THE INVARIANCE PRINCIPLE FOR RANDOM SUMS OF A DOUBLE RANDOM SEQUENCE

ZHENLONG GAO AND LIANG FANG

Abstract. In this paper, we extend Donsker’s invariance principle to the case of random partial sums processes based on a double sequence of row-wise i.i.d. random variables.

1. Introduction

The early studies of the invariance principle for partial sums of an i.i.d. random sequence are dated back to P. Erdös and M. Kac ([5, 6]). Various particular cases of the invariance principle are derived in their articles. The present paper deals with the general form of the invariance principle defined as following:

Definition 1.1. (Let \( \{Y_n, n \geq 1\} \) be a sequence of random variables and
\( \{g_n(a_1, \ldots, a_n), n \geq 1\} \)
be a sequence of Borel measurable functions. If the limit distribution

\[
\lim_{n \to \infty} P(g_n(Y_1, \ldots, Y_n) < \lambda), \quad -\infty < \lambda < \infty
\]

does not depend on the distributions of \( \{Y_n\} \), then it is said that \( \{Y_n\} \) satisfies the invariance principle of \( \{g_n\} \).)

The first general invariance principle for partial sums of i.i.d. random variables is due to M. Donsker ([4]). Let \( C = C[0, 1] \) be the space of continuous functions on \( [0, 1] \) and \( C \) be the Borel \( \sigma \)-field with respect to the uniform topology, that is, for any \( x, y \in C \),

\[
\rho(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|.
\]

Received June 2, 2012; Revised December 4, 2012.
2010 Mathematics Subject Classification.  60F05, 60F17.
Key words and phrases. double random sequence, invariance principle, weak convergence.

The first author is financially supported by the Youth Foundation and Doctor’s Initial Foundation of Qufu Normal University (XJ201113, BSQD20110127). The second author is financially supported by Scientific Research Fund of Hunan Provincial Education Department (No. 12C0027) and The Open Fund Project of Key Research Institute of Philosophies and Social Sciences in Hunan Universities (No.12FEFM006).

©2013 The Korean Mathematical Society

1539
Denote \( W \) to be the Wiener measure on \( (C, \mathcal{C}) \) and \( C^* \) to be the space of bounded continuous functions on \( (C, \mathcal{C}) \). Let \( X = X[0, 1] \) be the space of continuous functions except for finite points on \([0, 1]\) and \( X^* \) be the space of bounded continuous functions on \( X \) with respect to the uniform topology. M. Donsker ([4]) obtained the following result: Let \( \{X_n\} \) be a sequence of i.i.d. random variables. Define \( S_n = \sum_{i=1}^{n} X_i \) and

\[
x_n(t, a_1, \ldots, a_n) = \begin{cases} \frac{a_i}{\sqrt{n}} & t \in \left(\frac{i-1}{n}, \frac{i}{n}\right], \ i = 1, \ldots, n; \\ 0 & t = 0. \end{cases}
\]

For any \( f \in X^* \), define \( g_n(a_1, \ldots, a_n) = f(x_n(t, a_1, \ldots, a_n)) \), then \( \{S_n\} \) satisfies the invariance principle of \( \{g_n\} \) and

\[
\lim_{n \to \infty} P(g_n(S_1, \ldots, S_n) < \lambda) = W(x \in C : f(x) < \lambda), \ \lambda \in (-\infty, \infty).
\]

D. H. Hu ([9]) extended M. Donsker’s result to the case of random sums. Let \( \{Z_n, n \geq 1\} \) be a sequence of positive integer valued random variables and \( \{c_n, n \geq 1\} \) be a sequence of positive real numbers such that

\[
c_n \to \infty \text{ and } \frac{Z_n}{c_n} \to Z, \ n \to \infty,
\]

where \( Z \) is a positive random variable independent of \( \{Z_n, n \geq 1\} \), then (1.2) is changed into

\[
\lim_{n \to \infty} P(g_{Z_n}(S_1, \ldots, S_{Z_n}) < \lambda) = W(x \in C : f(x) < \lambda), \ \lambda \in (-\infty, \infty).
\]

In more recent literatures, (1.1) is often modified by

\[
x_n(t, a_1, \ldots, a_n) = \begin{cases} \frac{a_i}{\sqrt{n}} + n(t - \frac{i-1}{n})(\frac{a_i}{\sqrt{n}} - \frac{a_{i-1}}{\sqrt{n}}), & t \in \left(\frac{i-1}{n}, \frac{i}{n}\right], \ i = 1, \ldots, n; \\ 0 =: a_0, & t = 0. \end{cases}
\]

and (1.2) can be written as the following version:

\[
\psi_n(t) := \frac{1}{\sqrt{n}} (S_{[nt]} + (nt - [nt])X_{[nt]+1}), \ t \in [0, 1],
\]

then \( \{\psi_n(t)\}_{t \in [0, 1]} \overset{d}{\to} \text{SBM} \), where \([x]\) is the maximal integer that no more than \( x \) and \( \text{SBM} \) is a standard Brown motion on \([0, 1]\). (1.4) can be rewritten accordingly.

This general case of invariance principle has been widely studied for many topics (see P. Billingsley ([1]), P. Hall and C. C. Heyde ([8]), M. Peligrad ([11]), Q. M. Shao ([13])). Recently, many researchers investigated the corresponding results for triangular arrays of random variables. For example, A. De Acosta ([3]) derived the invariance principle for triangular arrays of row-wise i.i.d. \( B \)-valued random vectors, where each row has an infinitely divisible distribution (see also A. D’Aristotile ([2]), A. Rackauskas and C. Suquet ([12])).
We are interested in the invariance principle for random partial sums processes based on a double sequence of row-wise i.i.d. random variables \( \{\xi_{n,j}, n \geq 0, j \geq 1\} \), which arose from branching process in varying environment (see D. H. Fearn ([7])).

Throughout this paper we assume that \( E(\xi_{n,j}) \equiv 0, \text{Var}(\xi_{n,j}) \equiv 1, n \geq 0, j \geq 1 \). Define \( T^{(n)}_m = \sum_{i=1}^m \xi_{n,i}, m \geq 1 \) and

\[
\mu^{(n)}_m(t) = \frac{1}{\sqrt{m}} \left( T^{(n)}_{\lfloor mt \rfloor} + (mt - \lfloor mt \rfloor) \xi_{n,\lfloor mt \rfloor + 1} \right), \quad t \in [0, 1],
\]

then we have the following result:

**Theorem 1.1.** Let \( \{Z_n, n \geq 1\}, \{c_n, n \geq 1\} \) and \( Z \) satisfy (1.3), where \( Z \) is independent of \( \{\xi_{n,j}, n \geq 0, j \geq 1\} \), then we have the following result:

\[
Theorem 1.1. \text{Let} \{Z_n, n \geq 1\}, \{c_n, n \geq 1\} \text{and} Z \text{ satisfy (1.3), where} Z \text{ is independent of} \{\xi_{n,j}, n \geq 0, j \geq 1\}, \text{then}
\]

\[
\lim_{n \to \infty} P(E^{(n)}_{Z_n}) = \vartheta(E).
\]

For any \( n \geq 1, x \in C, j = 1, 2, \ldots, k \), define \( R_n = \{\omega \mid \omega \in \Omega, \alpha_j \leq \mu^{(n)}_{Z_n}(t) \leq \beta_j, t \in I_{k,j}, j = 1, \ldots, k\} \),

\[
R_n = \{\omega \mid \omega \in \Omega, \alpha_j \leq \mu^{(n)}_{Z_n}(t) \leq \beta_j, t \in I_{k,j}, j = 1, \ldots, k\},
\]

\[
p^{(n)}_j = \sup_{t \in I_{k,j}} \mu^{(n)}_{Z_n}(t), q^{(n)}_j = \inf_{t \in I_{k,j}} \mu^{(n)}_{Z_n}(t),
\]
The second step is to prove:

**Lemma 2.2.** If the conditions in Theorem 1.1 are satisfied, we have

\begin{equation}
\lim_{n \to \infty} P(R_n) = \mathcal{W}(E).
\end{equation}

For any bounded and Borel measurable function \( g : \mathbb{R}^{2k} \to \mathbb{R} \), one has

\begin{equation}
\nabla g := \lim_{n \to \infty} \int_{\Omega} g\left(p_1^{(n)}, \ldots, p_k^{(n)}, q_1^{(n)}, \ldots, q_k^{(n)}\right) dP
= \int_C g(p_1(x), \ldots, p_k(x), q_1(x), \ldots, q_k(x)) \mathcal{W}(dx).
\end{equation}

The last step is to prove:

**Lemma 2.3.** If the conditions in Theorem 1.1 are satisfied, for any \( h \in C^* \) one has

\begin{equation}
\lim_{n \to \infty} \int_{\Omega} h\left(p_{Z_n}^{(n)}(t)\right) dP = \int_C h(x) \mathcal{W}(dx).
\end{equation}

Theorem 1.1 follows from Lemma 2.3.

### 3. Proof of Lemma 2.1

We follow the notations introduced in above sections. The proof of Lemma 2.1 is divided into three steps. First, we prove that:

**Lemma 3.1.** Let \( \{l_n, n \geq 1\} \) be a sequence of positive integers with \( \lim_{n \to \infty} l_n = \infty \). Then one has

\begin{equation}
\lim_{n \to \infty} P\left(E_{l_n}^{(n)}\right) = \mathcal{W}(E).
\end{equation}

Second, using Lemma 3.1 we prove that:

**Lemma 3.2.** Let \( \{Z_n, n \geq 1\} \) be a sequence of positive integer valued random variables and \( \{c_n, n \geq 1\} \) be a sequence of positive real numbers such that

\begin{equation}
c_n \to \infty \quad \text{and} \quad \frac{Z_n}{c_n} \overset{P}{\to} c > 0, \quad n \to \infty,
\end{equation}

where \( c \) is a constant. Then we have

\begin{equation}
\lim_{n \to \infty} P\left(E_{Z_n}^{(n)}\right) = \mathcal{W}(E).
\end{equation}

Finally, we prove that Lemma 2.1 follows from Lemma 3.2.
3.1. Proof of Lemma 3.1

Lemma 3.3 (c.f. [10]). Let \( \{k_n, n \geq 0\} \) be a sequence of positive integers with \( k_n \to \infty \). Then one has
\[
\frac{X_{k_n}}{\sqrt{k_n}} \overset{d}{\to} N(0, 1), \quad n \to \infty.
\]

Lemma 3.4. Let \( \{(Y_{t_1}^{(n)}, Y_{t_2}^{(n)}, \ldots, Y_{t_m}^{(n)}), n \geq 1\} \) be a sequence of random vectors taking values in \( \mathbb{R}^m \) such that
\[
t_i > 0, \quad i = 1, \ldots, m, \quad \sum_{i=1}^m t_i \leq 1,
\]
for each \( n \), \( \{Y_{t_1}^{(n)}, Y_{t_2}^{(n)}, \ldots, Y_{t_m}^{(n)}\} \) are independent and for each \( i = 1, \ldots, m \), \( Y_{t_i}^{(n)} \overset{d}{\to} N(0, t_i) \) when \( n \to \infty \). For any vectors \( \{a_j, 1 \leq j \leq m\}, \{b_j, 1 \leq j \leq m\} \in \mathbb{R}^m \), write
\[
G_n = \left\{ \omega : a_j \leq \sum_{i=1}^j Y_{t_i}^{(n)} \leq b_j, 1 \leq j \leq m \right\},
\]
\[
G = \left\{ x \in C : a_j \leq x \left( \sum_{i=1}^j t_i \right) \leq b_j \right\}.
\]
Then one has
\[
(3.4) \quad \lim_{n \to \infty} P(G_n) = \nu(G).
\]

Proof. Since for each \( i = 1, \ldots, m \), \( Y_{t_i}^{(n)} \overset{d}{\to} N(0, t_i) \) when \( n \to \infty \), one has
\[
\frac{Y_{t_i}^{(n)}}{\sqrt{t_i}} \overset{d}{\to} N(0, 1), \quad n \to \infty, \quad i = 1, \ldots, m.
\]

Note that for each \( n \), \( \{Y_{t_1}^{(n)}, Y_{t_2}^{(n)}, \ldots, Y_{t_m}^{(n)}\} \) are independent, by (4.16) and (4.17) of P. Billingsley ([1, p. 26]) we know
\[
\left( \frac{Y_{t_1}^{(n)}}{\sqrt{t_1}}, \frac{Y_{t_2}^{(n)}}{\sqrt{t_2}}, \ldots, \frac{Y_{t_m}^{(n)}}{\sqrt{t_m}} \right) \overset{d}{\to} N(0, I_{m \times m}), \quad n \to \infty,
\]
where \( I_{m \times m} \) is the unit matrix of order \( m \times m \). Define
\[
A = \begin{pmatrix}
\sqrt{t_1} & \sqrt{t_1} & \cdots & \sqrt{t_1} \\
0 & \sqrt{t_2} & \cdots & \sqrt{t_2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sqrt{t_m}
\end{pmatrix},
\]

where for each \( n \), \( \{Y_{t_1}^{(n)}, Y_{t_2}^{(n)}, \ldots, Y_{t_m}^{(n)}\} \) are independent and for each \( i = 1, \ldots, m \), \( Y_{t_i}^{(n)} \overset{d}{\to} N(0, t_i) \) when \( n \to \infty \).
where $S (3.6)$

It is obvious that for any $1 \leq r \leq l_n$, there exists $\{1 \leq j_r \leq k\}$ such that

$$\eta_{j_r, i} < r \leq \eta_{j_r, l_n},$$

where $\eta_{j, m}$ is defined in (2.1). Define

$$E_{l_n, r}^{(n)} = \left\{ \omega : \alpha_j \leq S^{(n)}_i (l_n) \leq \beta_j, \eta_{j_r, l_n}, 1 \leq j \leq j_r - 1 \right\}$$

$$\cap \left\{ \omega : \alpha_{j_r} \leq S^{(n)}_i (l_n) \leq \beta_{j_r}, \eta_{j_r, l_n}, 1 \leq j \leq j_r - 1 \right\}$$

$$\cap \left\{ \omega : \alpha_{j_r} \leq S^{(n)}_i (l_n) \leq \beta_{j_r} \right\}^c.$$

It is obvious that

$$1 - P \left( E_{l_n}^{(n)} \right) = \sum_{r=1}^{l_n} P \left( E_{l_n, r}^{(n)} \right),$$

where $E_{l_n}^{(n)}$ is defined in (2.2). Let $\chi$ be any fixed positive integer and $\epsilon$ be any fixed positive real number. For any $1 \leq j \leq k$, $0 \leq d \leq \chi$, define

$$l_n (j, d) = \left\lceil \frac{(j-1)l_n}{k} + \frac{d}{\chi} \frac{l_n}{k} \right\rceil.$$

For any $1 \leq r \leq l_n$, there exists $0 \leq d_r \leq \chi$ such that

$$l_n (j_r, d_r) < r \leq l_n (j_r, d_r + 1).$$

It is obvious that

$$P \left( E_{l_n, r}^{(n)} \right) = P \left( E_{l_n, r}^{(n)} \cap \left( \left| S^{(n)}_{l_n (j_r, d_r + 1)} (l_n) - S^{(n)}_{l_n (j_r, d_r + 1)} (l_n) \right| \geq \epsilon \right) \right)$$

$$+ P \left( E_{l_n, r}^{(n)} \cap \left( \left| S^{(n)}_{l_n (j_r, d_r + 1)} (l_n) - S^{(n)}_{l_n (j_r, d_r + 1)} (l_n) \right| < \epsilon \right) \right),$$

where $S^{(n)}_i (m)$ is defined in (2.1). Note that

$$E(X_{n, j}) \equiv 0, \ Var(X_{n, j}) \equiv 1, \ n \geq 0, \ j \geq 1,$$

$$l_n (j_r, d_r + 1) - r \leq l_n (j_r, d_r + 1) - l_n (j_r, d_r) \leq \left\lfloor \frac{l_n}{k \chi} \right\rfloor,$$
according to Tchebychev’s inequality one has

\[ P \left( \left| \frac{S_{n}^{(n)}(i, j, d, r)}{l_n} - S_1^{(n)}(l_n) \right| \geq \epsilon \right) \leq \frac{1}{\epsilon^2 k^{\chi}}, \]

which means that the first term on the right of (3.6) is bounded by \(1/(\epsilon^2 k^{\chi})\).

For the second term on the right of (3.6), we define

\[ F_n^{(n)} = \left\{ \omega : \alpha_j \leq S_n^{(n)}(i, j, d, r) \leq \beta_j, j = 1, \ldots, k; d = 0, 1, \ldots, \chi \right\}, \]

\[ F_n^{(n), \epsilon} = \left\{ \omega : \alpha_j + \epsilon \leq S_n^{(n)}(i, j, d, r) \leq \beta_j - \epsilon, j = 1, \ldots, k; d = 0, 1, \ldots, \chi \right\}. \]

According to the definition of \(E_{i_n, r}^{(n)}\), for any \(1 \leq r \leq l_n\) one has

\[ \Delta_{i_n, r}^{(n)} = E_{i_n, r}^{(n)} \cap \left( S_n^{(n)}(i, j, d, r) - S_1^{(n)}(l_n) \right) < \epsilon \]

\[ \subseteq \left\{ \omega : \alpha_j \leq S_n^{(n)}(i, j, d, r) \leq \beta_j \right\} \cap \left( S_n^{(n)}(i, j, d, r) - S_1^{(n)}(l_n) \right) < \epsilon \]

\[ \subseteq \left\{ \omega : S_n^{(n)}(i, j, d, r) < \alpha_j + \epsilon \text{ or } S_n^{(n)}(i, j, d, r) > \beta_j - \epsilon \right\} \]

\[ \subseteq [F_n^{(n), \epsilon}]. \]

Note that \(r\) is arbitrary we know that

\[ \bigcup_{r=1}^{l_n} \Delta_{i_n, r}^{(n)} \subseteq [F_n^{(n), \epsilon}]. \]

On the other hand, \(E_{i_n}^{(n)} \subseteq F_n^{(n)}\). By (3.5) and (3.6) one has

\[ P \left( F_n^{(n), \epsilon} \right) \leq \frac{1}{\epsilon^2 k^{\chi}} \leq P \left( E_{i_n}^{(n)} \right) \leq P \left( F_n^{(n)} \right). \]

For any \(\chi = 2^T\), where \(T\) is positive integer, define

\[ D_{\chi} = \left\{ x \in C : \alpha_j \leq x \left( \frac{j - 1}{k^{\chi}} + d \right) \leq \beta_j, d = 1, 2, \ldots, k^{\chi}; j = 1, \ldots, k \right\}, \]

\[ D_{\chi, \epsilon} = \left\{ x \in C : \alpha_j + \epsilon \leq x \left( \frac{j - 1}{k^{\chi}} + d \right) \leq \beta_j - \epsilon, d = 1, 2, \ldots, k^{\chi}; j = 1, \ldots, k \right\}. \]

Taking

\[ t_i = \frac{i}{k^{\chi}}, \quad Y_{t_i} = \sum_{i=1}^{[t_i/k^{\chi}]} \ell_{n, i}, \quad i = 1, 2, \ldots, k^{\chi}, \]

one can obtain that for fixed \(N\), \(\{Y_{t_i}, i = 1, 2, \ldots, k^{\chi}\}\) are independent. According to Lemma 3.3 and Lemma 3.4 one has

\[ \lim_{n \to \infty} P \left( F_n^{(n), \epsilon} \right) = \mathcal{W}(D_{\chi, \epsilon}) \quad \text{and} \quad \lim_{n \to \infty} P \left( F_n^{(n)} \right) = \mathcal{W}(D_{\chi}). \]

Hold \(\chi, \epsilon\) fixed and let \(n \to \infty\) in (3.7) we have

\[ \mathcal{W}(D_{\chi, \epsilon}) - \frac{1}{\epsilon^2 k^{\chi}} \leq \liminf_{n \to \infty} P \left( E_{i_n}^{(n)} \right) \leq \limsup_{n \to \infty} P \left( E_{i_n}^{(n)} \right) \leq \mathcal{W}(D_{\chi}). \]
Note that
\[ \lim_{\chi \to \infty} \mathcal{W}(D_\chi) = \mathcal{W}(E) \quad \text{and} \quad \lim_{\chi \to \infty} \mathcal{W}(D_\chi, \epsilon) = \mathcal{W}(E, \epsilon), \]
where \( E \) is defined in (2.3) and
\[ E_\epsilon = \{ x \in C : \alpha_j + \epsilon \leq x(t) \leq \beta_j - \epsilon, \ t \in I_j, \ j = 1, \ldots, k \}. \]
Note that when \( \epsilon \to 0 \) one has \( E_\epsilon \uparrow E \), then (3.1) is obtained if we first let \( \chi \to \infty \) and then \( \epsilon \to 0 \) in (3.8). \hfill \Box

3.2. Proof of Lemma 3.2

In the case of Lemma 3.2, for any \( c > \epsilon > 0 \), with a large probability, \( \{Z_n\} \) is dominated in \((c - \epsilon)c_n, (c + \epsilon)c_n)\) when \( n \) is sufficiently large. Taking \( l_n = [(c - \epsilon)c_n] \) and \( l_n = [(c + \epsilon)c_n] \) respectively in Lemma 3.1, it is reasonable that we can obtain the conclusion of Lemma 3.2. Details are given below.

Proof of Lemma 3.2. Since \( Z_n/c_n \xrightarrow{L} c \) when \( n \to \infty \), we know that for any \( \epsilon > 0, \delta > 0 \), there exists \( N_0 = N_0(\epsilon, \delta) \) such that for any \( n \geq N_0 \) one has
\[ P(|Z_n - cc| \geq \epsilon c_n) < \delta. \]
By (3.10) and \( P(E_{Z_n}^{(n)}) = \sum_{m=1}^{\infty} P(E_m^{(n)}, Z_n = m) \) one has
\[ I_n := \sum_{|m - cc| < \epsilon c_n} P(E_m^{(n)}, Z_n = m) \leq P(E_{Z_n}^{(n)}) \leq \delta + I_n. \]
For any \( n \geq N_0 \), denote
\[ U_n = U_n(c) := [(c - \epsilon)c_n], \quad V_n = V_n(c) := [(c + \epsilon)c_n]. \]
According to the definition of \( \eta_{j,m} \) (see (2.1)), one has
\[ 0 \leq \eta_{j,V_n} - \eta_{j,U_n} \leq V_n - U_n, \ j = 1, \ldots, k; \]
\[ \eta_{j+1, U_n} - \eta_{j,U_n} \geq \left[ \frac{U_n}{k} \right] - 1 \geq \left[ \frac{(c - \epsilon)c_n}{k} \right] - 2, \ j = 1, \ldots, k - 1. \]
Then there exists a constant \( \epsilon_0 > 0 \) such that for any \( \epsilon < \epsilon_0 \) one has
\[ \eta_{j,U_n} \leq \eta_{j,V_n} < \eta_{j+1,U_n}, \ j = 1, \ldots, k - 1. \]
In addition,
\[ \eta_{j,U_n} \leq \eta_{j,m} < \eta_{j+1,U_n}, \ j = 1, \ldots, k - 1, \ U_n \leq m \leq V_n. \]
If \( U_n \leq m \leq V_n \), that is, \( |m - cc| < \epsilon c_n \), by the definition of \( E_m^{(n)} \) (see (2.2)),
\[ E_m^{(n)} \subset \left\{ \omega : \alpha_j \leq \alpha_i^{(n)}(m) \leq \beta_j, \ \eta_{j-1,m} < i \leq \eta_{j,U_n}, \ j = 1, \ldots, k \right\} \]
\[ = \left\{ \omega : \frac{m}{U_n} \alpha_j \leq S_i^{(n)}(U_n) \leq \frac{m}{U_n} \beta_j, \ \eta_{j-1,m} < i \leq \eta_{j,U_n}, \ 1 \leq j \leq k \right\} . \]
For any \( \eta \in \mathbb{R}, \gamma > 0 \) and real numbers \( a, b \), denote
\[ E_{m, \eta}^{(n)} = \left\{ \omega : \alpha_j - \eta \leq S_i^{(n)}(m) \leq \beta_j + \eta, \right\} \]
Note that for any $\eta(j+1), m, j = 1, \ldots, k$,

\begin{equation}
A_{m,j}^{(n)}(a, b) = \left\{ \omega : a \leq S_{i}^{(n)}(m) \leq b \right\},
\end{equation}

\begin{equation}
B_{i,j}^{(n)}(\gamma) = \left\{ \omega : |S_{\eta(j+1), \nu}^{(n)}(U_n) - S_{i}^{(n)}(U_n)| < \gamma \right\},
\end{equation}

Thus, $\eta(j-1), \nu < i \leq \eta(j-1), \nu$.

Note that for any $U_n \leq m \leq V_n$ one has

\begin{equation}
A_{U_n,i}^{(n)} \left( \sqrt{\frac{m}{U_n}} \alpha_j, \sqrt{\frac{m}{U_n}} \beta_j \right) \subset A_{U_n,i}^{(n)} (\alpha_j - d, \beta_j + d),
\end{equation}

where

\begin{equation}
d = d(c) := \left( \max_{n \geq N_0} \sqrt{\frac{V_n - U_n}{U_n}} \right) \cdot \max_{1 \leq j \leq k} \{|\alpha_j|, |\beta_j|\}
\end{equation}

\begin{equation}
\leq \left( \max_{n \geq N_0} \sqrt{\frac{2\varepsilon c_n}{(c - \varepsilon)c_n}} \right) \max_{1 \leq j \leq k} \{|\alpha_j|, |\beta_j|\}.
\end{equation}

Thus,

\begin{equation}
E_{m}^{(n)} \subset \bigcap_{j=1}^{k} \bigcap_{i=\eta(j-1), m+1}^{\eta(j), \nu} A_{U_n,i}^{(n)} \left( \sqrt{\frac{m}{U_n}} \alpha_j, \sqrt{\frac{m}{U_n}} \beta_j \right)
\end{equation}

\begin{equation}
\subset \bigcap_{j=1}^{k} \bigcap_{i=\eta(j-1), \nu}^{\eta(j), V_n} A_{U_n,i}^{(n)} (\alpha_j - d, \beta_j + d).
\end{equation}

Then

\begin{equation}
E_{m}^{(n)} \subset \left\{ \left[ \bigcap_{j=1}^{k} \bigcap_{i=\eta(j-1), m+1}^{\eta(j), \nu} A_{U_n,i}^{(n)} (\alpha_j - d, \beta_j + d) \right] \right. \cap \left[ \bigcap_{j=1}^{k} \bigcap_{i=\eta(j-1), \nu}^{\eta(j), V_n} B_{i,j}^{(n)}(\gamma) \right] \cup \left[ \bigcap_{j=1}^{k} \bigcap_{i=\eta(j-1), \nu}^{\eta(j), V_n} B_{i,j}^{(n)}(\gamma) \right] \cup \right.
\end{equation}

\begin{equation}
\left. \left[ \bigcap_{j=1}^{k} \bigcap_{i=\eta(j-1), \nu}^{\eta(j), V_n} A_{U_n,i}^{(n)} (\alpha_j - d - \gamma, \beta_j + d + \gamma) \right] \cup \left[ \bigcap_{j=1}^{k} \bigcap_{i=\eta(j-1), \nu}^{\eta(j), V_n} B_{i,j}^{(n)}(\gamma) \right] \subset E_{U_n, \rho}^{(n)} \cup G_{\gamma}^{(n)}, \right.
\end{equation}

where

\begin{equation}
\rho = \rho(c) := d(c) + \gamma,
\end{equation}
\[ G_\gamma^{(n)} = \bigcup_{j=1}^k \bigcup_{i=\eta(j-1),v_n+1}^{\eta(j),v_n} [B_{i,j}^{(n)}(\gamma)]^c \]

\[ = \left\{ \omega : \max_{\eta(j-1),v_n \leq i \leq \eta(j),v_n, 1 \leq j \leq k} |S_{\eta(j-1),v_n+i}^{(n)}(U_n) - S_i^{(n)}(U_n)| \geq \gamma \right\}. \]

Similarly, when \( U_n \leq m \leq V_n \), that is, \(|m - cc_n| < \varepsilon c_n\), one has

\[ E_m^{(n)} \supset \left\{ \omega : \alpha_j \leq S_i^{(n)}(m) \leq \beta_j, \eta(j-1),m < i \leq \eta(j),v_n, j = 1, \ldots, k \right\} \supset E_{U_n, -\rho} - G_\gamma^{(n)}. \]

Note that \( E(\xi_{n,j}) \equiv 0, \ Var(\xi_{n,j}) \equiv 1, n \geq 0, j \geq 1 \), according to Kolmogorov's inequality one has

\[ P(G_\gamma^{(n)}) \leq \sum_{j=1}^k P\left( \max_{\eta(j-1),v_n \leq i \leq \eta(j),v_n} |S_{\eta(j-1),v_n+i}^{(n)}(U_n) - S_i^{(n)}(U_n)| \geq \gamma \right) \leq \sum_{j=1}^k \frac{\eta(j-1),v_n - \eta(j-1),U_n}{U_n \gamma^2} \leq \frac{k(V_n - U_n)}{U_n \gamma^2} \leq \frac{2kc_n \varepsilon}{U_n \gamma^2}. \]

According to the definition of \( I_n \) (see (3.11)) one has

\[ (3.18) \quad I_n \leq \sum_{|m - cc_n| < \varepsilon c_n} P(E_{U_n, -\rho}^{(n)} \cup G_\gamma^{(n)}, Z_n = m) \leq \sum_{|m - cc_n| < \varepsilon c_n} P(E_{U_n, -\rho}^{(n)}, Z_n = m) + 2kc_n \varepsilon \leq \frac{2kc_n \varepsilon}{U_n \gamma^2}. \]

Similarly, when \( U_n \leq m \leq V_n \), one has

\[ (3.19) \quad I_n \geq P(E_{U_n, -\rho}^{(n)}) - \delta - \frac{2kc_n \varepsilon}{U_n \gamma^2}. \]

By (3.11), (3.18) and (3.19) one has

\[ (3.20) \quad P(E_{U_n, -\rho}^{(n)}) - \delta - \frac{2kc_n \varepsilon}{U_n \gamma^2} \leq P(E_{Z_n}^{(n)}) \leq \delta + P(E_{U_n, -\rho}^{(n)}) + 2kc_n \varepsilon \leq \frac{2kc_n \varepsilon}{U_n \gamma^2}. \]

According to Lemma 3.1, if we let \( n \to \infty \) in (3.20) we have

\[ (3.21) \quad \mathcal{W}(E_{-\rho}) - \delta - \frac{2k\varepsilon}{(c - \varepsilon)\gamma^2} \leq \liminf_{n \to \infty} P(E_{Z_n}^{(n)}) \leq \limsup_{n \to \infty} P(E_{Z_n}^{(n)}) \leq \mathcal{W}(E_{\rho}) + \delta + \frac{2k\varepsilon}{(c - \varepsilon)\gamma^2}. \]
where $E_\rho$ is defined in (3.9). Note that when $\epsilon \to 0$ one has $\rho \downarrow \gamma$, then $E_\rho \uparrow E_\gamma$, if we first let $\delta \to 0$ and then $\epsilon \to 0$ in (3.21) we have

$$\mathbb{W}(E) \leq \liminf_{n \to \infty} P(E_{Z_n}^{(n)}) \leq \limsup_{n \to \infty} P(E_{Z_n}^{(n)}) \leq \mathbb{W}(E),$$

which implies (3.3).  

### 3.3. Proof of Lemma 2.1

In the case of Lemma 2.1, $Z$ is positive almost everywhere. Lemma 3.2 works on each $\{Z = c\}$. Finally, we can get Lemma 2.1. Details are given below.

**Proof of Lemma 2.1.** Denote the distribution function of $Z$ is $G(x)$, according to the definition of conditional expectation one has

$$P(E_{Z_n}^{(n)}|Z) = \int_\Omega P(E_{Z_n}^{(n)}|Z)(\omega)P(d\omega) = \int_0^\infty P(E_{Z_n}^{(n)}|Z = c)dG(c).$$

It is obvious that

$$P(E_{Z_n}^{(n)}|Z = c) = \sum_{|m-cn| < c\epsilon_n} P(E_{m}^{(n)}, Z_n = m|Z = c)
+ \sum_{|m-cn| \geq c\epsilon_n} P(E_{m}^{(n)}, Z_n = m|Z = c)
=: I_1^{(n)}(c) + I_2^{(n)}(c).$$

Note that $I_2^{(n)} \leq P(|Z_n/c_n - c| \geq \epsilon|Z = c)$ and $Z_n/c_n \xrightarrow{P} Z$ we know that

$$\int_0^\infty I_2^{(n)}(c)dG(c) \leq \int_0^\infty \int_0^\infty P\left(\left|\frac{Z_n}{c_n} - c\right| \geq \epsilon|Z = c\right) dG(c)
= \int_0^\infty P\left(\left|\frac{Z_n}{c_n} - Z\right| \geq \epsilon|Z = c\right) dG(c)
= P\left(\left|\frac{Z_n}{c_n} - Z\right| \geq \epsilon\right) \to 0.$$
Note that all the terms in (3.25) are bounded and Borel measurable with respect to $c$, so all the terms are integral. The number of the possible values of $(U_n(c), \rho(c))$ is countable, then $U_n(c), \rho(c)$ are measurable with respect to $c$, so if $P(E_{U_n(c),\rho(c)}^{(n)})$ is viewed as the function of $c$, it is Borel measurable. According to the independence of $Z$ and $\{\zeta_{n,j}, n \geq 0, j \geq 1\}$ one has

$$
\lim_{n \to \infty} \int_0^\infty P(E_{U_n(Z),\rho(Z)}^{(n)}|Z = c)dG(c) = \int_0^\infty P(E_{U_n(c),\rho(c)}^{(n)}) dG(c).
$$

Let $n \to \infty$ in (3.26), by Lemma 3.2 and Lebesgue’s denominating convergence theorem,

$$
\lim_{n \to \infty} \int_0^\infty P(E_{U_n(Z),\rho(Z)}^{(n)}|Z = c)dG(c) = \int_0^\infty W(E_{\rho(c)}) dG(c),
$$

where $E_{\rho(c)}$ is defined in (3.9). Note that when $\varepsilon \to 0$ and $\gamma \to 0$ one has $E_{\rho(c)} \uparrow \bar{E}$, by (3.27) and Lebesgue’s denominating convergence theorem,

$$
\lim_{\gamma \to 0, \varepsilon \to 0} \lim_{n \to \infty} \int_0^\infty P(E_{U_n(Z),\rho(Z)}^{(n)}|Z = c)dG(c) = \lim_{\gamma \to 0, \varepsilon \to 0} \lim_{n \to \infty} W(E_{\rho(c)}) dG(c) = W(E).
$$

Similarly, we have

$$
\lim_{\gamma \to 0, \varepsilon \to 0} \lim_{n \to \infty} \int_0^\infty P(E_{V_n(Z),\rho(Z)}^{(n)}|Z = c)dG(c) = W(E),
$$

$$
\lim_{\gamma \to 0, \varepsilon \to 0} \lim_{n \to \infty} \int_0^\infty P(G_{\gamma}^{(n)}|Z = c)dG(c) = 0.
$$

Take the integrations of all the terms in (3.25) with respect to $G(c)$ and let first $n \to \infty$, then $\varepsilon \to 0$ and finally $\gamma \to 0$, one has

$$
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_0^\infty I_1^{(n)} dG(c) = W(E).
$$

By (3.22), (3.23), (3.24) and (3.28) one has (2.4).

\section{4. Proofs of Lemma 2.2 and Lemma 2.3}

In this section, we prove the last two lemmas. For Lemma 2.2, our main idea is to prove that $|P(R_n) - P(E_{Z_n}^{(n)})|$ is small when $n$ is sufficiently large, so Lemma 2.2 follows from Lemma 2.1. Lemma 2.2 and an approximation theorem of M. Donsker guarantee the correctness of Lemma 2.3.

\textbf{Proof of Lemma 2.2.} Note that for any $j = 1, \ldots, k,$

$$
\frac{|(j - 1)\bar{Z}_n| + 2 - 1}{Z_n} \geq \frac{(j - 1)\bar{Z}_n - 1 + 2 - 1}{Z_n} = \frac{j - 1}{k} \quad \text{and} \quad \frac{|j\bar{Z}_n|}{Z_n} \leq \frac{j}{k}.
$$
one has \( I_{Z_n, ((j-1)\frac{k}{n}+2)} \subset I_{k,j}, j = 1, \ldots, k \), where \( I_{k,j} \) is defined in (2.1). Define

\[
J_{k,j}^{(n)} = I_{k,j} - \left( \bigcup_{i=([j-1] \frac{k}{n}+2)}^{([j] \frac{k}{n})} I_{Z_{n,i}} \right), \quad \Pi_n := \bigcap_{j=1}^{k} \{ \alpha_j \leq \mu_{Z_n}^{(n)}(t) \leq \beta_j, t \in J_{k,j}^{(n)} \},
\]

we have

\[
R_n = \Pi_n \cap \bigcap_{j=1}^{k} \left\{ \alpha_j \leq S_{n,j}^{(n)}(Z_n) \leq \beta_j, \left[ (j-1) \frac{Z_n}{k} \right] < i \leq \left[ j \frac{Z_n}{k} \right] \right\}
\]

\[
= \Pi_n \cap E_{Z_n}^{(n)}.
\]

But

\[
J_{k,j}^{(n)} = \left( \frac{j-1}{k}, \frac{[j-1]Z_n/k + 2 - 1}{Z_n} \right] \cup \left( \frac{[j]Z_n/k}{Z_n}, \frac{j}{k} \right]
\]

\[
\subseteq \left( \frac{[j-1]Z_n/k}{Z_n}, \frac{[j-1]Z_n/k + 1}{Z_n} \right] \cup \left( \frac{[j]Z_n/k}{Z_n}, \frac{[j]Z_n/k + 1}{Z_n} \right]
\]

\[
= I_{Z_n, ((j-1)\frac{k}{n})+1} \cup I_{Z_n, ([j] \frac{k}{n})+1}.
\]

Therefore,

\[
\Pi_n \supset \bigcap_{j=1}^{k} \left\{ \alpha_j \leq \mu_{Z_n}^{(n)}(t) \leq \beta_j, t \in I_{Z_n, ((j-1)\frac{k}{n})+1} \cup I_{Z_n, ([j] \frac{k}{n})+1} \right\}
\]

\[
= \bigcap_{j=1}^{k} \left\{ \alpha_j \leq S_{n,j-1,x_n+1}^{(n)}(Z_n) \leq \beta_j, \alpha_j \leq S_{n,j,x_n+1}^{(n)}(Z_n) \leq \beta_j \right\}
\]

\[
\cap \bigcap_{j=1}^{k} \left\{ \alpha_j \leq S_{n,j-1,x_n}^{(n)}(Z_n) \leq \beta_j, \alpha_j \leq S_{n,j,x_n}^{(n)}(Z_n) \leq \beta_j \right\}.
\]

For any \( \eta > 0 \), define

\[
T_{n,\eta} = \{ \omega \mid \max_{1 \leq j \leq k} \{ |S_{n,j,x_n+1}^{(n)} - S_{n,j,x_n}^{(n)}| \} \geq \eta \}.
\]

Thus,

\[
E_{Z_n}^{(n)} \supset R_n \supset \bigcap_{j=1}^{k} \{ \alpha_j \leq S_{n,j-1,x_n+1}^{(n)}(Z_n) \leq \beta_j, \eta_{(j-1),x_n} \leq i \leq \eta_{j,x_n+1} \}
\]

\[
\supset E_{Z_n, -\eta}^{(n)} \cap T_{n,\eta},
\]

where \( E_{Z_n, -\eta}^{(n)} \) is defined in (3.13). Note that

\[
P(T_{n,\eta}) \leq \sum_{j=1}^{k} \left( \sum_{i=1}^{\eta_{j,x_n+1}^{(n)} - \eta_{(j-1),x_n}^{(n)}} \right) P(|S_{n,j,x_n+1}^{(n)} - S_{n,j,x_n}^{(n)}(Z_n)| \geq \eta)
\]
and
\[
P \left( |S_{n, x_n}^{(n)}(Z_n) - S_{q, x_n}^{(n)}(Z_n)| \geq \eta \right)
\]
\[
= \int_0^{\infty} P(|S_{q, x_n}^{(n)}(Z_n)| \geq \eta) dG(c)
\]
\[
\leq \int_0^{\infty} \sum_{|m - cc_n| < cc_n} P(|S_{q, x_n}^{(n)}(Z_n)| \geq \eta) dG(c)
\]
\[
+ \int_0^{\infty} \sum_{|m - cc_n| \geq cc_n} P(Z_n = m) dG(c) =: I_1 + I_2,
\]
where
\[
I_1 = \int_0^{\infty} \sum_{|m - cc_n| < cc_n} P(\xi_{n, l} \geq \eta) dG(c)
\]
\[
\leq \int_0^{\infty} 2c_c P(\xi_{n, l} \geq \eta) dG(c)
\]
\[
= 2c_c P(\xi_{n, l} \geq \eta) \leq \frac{2c_c}{\eta^2 (c - \epsilon) c_n},
\]
and
\[
I_2 \leq \int_0^{\infty} P(|Z_n| \geq \epsilon) dG(c) = P(\frac{Z_n}{c_n} - Z \geq \epsilon),
\]
so we have
\[
(4.3) \quad P(T_{n, \eta}) \leq k \left[ \frac{2c_c}{\eta^2 (c - \epsilon) c_n} + P(\frac{Z_n}{c_n} - Z \geq \epsilon) \right].
\]
Note that \( Z_n/c_n \xrightarrow{p} Z \), by (4.1), (4.3) and Lemma 2.2 we have
\[
(4.4) \quad W(E) = \lim_{n \to \infty} P(E_Z^{(n)}) \geq \limsup_{n \to \infty} P(R_n) = \liminf_{n \to \infty} P(R_n)
\]
\[
\geq W(E_{- \eta}) = \frac{2c_c}{\eta^2 (c - \epsilon)}.
\]
Let first \( \epsilon \to 0 \), then \( \eta \to 0 \) in (4.4) one has (2.8).

Define \( B = \{(t_1, \ldots, t_{2k}) : -\infty < t_i \leq \beta_i, \ t_i \leq t_{i+k} < \infty, \ i = 1, \ldots, k\} \).

Note that by (2.3), (2.5), (2.6) and (2.7) we have
\[
(4.5) \quad R_n = \{ \omega \mid \omega \in \Omega, \ \alpha_j \leq \beta_j, \ t \in I_{k,j}, \ j = 1, \ldots, k\}
\]
\[
= \{ \omega \mid (p_1^{(n)}, \ldots, p_k^{(n)}), (q_1^{(n)}, \ldots, q_k^{(n)}) \in B)\},
\]
\[
(4.6) \quad E = \{ x \in C \mid \alpha_j \leq x(t) \leq \beta_j, \ t \in I_{k,j}, \ j = 1, \ldots, k\}
\]
\[
= \{ x \in C \mid (p_1(x), \ldots, p_k(x), q_1(x), \ldots, q_k(x)) \in B)\}.
\]
Let $I_B$ be the indicator function of $B$, by (4.5) and (4.6), we know that (2.8) is equivalent to
\begin{equation}
\nabla I_B := \lim_{n \to \infty} \int_{\Omega} I_B \left( p_1^{(n)}, \ldots, p_k^{(n)}, q_1^{(n)}, \ldots, q_k^{(n)} \right) dP
= \int_{C} I_B(p_1(x), \ldots, p_k(x), q_1(x), \ldots, q_k(x)) \mathbb{W}(dx).
\end{equation}

According to the proof of Theorem 2.3 of D. H. Hu ([9]) we know that the $\sigma$-field generated by all the sets like $B$ is the Borel $\sigma$-field of $\mathbb{R}^{2k}$. By the monotone class theorem one has (2.9). □

**Lemma 4.1** (c.f. [4]). For any $h \in C^\ast$ and $\epsilon > 0$, there exist $h_1, h_2 \in X^\ast$ such that
\begin{equation}
h_1(x) \leq h(x) \leq h_2(x), \ \forall x \in X, \ \int_{C} [h_2(x) - h_1(x)] \mathbb{W}(dx) \leq \epsilon
\end{equation}
and $h_i(x), \ i = 1, 2$ can be rewritten by
\begin{equation}
h_i(x) = f_i(p_1(x), \ldots, p_k(x), q_1(x), \ldots, q_k(x)),
\end{equation}
where $f_i, \ i = 1, 2$ are two bounded and Borel measurable functions on $\mathbb{R}^{2k}$.

**Proof of Lemma 2.3.** Lemma 2.3 follows from (2.9) and Lemma 4.1. □

**Acknowledgement.** The authors would like to thank the referee for his/her valuable suggestions, which essentially improved the manuscript. We also thank Professor Xiaoyu Hu for her help and carefully reading of several early manuscripts.

**References**


Zhenlong Gao  
School of Mathematical Sciences  
Qufu Normal University  
Qufu 273165, P. R. China  
E-mail address: gzlyb@mail.qfnu.edu.cn

Liang Fang  
College of Mathematic and Computer Sciences  
Changsha University of Science and Technology  
Changsha 410004, P. R. China  
and  
College of Mathematic and Computer Sciences  
Hunan Normal University  
Changsha 410081, P. R. China  
E-mail address: FL8629758@163.com