ on $\mathbb{R}^k$, $k \geq 3$, for $0 \leq t \leq \varepsilon$ for some number $\varepsilon > 0$ with the following property: $g_0$ is the Euclidean metric on $\mathbb{R}^k$, the scalar curvatures of $g_t$ are strictly decreasing in $t$ in the open unit ball and $g_t$ is isometric to the Euclidean metric in the complement of the ball. Furthermore we extend the discussion to the Fubini-Study metric in a similar way.

1. Introduction

In a remarkable paper [11], Lohkamp has made the following conjecture in Riemannian geometry.

Conjecture. Let $(M^k, g_0)$, $k \geq 3$, be a manifold and $B \subset M$ a ball. Then there is a $C^\infty$-continuous path of Riemannian metrics $g_t$, $0 \leq t \leq \varepsilon$, on $M$ with

(i) Ricci curvature of $g_t$ is strictly decreasing in $t$ on $B$.

(ii) $g_t \equiv g_0$ on $M \setminus B$.

If such a path $g_t$ exists, we call it a Ricci-curvature melting of $g_0$ on $B$. This conjecture, if true, would certainly imply a scalar-curvature melting, meaning a path $g_t$ as above but with scalar curvature replacing the Ricci curvature in the condition (i). We note that common metric-surgery arguments do not seem to yield a scalar-curvature melting. If one considers the scalar curvatures $s(g_t)$ for a scalar-curvature melting $g_t$, then $\frac{ds(g_t)}{dt}|_{t=0} \leq 0$ on $B$. In this way, the scalar-curvature melting is related to the deformation theory of the scalar curvature functional [4, Chapter 4]. A remarkable approach is the theory of local scalar curvature deformation of J. Corvino [6, Theorem 4]. He considered the formal adjoint $L^*_g$ of the linearization $L_g$ of the scalar curvature functional on the space of Riemannian metrics restricted to a domain. According to his work, a scalar-curvature melting of $g$ seems to exist when $L^*_g$ is injective. Years
later, this injectivity condition of $L_g^*$ on domains was shown to be a generic one by Beig, Chruściel and Schoen (see Theorem 6.1 and Theorem 7.4 in [3]). Now the question is how to melt a Riemannian metric which does not satisfy this condition.

In this context, Euclidean metrics arise importantly because they are outstanding ones, not satisfying this condition. In a recent paper [8], we explained the scalar-curvature melting of Euclidean metric in 3 dimension. The purpose of this article is to complete the scalar-curvature melting of Euclidean metrics in any dimension $\geq 3$ and then extend the discussion to the Fubini-Study metric in a similar way.

We shall first construct a family of Riemannian metrics on $\mathbb{R}^k, k \geq 3$ which have negative scalar curvatures on a pre-compact (open) set and are Euclidean away from it. In even dimension we already have such a family of metrics [7]. In odd dimension, we use the coordinates $(r_1, \theta_1, \ldots, r_n, \theta_n, z)$ on $\mathbb{R}^{2n+1}$ where $(r_i, \theta_i)$ are the polar coordinates on the $i$-th direct summand of $\mathbb{R}^{2n+1} := \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}$ and $z$ is the coordinate for the last summand $\mathbb{R}$. We express the Euclidean metric as $g_0 = \sum_{i=1}^n (dr_i^2 + r_i^2 d\theta_i^2) + dz^2$. We deform it to $g = \sum_{i=1}^n (f_i^2 dr_i^2 + \frac{r_i^2}{f_i^2} d\theta_i^2) + dz^2$ and choose smooth functions $f_i$ so that $g$ has negative scalar curvature on a pre-compact set near origin and is Euclidean away from it.

Then by conformal change of $g$ (also for the even dimensional metrics mentioned above), we spread the negativity inside the pre-compact set over to a larger ball. In the process, we found a natural choice of parameter $t$ to get $g_t$.

In this way we get a scalar-curvature melting:

**Theorem 1.1.** There exists a $C^\infty$-continuous path of Riemannian metrics $g_t$ on $\mathbb{R}^k, k \geq 3$ which exists for $0 \leq t \leq \varepsilon$ for some number $\varepsilon$ with the following property: $g_0$ is the Euclidean metric on $\mathbb{R}^k$, $s(g_t) < s(g_t)$ for $0 \leq t < \tilde{t} \leq \varepsilon$ in the open unit ball and $g_t$ is the Euclidean metric in the complement of the ball.

In Section 2, we construct Riemannian metrics on $\mathbb{R}^{2n+1}$ that have negative scalar curvatures on a pre-compact set and are Euclidean away from it. In Section 3, we demonstrate a $C^\infty$-continuous path of metrics $g_t$ such that the scalar curvature $s(g_t)$ is monotonically decreasing in $t$. In Section 4, by a conformal deformation we get a genuine scalar-curvature melting on the unit ball in $\mathbb{R}^{2n+1}$. We also observe that similar argument works for even dimensions. In Section 5 we discuss the Fubini-Study metric in a similar way.

**2. Construction of the metric**

We will deform the Euclidean metric $g_0 = \sum_{i=1}^n (dr_i^2 + r_i^2 d\theta_i^2) + dz^2$ on $\mathbb{R}^{2n+1} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}$ to a metric of the form

$$
\tilde{g} = \sum_{i=1}^n \left( f_i^2 dr_i^2 + \frac{r_i^2}{f_i^2} d\theta_i^2 \right) + dz^2,
$$

(1)
where $f_i$'s are smooth positive functions on $\mathbb{R}^{2n+1}$ depending only on the variables $r_1, \ldots, r_n, z$. $\tilde{g}$ is a metric on $\mathbb{R}^{2n+1} \backslash \{(r_1, \theta_1, \ldots, r_n, \theta_n, z) \mid r_1 = 0 \text{ for some } i\}$. Below we shall choose $f_i$ so that $\tilde{g}$ is smooth on $\mathbb{R}^{2n+1}$. Let $e_{2i-1} = \frac{\partial}{\partial r_i}, e_{2i} = \frac{\partial}{\partial \theta_i}, i = 1, 2, \ldots, n, e_{2n+1} = \frac{\partial}{\partial z}$.

Let $\omega_i$ be the dual co-frame fields of $e_i$: $\omega_{2i-1} = f_idr_i, \omega_{2i} = \frac{\partial}{\partial \theta_i}, \omega_{2n+1} = dz$. We compute the connection 1-forms $\omega_{ij}$ with respect to $\omega_i$:

$$\omega_i = \sum_{j=1}^{2n+1} \omega_i \wedge \omega_j, \text{ with } \omega_{ij} = -\omega_{ji};$$

one may compute

$$2\omega_{ijk} = \langle d\omega_k, \omega_i \wedge \omega_j \rangle - \langle d\omega_i, \omega_j \wedge \omega_k \rangle - \langle d\omega_j, \omega_k \wedge \omega_i \rangle,$$

where $\omega_{ij} = \sum_{k=1}^n a_{ijk} \omega_k$. We get

$$d\omega_{2n+1} = 0,$$

$$d\omega_{2i-1} = \frac{f_{2i, 2n+1}}{f_i} \omega_{2n+1} \wedge \omega_{2i-1} + \sum_{j=1}^{n} \frac{f_{i,j}}{f_j} \omega_{2j-1} \wedge \omega_{2i-1} \text{ and}$$

$$d\omega_{2i} = -\frac{f_{2i, 2n+1}}{f_i} \omega_{2n+1} \wedge \omega_{2i} + \sum_{j=1}^{n} \frac{\delta_{ij} f_i - r_j f_{i,j}}{r_j f_{i,j}} \omega_{2j-1} \wedge \omega_{2i}$$

for $i = 1, 2, \ldots, n$.

Here we write $f_{i,j} = \frac{\partial f_i}{\partial r_j}, f_{i,j,k} = \frac{\partial^2 f_i}{\partial r_j \partial r_k}$. Then we can get $\omega_{2i-1} \omega_{2j-1} = -\frac{f_{i,j}}{f_j} \omega_{2i-1} + \frac{f_{j,i}}{f_i} \omega_{2j-1}$, $\omega_{2i-1} \omega_{2j} = \frac{\delta_{ij} f_i - r_j f_{i,j}}{r_j f_{i,j}} \omega_{2j}$. $\omega_{2i} = 0$ for $i, j = 1, 2, \ldots, n$ and $n + 1$.

We use the formula $d\omega_{j} = -\omega_{i,k} \wedge \omega_{k,j} = \sum_{k<i}^{2n+1} R_{i,j,k} \omega_k \wedge \omega_j$ to compute the curvature components:

$$R_{2i-1} \omega_{2j-1} \omega_{2j-2} = -d\omega_{2i-1} \omega_{2j-1} + \omega_{2i-1} \omega_{2j-2} \wedge \omega_{2j-1}$$

$$= -\frac{f_{i,j,k}}{f_j f_k} + \frac{f_{j,i,k}}{f_i f_k} - \frac{f_{i,i,k}}{f_j f_k} + \frac{f_{j,j,k}}{f_i f_k} - \sum_{k=1}^{n} \frac{f_{i,j,k}}{f_j f_k} - \frac{f_{i,2n+1}}{f_i} \frac{f_{j,2n+1}}{f_j},$$

$$R_{2i-1} \omega_{2j} \omega_{2j-1} = -d\omega_{2i-1} \omega_{2j} + \omega_{2i-1} \omega_{2j} \wedge \omega_{2j-1}$$

$$= \frac{f_{i,i}}{f_j f_i} + \frac{f_{i,j}}{r_j f_j f_i} - \sum_{k=1}^{n} \frac{f_{i,j,k}}{f_i f_k f_j} - \frac{f_{i,2n+1}}{f_i} \frac{f_{j,2n+1}}{f_j},$$

$$R_{2i-1} \omega_{2j} \omega_{2j} = -d\omega_{2i-1} \wedge \omega_{2j} + \omega_{2i-1} \omega_{2j} \wedge \omega_{2j}$$

$$= \frac{\delta_{ij} f_{i,j}}{r_j f_j} + \frac{f_{i,j,j}}{f_j f_j} + \frac{f_{i,j,i}}{f_j f_j} - \frac{2 f_{i,j,j}}{f_j f_j} + \sum_{k=1}^{n} \frac{\delta_{ij} f_{i,k}}{r_j f_j f_k} + \sum_{k=1}^{n} \frac{f_{i,j,k}}{f_i f_k f_j},$$

$$+ \frac{f_{i,2n+1}}{f_i} \frac{f_{j,2n+1}}{f_j}. $$
\[ R_{2n+1} 2i-1 2i-1 2n+1 = -(f_{i,2n+1}^2)_{2n+1} - (f_{i,2n+1})_i^2, \]
\[ R_{2n+1} 2i 2i 2n+1 = (f_{i,2n+1}^2)_{2n+1} - (f_{i,2n+1})_i^2. \]

The scalar curvature is as follows:
\[ \frac{s^g}{2} = \sum_{1 \leq s < t} R_{sts} \]
\[ = \sum_{1 \leq t} R_{2n+1 t t 2n+1} + \sum_{1 \leq i < j} (R_{2i-1 2j-1 2j-1 2i-1} + R_{2i 2j 2j 2i}) \]
\[ + \sum_{1 \leq i < j} (R_{2i-1 2j 2i-1} + R_{2j-1 2i 2i-1}) + \sum_{i=1}^n R_{2i-1 2i 2i-1} \]
\[ = -\sum_{i=1}^n (f_{i,2n+1})_i^2 + \sum_{i=1}^n (f_{i,i}^2)_{i>2} + 3 f_{i,i}^2 \sum_{i=1}^n f_{i,i} - 3 f_{i,i}^2 \sum_{i<j} f_{i,j}^2 f_{j,i}^2 - \sum_{i<j} f_{i,j}^2 + f_{j,i}^2 \]
\[ = -\frac{1}{2} \sum_{i=1}^n \left( (f_{i,i}^2)_i + \frac{3}{r_i} (f_{i,i}^2)_i \right) - \sum_{i<j} f_{i,j}^2 f_{j,i}^2 - \sum_{i=1}^n (f_{i,2n+1})_i^2. \]

Set \( F_i = f_i^{-2}, i = 1, \ldots, n. \) We shall find the functions \( F_i \) so that they satisfy
\[ (2) \quad \sum_{i=1}^n (F_{i,ii} + \frac{3}{r_i} F_{i,i}) = 0. \]

We consider smooth functions \( \beta(z) \) and \( \alpha_j^i(r), i = 1, \ldots, n - 1, j = 1, \ldots, n \) on \( \mathbb{R} \) which satisfy at least
\[ \beta(z) = 0 \quad \text{for} \quad z \leq -1, \quad \text{or} \quad z \geq 1, \quad \text{and} \quad \beta(z) > 0 \quad \text{on} \quad -1 < z < 1, \]
\[ \alpha_j^i(r) = 0 \quad \text{for} \quad r \leq 0, \quad \text{or} \quad r \geq 1. \]

The functions \( \alpha_j^i \)'s need to be specified more. Let \( k_j^i(r) \) be smooth functions on \( \mathbb{R} \) satisfying
\[
\begin{align*}
\text{a)} & \quad k_j^i(r) = 0 \quad \text{for} \quad r \leq 0, \quad r \geq 1, \\
\text{b)} & \quad |(k_j^i)'(r)|_{C_0} \ll |r^3|_{C_0}, \\
\text{c)} & \quad \int_0^1 \frac{k_j^i(r)}{r^3} \, dr = 0, \\
\text{d)} & \quad 0 < \int_0^c \frac{k_j^i(r)}{r^3} \, dr < 1 \quad \text{for any} \ c \quad \text{with} \ 0 < c < 1.
\end{align*}
\]

Set \( \alpha_j^i(r) = \frac{1}{r^3} \frac{d}{dr} k_j^i(r) \), which will be smooth on \( \mathbb{R} \). Graphs of typical \( \alpha_j^i \) and \( \beta \) are given in Figures 1 and 2 below.
Define the functions $F_i$, $i = 1, \ldots, n - 1$, and $F_n$ by

$$F_i(r_1, \ldots, r_n, z) = 1 + \beta(z) \cdot \alpha^1(r_1) \cdots \alpha_{i-1}^i(r_i) \cdots \alpha_n^i(r_n) \int_0^{r_i} \left( \frac{1}{y^3} \int_0^y x^3 \alpha_1^i(x) \, dx \right) \, dy,$$

where $\hat{}$ denotes the missing factor in that position,

$$F_n(r_1, \ldots, r_n, z) = 1 - \beta(z) \cdot \sum_{i=1}^{n-1} \alpha_1^i(r_1) \cdots \alpha_{i-1}^i(r_{i-1}) \cdots \alpha_n^i(r_n-1) \int_0^{r_n} \left( \frac{1}{y^3} \int_0^y x^3 \alpha_1^i(x) \, dx \right) \, dy.$$

We consider $F_i$’s and $F_n$ defined on $\mathbb{R}^{2n+1} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}$. Then they satisfy the equation (2) and

- $F_i, F_n \equiv 1$ if $r_k \leq 0$ or $r_k \geq 1$ for some $k$, or $|z| > 1$,
- $F_i, F_n > 0$ everywhere.

We set $C = \{(r_1, \theta_1, \ldots, r_n, \theta_n, z) \mid |z| < 1, 0 \leq r_i < 1, 0 \leq \theta_i < 2\pi\}$. We now see that $\tilde{g}$ is Euclidean away from $C$ and that its scalar curvature $s_3$ is negative inside $C$ except the thin subset $\mathcal{C} := \{(r_1, \theta_1, \ldots, r_n, \theta_n, z) \in C \mid F_{i,j} = 0, F_{i,2n+1} = 0, 1 \leq i \neq j \leq n\}$.

**Proposition 2.1.** There exist Riemannian metrics on $\mathbb{R}^{2n+1}$, $n \geq 2$ such that their scalar curvatures are negative on the pre-compact subset $C \setminus \mathcal{C}$ and they are Euclidean away from $C$.

We need to recall the similar result in even dimensions from Sections 3 and 5 of [7].
Proposition 2.2. There exist Riemannian metrics on \( \mathbb{R}^{2n} \), \( n \geq 2 \) such that their scalar curvatures are negative on a pre-compact subset \( K \) and they are Euclidean away from \( K \).

3. Decreasing property of the scalar curvature of metrics

We are going to show that there is a \( C^\infty \)-continuous path \( \tilde{g}_t \) among the metrics in the previous section such that its scalar curvature \( s(\tilde{g}_t) \) is decreasing in \( C \setminus \Sigma \) and \( \tilde{g}_t \) is Euclidean in the complement of \( C \).

We set

\[
F_i^t(r_1, \ldots, r_n, z) = 1 + t \cdot \beta(z) \cdot \sum_{i=1}^{n-1} \alpha_i^1(r_1) \cdots \alpha_i^i(r_i) \cdot \alpha_i^{i+1}(r_{i+1}) \int_0^{r_i} \frac{1}{y^2} \int_0^y x^3 \alpha_i^j(x) \, dx \, dy,
\]

where \( \hat{\cdot} \) denotes the missing factor in that position,

\[
F_n^t(r_1, \ldots, r_n, z) = 1 - t \cdot \beta(z) \cdot \sum_{i=1}^{n-1} \alpha_n^1(r_1) \cdots \alpha_n^{i-1}(r_{i-1}) \cdot \alpha_n^i(r_n) \int_0^{r_n} \frac{1}{y^2} \int_0^y x^3 \alpha_n^j(x) \, dx \, dy.
\]

Still under the relation \( F_i^t = (f_i^t)^2 \), \( i = 1, \ldots, n \), we let

\[
(3) \quad \tilde{g}_t = dz^2 + \sum_{i=1}^{n} (f_i^t)^2 dr_i^2 + \frac{r_i^2}{(f_i^t)^2} d\theta_i^2.
\]

The scalar curvature is

\[
s_{\tilde{g}_t}(r_1, \ldots, r_n) = -\frac{1}{4} \sum_{i<j} \left( \frac{F_{i,j}^t}{F_i^t} \right)^2 + \frac{1}{4} \sum_{i=1}^{n} \frac{F_{i,2n+1}^t}{F_i^t}.
\]

One can easily check \( \frac{d(s(\tilde{g}_t))}{dt}|_{t=0} = 0 \) and

\[
\frac{d^2(s(\tilde{g}_t))}{dt^2}|_{t=0} = -\frac{1}{4} \sum_{i<j} \left( \frac{d^2(F_{i,j}^t)}{dt^2} \right)^2 \bigg|_{t=0} + \frac{d^2(F_{i,i}^t)}{dt^2} \bigg|_{t=0}
\]
\[
- \frac{1}{4} \sum_{i=1}^{n} \frac{d^2(F_{i,2n+1}^t)}{dt^2} \bigg|_{t=0}
\]
\[
= -\frac{1}{2} \sum_{i<j} \{(F_{i,j})^2 + (F_{j,i})^2\} - \frac{1}{2} \sum_{i=1}^{n} (F_{i,2n+1})^2 \leq 0.
\]

Note that inside \( C \) the set of points with \( \frac{d^2(s(\tilde{g}_t))}{dt^2}|_{t=0} = 0 \) is identical to the set \( \Sigma \). We see that \( s(\tilde{g}_t) \) is strictly decreasing only on \( C \setminus \Sigma \). In order to have the right decreasing property, we need to diffuse the negativity (of scalar curvature) onto a ball containing \( C \setminus \Sigma \).
4. Diffusion of negative scalar curvature onto a ball

Our argument in this section is similar to that in [8, Section 4], so we avoid some details. We use the following functions: $F_{t,m}(\rho) \in C^{\infty}(\mathbb{R}, \mathbb{R}^{\geq 0})$ for $m > 0$, $t \geq 0$ defined by $F_{t,m}(\rho) = m \cdot t^2 \cdot \exp(-\frac{m^2}{\rho})$ on $\mathbb{R}^{\geq 0}$ and $F_{t,m} = 0$ on $\mathbb{R}^{\leq 0}$. Also choose an $H \in C^{\infty}(\mathbb{R}, [0,1])$ with $H = 0$ on $\mathbb{R}^{\geq 1}$, $H = 1$ on $\mathbb{R}^{\leq 0}$ and $H^b_{\rho}(\rho) = H(\frac{1}{c}(\rho - b))$ for $b > 0$, $\epsilon > 0$. We consider the Ricci-curvature melting.

Let $B_r(x)$ be the open ball of radius $r$ with respect to $g_0$ centered at $x$. We choose a point $p$ and a number $\epsilon_1 < 0.1$ so that $B_{2\epsilon_1}(p) \subset C \setminus \Xi$. Then $s(\tilde{g}_t) < 0$ on $B_{\epsilon_1}(p)$ when $0 < t < \epsilon$ for some number $\epsilon$.

Let $f_{t,m} \in C^{\infty}(\mathbb{R}^{2n+1}, \mathbb{R}^{\geq 0})$ be $f_{t,m}(q) = F_{t,m}(\rho(q))$, where $\rho$ is the $g_0$-distance from the above point $p$ to $q \in \mathbb{R}^{2n+1}$ and let $h^b_{\rho} \in C^{\infty}(\mathbb{R}^{2n+1}, \mathbb{R}^{\geq 0})$ be $h^b_{\rho}(q) = H^b_{\rho}(\rho(q))$. We choose $b = 9$ and $\epsilon = \epsilon_1$. We consider the Riemannian metric $e^{2\phi_t} \tilde{g}_t$, where

$$\phi_t(\rho) = f_{t,m}(9 + \epsilon_1 - \rho) \cdot h^b_{\rho}(9 + \epsilon_1 - \rho) = m t^2 e^{-\frac{m^2}{\rho}} h^b_{\rho}(9 + \epsilon_1 - \rho).$$

We consider the scalar curvature $s(e^{2\phi_t} \tilde{g}_t)$. We easily get \( \frac{ds(e^{2\phi_t} \tilde{g}_t)}{dt}|_{t=0} = 0 \). Using the conformal deformation formula \( s(e^{2\phi_t} g_t) = e^{-2\phi_t} (s_{\tilde{g}_t} + 4n \Delta_t \phi_t - 2n(2n-1) \nabla_{\tilde{g}_t} \phi_t^2) \), we calculate as in [8, Section 4] to show that \( \frac{ds(e^{2\phi_t} \tilde{g}_t)}{dt}|_{t=0} < 0 \) on $B_{9+\epsilon_1}(p)$ for small $m > 0$. Note that $e^{2\phi_t} \tilde{g}_t = g_0$ on $\mathbb{R}^{2n+1} \setminus B_{9+\epsilon_1}(p)$.

But due to the boundary $\partial B_{9+\epsilon_1}(p)$, we can not yet conclude the existence of a constant $\epsilon$ such that $s(e^{2\phi_t} \tilde{g}_t)$ is strictly decreasing in the ball $B_{9+\epsilon_1}(p)$ for $0 \leq t \leq \epsilon$.

We continue to follow the argument in [8, Section 4] to show that $\frac{ds(e^{2\phi_t} \tilde{g}_t)}{dt}|_{t=0} < 0$ on $B_{9+\epsilon_1}(p) \setminus B_{9}(p)$ when $0 < t < t_0$ for some number $t_0 > 0$. This yields a scalar-curvature melting $g_t = e^{2\phi_t} \tilde{g}_t$ on $B_{9+\epsilon_1}(p)$. By pulling it back by an affine transformation, we can get a scalar-curvature melting on the unit ball.

In even dimensions, we start with the metrics in Proposition 2.2 and proceed similarly as in Section 3 and Section 4. Then we can get a scalar-curvature melting on the unit ball in $\mathbb{R}^{2n}$, $n \geq 2$. This proves Theorem 1.1.

Remark 4.1. The odd dimensional metric in Proposition 2.1 is in fact a contact metric compatible with the standard contact structure on $\mathbb{R}^{2n+1}$. We suspect our melting can be done in the space of contact metrics. It is very interesting to find a scalar curvature melting of a general metric on a ball, not to mention a Ricci-curvature melting.

5. Fubini-Study metric

In this section we demonstrate that the arguments for Euclidean metrics can work similarly for the Fubini-Study metric.

We need to discuss in the context of almost Kähler metrics, which are Riemannian metrics $g$ compatible with a symplectic structure $\omega$, i.e., $\omega(X, Y) =$
$g(X, Y)$ for an almost complex structure $J$, where $X, Y$ are tangent vectors. Here $\omega$ and $g$ determine $J$. One may refer to [2] for some knowledge of almost Kähler geometry needed in this section. In this geometry, for the canonical hermitian connection $\nabla$ determined by $J$ we have the corresponding hermitian scalar curvature $s^\nabla$. It proves to be equal to $\frac{1}{2}(s^* + s)$, where $s^*$ is the star-scalar curvature. It is known that $s^* - s = \frac{1}{2}|DJ|^2$, where $D$ is the Levi-Civita connection. So $s^\nabla \geq s$, with equality if and only if $(\omega, g)$ is Kähler.

In [9, Subsection 4.1], for a toric symplectic manifold $(M^{2n}, \omega)$, i.e., a symplectic manifold equipped with an effective Hamiltonian action of an $n$-dimensional torus $T$, M. Lejmi considered $\omega$-compatible $T$-invariant almost Kähler metrics $g$ which have the local expression

\begin{equation}
(4) \quad g = \sum_{i,j=1}^{n} G_{ij}(z) dz_i \otimes dz_j + H_{ij}(z) dt_i \otimes dt_j,
\end{equation}

where $z_1, \ldots, z_n$ are moment coordinates corresponding to Hamiltonian vector fields generating $T$ action and $H = (H_{ij})$ is a symmetric positive-definite matrix-valued function and $G = (G_{ij})$ is the inverse matrix of $H$. In $z, t$ coordinates, $\omega = \sum dz_i \wedge dt_i$. Any metric of the form $(4)$ is $\omega$-compatible almost Kähler. He computed that $s^\nabla = \frac{1}{2}(s + s^*) = -\sum_{i,j=1}^{n} H_{ij,ij}$, where $(i, ij) = \frac{\partial^2 (\cdot)}{\partial z_i \partial z_j}$.

**Example** ([1]). Consider the complex projective space $\mathbb{CP}_n$ with the Fubini-Study metric $g_{FS}$ in homogeneous coordinates $[z_0, z_1, \ldots, z_n]$. We denote the Kähler form by $\omega_{FS}$. The $T^n$-action on $\mathbb{CP}_n$ given by $(y_1, \ldots, y_n)[z_0, z_1, \ldots, z_n] = [z_0, e^{-y_1}z_1, \ldots, e^{-y_n}z_n]$, is Hamiltonian, with moment map $\mu: \mathbb{CP}_n \to \mathbb{R}^n$ given by $\mu([z_0, z_1, \ldots, z_n]) = \frac{1}{\sqrt{\mu}}(\|z_1\|^2, \ldots, \|z_n\|^2)$.

Set $S_1 := \{ (x_1, \ldots, x_n) \mid \text{each } x_i > 0, \sum_{i=1}^{n} x_i < t \} \subset \mathbb{R}^n$. Then the image of $\mu$ is the closure of $S_1$. $g_{FS}$ can be expressed as (4) with some $H_{ij}^0(z)$.

**Proposition 5.1.** Given an open set $S_c$, $0 < c < 1$, there exists a family of $T^n$-invariant almost-Kähler metrics $(\omega_{FS}, \tilde{g}_t)$ on $\mathbb{CP}_n$, $0 \leq t < \epsilon_2$ for some number $\epsilon_2$ such that

(i) on $\mathbb{CP}_n - \mu^{-1}(S_c)$: $\tilde{g}_t = g_{FS}$ for $0 \leq t < \epsilon_2$,

(ii) on $\mathbb{CP}_n$: $\tilde{g}_0 = g_{FS}$, $s^{\nabla_{\tilde{g}_t}} = s^{\nabla_{g_{FS}}}$ and $s(\tilde{g}_t) \leq s(\tilde{g}_0)$ for $0 \leq t < \epsilon_2$,

(iii) $s(\tilde{g}_t) < s(\tilde{g}_0)$ for $0 < t < \epsilon_2$ on some open subset $W$ of $\mu^{-1}(S_c)$.

**Proof.** Set $H_{ij}^0(z) = H_{ij}^0(\tilde{g}_t) + tU_{ij}(z)$ and we denote the corresponding metric in $(4)$ by $\tilde{g}_t$. The condition $s^{\nabla_{\tilde{g}_t}} = s^{\nabla_{g_{FS}}}$ is equivalent to $\sum_{i,j=1}^{n} U_{ij}(z) = 0$. For its solution, choose $U = (U_{ij})$ as the diagonal matrix with diagonal entries $U_{ii}(z) = \alpha_1^i(z_1) \cdots \alpha_n^i(z_n) \int_0^{z_i} \left( \int_0^{y_i} \alpha_i^j(x) \, dx \right) \, dy$ for $i = 1, \ldots, n - 1$,
where \( \cdot \) denotes the missing factor in that position,

\[
U_{nn}(z) = -\sum_{i=1}^{n-1} \alpha_i^j(z_1) \cdots \alpha_i^{n-1}(z_{n-1}) \int_0^y (\int_0^{\alpha_i^j(x)} dx) dy,
\]

where \( \alpha_i^j(z_j), i = 1, \ldots, n-1, j = 1, \ldots, n \) are smooth functions on \( \mathbb{R} \) which satisfy at least \( \alpha_i^j(z_j) = 0 \) for \( z_j \leq 0 \), or \( z_j \geq \tilde{c} \) for some \( \tilde{c} > 0 \). This is similar to the solution of the equation (2). Again, one can properly choose \( \tilde{c} \) small and \( \alpha_i^j \) so that \( U_{ij} \) become smooth functions with compact support in \( \mu^{-1}(S_c) \) and that \( g_t, t > 0 \), is an almost Kähler metric which is non-Kähler, i.e., \( \frac{1}{2} |DJ|^2 = s^* - s \neq 0 \) somewhere. Indeed, either by direct computation on a component of \( DJ \) or by an argument using [5, Section 4], one can find \( \{U_{ij}\} \) so that near some chosen point \( \hat{g}_t \) is non-Kähler for any small \( t \).

As \( (\omega_{FS}, g_{FS}) \) is Kähler, \( s(g_{FS}) = s^{\nabla_{g_0}} \). But then, \( s^{\nabla_{g_0}} = s^{\nabla_{g_t}} \geq s(\hat{g}_t) \) with equality exactly where \( (\omega, \hat{g}_t) \) is Kähler. This proves that \( s(\hat{g}_t) < s(g_{FS}) \) for \( 0 < t < \epsilon \) on an open pre-compact subset \( W \) of \( \mu^{-1}(S_c) \). \[ \square \]

The metrics \( \hat{g}_t \) play the same role as those in Propositions 2.1 or 2.2.

**Theorem 5.2.** Suppose we are given a point \( p_0 \in \mathbb{C}P_n \) and a number \( r_0 \) with \( 0 < r_0 < \frac{1}{2} \text{diameter}(g_{FS}) \). Then there exists a \( C^\infty \)-continuous path of Riemannian metrics \( g_t \) on \( \mathbb{C}P_n \), which exists for \( 0 \leq t < \epsilon \) for some number \( \epsilon \) with the following property: \( g_0 = g_{FS} \), \( s(g_t) < s(g_{FS}) \) for \( 0 \leq t < \epsilon \) in the ball \( B_{FS}^{g_{FS}}(p_0) \) of \( g_{FS} \)-radius \( r_0 \) centered at \( p_0 \) and \( g_t \) is isometric to \( g_{FS} \) in the complement of the ball.

**Proof.** Since \((\mathbb{C}P_n, g_{FS}) \) is homogeneous, we may choose the coordinates and hamiltonian \( T^n \) action so that \( p_0 = \mu^{-1}(0, \ldots, 0) \). We choose \( c \) so that \( \mu^{-1}(S_c) \subset B_{FS}^{g_{FS}}(p_0) \) and get \( \hat{g}_t \) in Proposition 5.1. Choose the smallest natural number \( k \) such that \( \frac{d^k s_{g_t}}{dt^k} \big|_{t=0} \) is not identically zero. This \( k \) exists because at each point \( s_{g_t} \) is a rational function of \( t \). Then \( \frac{d^k s_{g_t}}{dt^k} \big|_{t=0} = 0 \) for \( j = 1, \ldots, k-1 \).

We consider a smooth coordinates system \( y := y_1, \ldots, y_{2n} \) on \( B_{2r_0}^{g_{FS}}(p_0) \), which is a topological ball, such that \( y(0) = p \) and \( B_{2r_0}^{g_{FS}}(p_0) \) becomes a \( y \)-coordinates ball of radius, say \( R \). Let \( g^0 \) be the Euclidean metric \( g^0 = dy_1^2 + \cdots + dy_{2n}^2 \) and \( \rho = \sqrt{\sum_{i=1}^{2n} y_i^2} \).

From now on, \( B_r(\cdot) \) means a ball of \( g^0 \)-radius \( r \) with center at \( \cdot \). For some positive number \( \epsilon < \frac{\rho}{n} \), \( B_{2\epsilon}(p) \) should satisfy \( B_{2\epsilon}(p) \cap \{q \mid \frac{d^k s_{g_t}}{dt^k}(q) \big|_{t=0} = 0\} = \emptyset \). Choosing \( \epsilon \) further small if necessary, we assume that \( B_{R-\epsilon}(p) \supset B_{g_{FS}}^{2r_0}(p_0) \).

Define \( F_{t,m}^d(x) = mt^k e^{-\frac{\rho}{n}} \). We consider \( g_t := e^{2\phi_t} \hat{g}_t \), where \( \phi_t(p) = F_{t,m}^d(b + \epsilon - \rho) \cdot h_b^0(b + \epsilon - \rho) \). We set \( b = R - \epsilon \). \( m \) and \( d \) shall be determined below.
The scalar curvature is as follows; \( s(g_t) = e^{-2\phi_t} B \), where \( B = s_{g_t} + a_n \Delta_{g_t} \phi_t - b_n |\nabla_{g_t} \phi_t|^2 \) for some positive numbers \( a_n, b_n \) depending on \( n \). Then

\[
(5) \quad \frac{ds(g_t)}{dt} = e^{-2\phi_t} \left( -2 \frac{d\phi_t}{dt} B + \frac{ds_{g_t}}{dt} + a_n \frac{d\Delta_{g_t} \phi_t}{dt} - b_n \frac{d|\nabla_{g_t} \phi_t|^2}{dt} \right).
\]

We easily get \( \frac{ds(g_t)}{dt} |_{t=0} = 0 \) for \( j = 1, \ldots, k - 1 \) and

\[
\frac{d^k s(g_t)}{dt^k} |_{t=0} = -2k! m s_{g_0} e^{-\frac{d}{4} \phi} h_\nu^k(\beta + \epsilon - \rho) + \frac{d^k s_{g_t}}{dt^k} |_{t=0} = a_n \frac{d^k \Delta_{g_t} \phi_t}{dt^k} |_{t=0}. \]

On \( B_{b+\tau}(p) - B_{c}(p) \), since \( h_\nu^k(\beta + \epsilon - \rho) = 1 \) we have

\[
\frac{d^k s(g_t)}{dt^k} |_{t=0} \leq -2k! m s_{g_0} e^{-\frac{d}{4} \phi} + a_n \frac{d^k \Delta_{g_t} \phi_t}{dt^k} |_{t=0}
\]

\[
= mk!(-2s_{g_0} e^{-\frac{d}{4} \phi} + a_n \Delta_{g_0} e^{-\frac{d}{4} \phi})
\]

\[
\leq mk!(-2s_{g_0} G - \alpha_1 G'' - \alpha_2 G') < 0, \quad \text{when } d \text{ is large},
\]

where \( G(\rho) = e^{-\frac{d}{4} \phi} \) and \( \alpha_1, \alpha_2 \) are some positive numbers and we used Lemmas 5.3 and 5.4 below. On \( B_{c}(p) \), \( \frac{d^k s(g_t)}{dt^k} |_{t=0} < -c_1 < 0 \) for some number \( c_1 > 0 \), so choose \( m > 0 \) small so that \(-2k! m s_{g_0} e^{-\frac{d}{4} \phi} + \frac{d^k s_{g_t}}{dt^k} |_{t=0} + a_n \frac{d^k \Delta_{g_t} \phi_t}{dt^k} |_{t=0} < 0 \).

In sum, we have \( \frac{ds(g_t)}{dt} |_{t=0} = 0 \) for \( j = 1, \ldots, k - 1 \) and \( \frac{ds(g_t)}{dt} |_{t=0} < 0 \) on \( B_{b+\tau}(p) \) and \( g_t = g_0 \) on \( M - B_{b+\tau}(p) \). On \( B_{\rho}(p) \), there exists \( \epsilon_3 > 0 \) such that \( s(g_t) \) is strictly decreasing for \( 0 \leq t \leq \epsilon_3 \).

On \( B_{b+\tau}(p) - B_{\rho}(p) \), \( g_t = g_0 \). From (5), Lemmas 5.3, 5.4 and 5.5, for large \( d \),

\[
e^{2\phi_t} \frac{ds(g_t)}{dt}
\]

\[
= -2 \frac{d\phi_t}{dt} (s_{g_0} + a_n \Delta_{g_0} \phi_t - b_n |\nabla_{g_0} \phi_t|^2) + a_n \frac{d\Delta_{g_0} \phi_t}{dt} - b_n \frac{d|\nabla_{g_0} \phi_t|^2}{dt}
\]

\[
\leq km^{k-1}(-2s_{g_0} G - \alpha_1 G'' - \alpha_2 G') \quad \text{for numbers } \alpha_1, \alpha_2 > 0,
\]

while \( 0 < t < \epsilon_4 \) for some \( \epsilon_4 \). This implies that \( s(g_t) \) is strictly decreasing for \( 0 \leq t < \epsilon_4 \) on \( B_{b+\tau}(p) - B_{\rho}(p) \). So, \( s(g_t) \) is strictly decreasing for \( 0 \leq t < \epsilon = \min\{\epsilon_3, \epsilon_4\} \) on \( B_{b+\tau}(p) \). This proves Theorem 5.2. \( \square \)
For the function $F(t) = e^{-\frac{t}{2}}$ on $\mathbb{R}^{>0}$, one can modify easily Lemma 1.2 in [10] as follows: for $m_0, m_1 \in \mathbb{R}$ and $m_2, b \in \mathbb{R}^{>0}$ there exist numbers $d_0(b) > 0$ and $d_1(m_0, m_1, m_2, b) > 0$ such that $F^{(j)} := \frac{d_1}{m_j} > 0$ on $(0, b)$ for $j = 0, 1, 2, 3$ if $d \geq d_0(b)$ and $m_2 F'' + m_1 F' + m_0 F > 0$ on $(0, b)$ if $d \geq d_1(m_0, m_1, m_2, b)$. Since $G^{(j)}(\rho) = (-1)^j F^{(j)}(b + \epsilon - \rho)$, we get:

**Lemma 5.3.** For $m_0, m_1 \in \mathbb{R}$ and $m_2, b \in \mathbb{R}^{>0}$, there exists $d_2(m_0, m_1, m_2, b) > 0$ such that $m_2 G'' + m_1 G' + m_0 G > 0$ on $(\epsilon, b + \epsilon)$ if $d \geq d_2(m_0, m_1, m_2, b)$. And $(-1)^j G^{(j)} > 0$ on $(\epsilon, b + \epsilon)$ for $j = 0, 1, 2, 3$ if $d \geq d_0(b)$.

Next, we modify Corollary 2.3 in [10] as follows. Assume that $g$ on a domain $D \subset \mathbb{R}^{n+1}$ fulfill the following two conditions for some $k > 1$: (i) $g_{Euc}(\nu, \nu) \leq k^2 \cdot g(\nu, \nu)$. (ii) The $C^3$-norm $\|g\|_{C^3_{Euc}}(D) \leq k$. Let $H \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function with $H' \leq 0$, $H'' \geq 0$. Then there are constants $a_1, a_2 > 0$ depending only on $n$ and $k$ such that $(a_1 H'' + a_2 H') \circ \pi \leq -\Delta_g (H \circ \pi)$ on $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the projection. This can be easily verified, following the argument in pp. 660–661 in [10].

We can choose a coordinates system $(u_1, \ldots, u_{2n})$ with $u_1 = \rho$ on a proper subdomain $\tilde{D}$ of $B_{b+\epsilon}(p) - B_\epsilon(p)$ so that (i) and (ii) holds with $g_{Euc} := d\rho^2 + du_2^2 + \cdots + du_{2n}^2$. Applying the above paragraph to $g_0|_{\tilde{D}}$ and $G$, we get:

**Lemma 5.4.** If $d \geq d_0(b)$, there are constants $a_1, a_2 > 0$ such that $\Delta_g G(\rho) \leq -a_1 G'' - a_2 G'$ on $B_{b+\epsilon}(p) - B_\epsilon(p)$.

Putting Lemmas 5.3 and 5.4 together;

**Lemma 5.5.** $\Delta_g G(\rho) < 0$ on $B_{b+\epsilon}(p) - B_\epsilon(p)$ if $d$ is large.

**Remark 5.6.** For the Fubini-Study metric, the kernel of $L^*_g$ on $\mathbb{CP}_n$ is trivial. But we do not know if the kernel of $L^*_g$ is trivial when restricted to a ball. In any case, our construction gives a large amount of deformation, compared to the small deformation of Corvino’s, as the latter is based on Implicit Function Theorem.

**References**


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