THE COMPETITION INDEX OF A NEARLY REDUCIBLE BOOLEAN MATRIX

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Abstract. Cho and Kim [4] have introduced the concept of the competition index of a digraph. Similarly, the competition index of an $n \times n$ Boolean matrix $A$ is the smallest positive integer $q$ such that $A^{q+i}(A^T)^{q+i} = A^{q+r}(A^T)^{q+r+i}$ for some positive integer $r$ and every nonnegative integer $i$, where $A^T$ denotes the transpose of $A$. In this paper, we study the upper bound of the competition index of a Boolean matrix. Using the concept of Boolean rank, we determine the upper bound of the competition index of a nearly reducible Boolean matrix.

1. Preliminaries and notations

In this paper, we follow the terminology and notation used in [3, 7]. A Boolean matrix is a matrix over the binary Boolean algebra $\{0, 1\}$. For $m \times n$ Boolean matrices $A = (a_{ij})$ and $B = (b_{ij})$, we say that $B$ is dominated by $A$ (denoted by $B \leq A$) if $b_{ij} \leq a_{ij}$ for all $i$ and $j$. We denote the $m \times n$ all-ones Boolean matrix by $J_{m,n}$ (and by $J_n$ if $m = n$), the $m \times n$ all-zeros Boolean matrix by $O_{m,n}$ (and by $O_n$ if $m = n$), and the $n \times n$ identity Boolean matrix by $I_n$. The subscripts $m$ and $n$ will be omitted whenever their values are clear from the context.

Let $D = (V, E)$ denote a digraph (directed graph) with vertex set $V = V(D)$ and arc set $E = E(D)$. Loops are permitted but multiple arcs are not. An $x \to y$ walk in a digraph $D$ is a sequence of vertices $x, v_1, \ldots, v_t, y \in V(D)$ and a sequence of arcs $(x, v_1), (v_1, v_2), \ldots, (v_t, y) \in E(D)$, where the vertices and arcs are not necessarily distinct. A closed walk is an $x \to y$ walk where $x = y$. A cycle is a closed $x \to y$ walk in which all vertices except $x$ and $y$ are distinct. The length of a walk $W$ is the number of arcs in $W$. The notation $x \xrightarrow{k} y$ is used to indicate that there is a $x \to y$ walk of length $k$. An $l$-cycle is a cycle of length $l$. 

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2001
length $l$. If the digraph $D$ has at least one cycle, the length of a shortest cycle in $D$ is called the \textit{girth} of $D$, denoted by $s(D)$.

For an $n \times n$ Boolean matrix $A = (a_{ij})$, its digraph, denoted by $D(A)$, is the digraph with vertex set $V(D(A)) = \{v_1, v_2, \ldots, v_n\}$, and $(v_i, v_j)$ is an arc of $D(A)$ if and only if $a_{ij} = 1$. Using Boolean arithmetic $(1 + 1 = 1, 0 + 0 = 0, 1 + 0 = 1)$, $AB$ and $A + B$ are Boolean matrices if $A$ and $B$ are Boolean matrices. Note that for a positive integer $k$, the (Boolean) $k$-th power $A^k = [b_{ij}]$ of $A$ is a Boolean matrix such that $b_{ij} = 1$ if and only if there is a directed walk of length $k$ from $v_i$ to $v_j$ in $D(A)$.

A digraph $D$ is called \textit{strongly connected} if for each pair of vertices $x$ and $y$ in $V(D)$, there is a walk from $x$ to $y$. For a strongly connected digraph $D$, the \textit{index of imprimitivity} of $D$ is the greatest common divisor of the lengths of the cycles in $D$, and it is denoted by $p(D)$. If $D$ is a trivial digraph of order 1, $p(D)$ is undefined. A strongly connected digraph $D$ is \textit{primitive} if $p(D) = 1$. If $D$ is primitive, there exists some positive integer $l$ such that there is a walk of length exactly $l$ from each vertex $x$ to each vertex $y$. The smallest such $l$ is called the \textit{exponent} of $D$, denoted by $\exp(D)$. Exponents have been studied by several researchers [3, 7, 8, 9, 10].

We say that a Boolean matrix $A$ is \textit{permutationally similar} to a Boolean matrix $B$ if there exists a permutation Boolean matrix $P$ satisfying $B = PAP^T$, where $P^T$ denotes the transpose of $P$. The Boolean matrix $A$ is called \textit{reducible} if $A$ is permutationally similar to a Boolean matrix of the form

$$
\begin{bmatrix}
  A_1 & O \\
  A_{21} & A_2
\end{bmatrix},
$$

where $A_1$ and $A_2$ are square Boolean matrices of order at least one. If $A$ is not reducible, it is called \textit{irreducible}. $A$ is irreducible if and only if $D(A)$ is strongly connected (see [3]). The Boolean matrix $A$ is called \textit{primitive} if $D(A)$ is primitive.

Let $D$ be a digraph (with or without loops) with the vertex set $\{v_1, v_2, \ldots, v_n\}$. Given a positive integer $m$, we say that a vertex $v_k$ of $D$ is an $m$-\textit{step common prey} of $v_i$ and $v_j$ if $v_i \rightarrow^{m} v_k$ and $v_j \rightarrow^{m} v_k$. Then, the $m$-\textit{step competition graph} of $D$, denoted by $C^m(D)$, has the same vertex set as $D$, and there is an edge between vertices $v_i$ and $v_j$ ($v_i \neq v_j$) if and only if $v_i$ and $v_j$ have an $m$-step common prey in $D$. The $m$-step digraph of $D$, denoted by $D^m$, has the same vertex set as $D$ and an arc $(v_i, v_j)$ if and only if $v_i \rightarrow^m v_j$. Then, we have $C^m(D) = C(D^m)$ for each positive integer $m$ (see [5]).

Consider the sequence

$$D, D^2, D^3, \ldots, D^m, \ldots$$

Then, there exists the smallest positive integer $q$ such that $D^q = D^{q+r}$ for some positive integer $r$. Such an integer $q$ is called the \textit{index} of $D$, and it is denoted by $\text{index}(D)$. There also exists the smallest positive integer $p$ such
that \( D^q = D^{q+p} \); such an integer is called the \textit{period} of \( D \), and it is denoted by \text{period}(D).

Now, consider the competition graph sequence 
\[
C(D), C(D^2), C(D^3), \ldots, C(D^m), \ldots.
\]
There exists the smallest positive integer \( q \) such that \( C(D^q+i) = C(D^q+r+i) \) for some positive integer \( r \) and every nonnegative integer \( i \). Such an integer \( q \) is called the \textit{competition index} of \( D \), and it is denoted by \text{cindex}(D).

Let \( q = \text{cindex}(D) \). Then, there exists the smallest positive integer \( p \) such that \( C(D^q+i) = C(D^{q+p+i}) \) for every nonnegative integer \( i \). Such an integer \( p \) is called the \textit{competition period} of \( D \), and it is denoted by \text{cperiod}(D).

An analogous definition for the competition index and competition period can be given for a Boolean matrix. The \textit{competition index} of a Boolean matrix \( A \), denoted by \text{cindex}(A), is the smallest positive integer \( q \) such that 
\[
A^q(A^T)^q+i = A^q(A^T)^q+r+i
\]
for some positive integer \( r \) and every nonnegative integer \( i \). The \textit{competition period} of a Boolean matrix \( A \), denoted by \text{cperiod}(A), is the smallest positive integer \( p \) such that 
\[
A^q(A^T)^q+i = A^{q+p}(A^T)^{q+p+i}
\]
for \( q = \text{cindex}(A) \) and every nonnegative integer \( i \). If \( A \) is the adjacency matrix of a digraph \( D \), then we have \text{cindex}(A) = \text{cindex}(D) \) and \text{cperiod}(A) = \text{cperiod}(D) \). As a result, throughout the paper, as long as no confusion occurs, we use the digraph \( D \) and the adjacency matrix \( A(D) \) interchangeably.

Akelbek and Kirkland [2] introduced the \textit{scrambling index} of a primitive digraph. The scrambling index is the smallest positive integer \( k \) such that for every pair of vertices \( u \) and \( v \), there exists a vertex \( w \) such that \( u \xrightarrow{k} w \) and \( v \xrightarrow{k} w \) in \( D \). Akelbek and Kirkland’s definition of the scrambling index is the same as our definition of the competition index in the case of a primitive digraph (see [6]). In [2], they presented the following result regarding the scrambling index.

**Proposition 1.1** (Akelbek and Kirkland [2]). Let \( D \) be a primitive digraph of order \( n \) and girth \( s \). Then,
\[
\text{cindex}(D) \leq \begin{cases} 
n - s + \left\lfloor \frac{s-1}{2} \right\rfloor n, & \text{when } s \text{ is odd}, \\
n - s + \frac{s-1}{2} s, & \text{when } s \text{ is even}.
\end{cases}
\]

For a positive integer \( n \geq 3 \), we define
\[
W_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \end{bmatrix},
\]
and \( \omega_n = \left\lfloor \frac{(n-1)^2+1}{2} \right\rfloor \), \( \omega_1 = 1 \), \( \omega_2 = 2 \).
Cho and Kim [4] presented the following result regarding the upper bound of the competition index of a strongly connected digraph.

**Proposition 1.2** (Cho and Kim [4]). Let $A$ be an irreducible $n \times n$ Boolean matrix, where $n \geq 3$. Then, we have

$$ cindex(A) \leq \omega_n. $$

The equality holds if and only if $A$ is permutationally similar to $W_n$.

2. A bound on the competition index of an irreducible Boolean matrix using Boolean rank

For a pair of vertices $u$ and $v$, let $cindex(D : u, v)$ denote the smallest positive integer $m$ such that $u$ and $v$ have an $l$-step common prey whenever $l \geq m$. If there is no such positive integer $m$, we let $cindex(D : u, v) = 1$. Further, we let $cindex(D : u, u) = 1$. If $D$ is strongly connected,

$$ cindex(D) = \max\{cindex(D : u, v) \mid u, v \in V(D)\}. $$

**Theorem 2.1.** Let $A$ be an $n \times n$ irreducible Boolean matrix, with $p(A) = p$. If we denote $r = \lfloor n/p \rfloor$ and $s = n - pr$, we have

$$ cindex(A) \leq \begin{cases} p \cdot \omega_r + s, & \text{when } r > 1, \\ s, & \text{when } r = 1 \text{ and } s > 0, \\ 1, & \text{when } r = 1 \text{ and } s = 0. \end{cases} $$

**Proof.** Let $D = D(A)$ and $V_0, V_1, \ldots, V_{p-1}$ be $p$ nonempty sets, with $V_p = V_0$, where each arc of $D$ issues from $V_i$ and enters $V_{i+1}$ for some $i$ with $0 \leq i \leq p-1$. Let $E_i$ be the subgraph of $D^p$ induced by $V_i$, where $0 \leq i \leq p-1$. Then, $E_i$ is primitive.

If $r = 1$ and $s = 0$, we have $cindex(A) = 1$. Further, if $r = 1$ and $s > 0$, we have $cindex(D : u, v) \leq s$. Suppose that $r > 1$. We claim that $cindex(D : u, v) \leq p \cdot \omega_r + s$ for any two vertices $u$ and $v$. If $u \in V_i$ and $v \in V_j$ where $i \neq j$, $u$ and $v$ do not have an $l$-step common prey for any positive integer $l$. Thus, $cindex(D : u, v) = 1$. We may suppose that $u, v \in V_j$ for some $0 \leq j \leq p-1$. Then, there exists $V_q$ such that $|V_q| \leq r$, and there exist walks

$$ u \rightarrow f \rightarrow u' \in V_q \text{ and } v \rightarrow f \rightarrow v' \in V_q, $$

where $0 \leq f \leq s$. Since $cindex(D^p : u', v') \leq \omega_{|V_q|} \leq \omega_r$, we have

$$ cindex(D : u, v) \leq f + cindex(D : u', v') \leq s + p \cdot cindex(D^p : u', v') \leq s + p \cdot \omega_r. $$

Thus, we have $cindex(D) = \max\{cindex(D : u, v) \mid u, v \in V(D)\} \leq p \cdot \omega_r + s$.

This establishes the result. \hfill \Box
For an \( m \times n \) Boolean matrix \( A \), we define its Boolean rank \( b(A) \) to be the smallest positive integer \( b \) such that for some \( m \times b \) Boolean matrix \( X \) and \( b \times n \) Boolean matrix \( Y \), \( A = XY \). The Boolean rank of the zero matrix is defined to be zero. \( A = XY \) is called a Boolean rank factorization of \( A \).

**Proposition 2.2** (Akelbek, Fital, and Shen [1]). Suppose that \( X \) and \( Y \) are \( n \times m \) and \( m \times n \) Boolean matrices, respectively, and that neither has a zero line (i.e., row or column).

(i) \( XY \) is primitive if and only if \( YX \) is primitive.

(ii) If \( XY \) and \( YX \) are primitive,

\[
|\text{cindex}(XY) - \text{cindex}(YX)| \leq 1.
\]

**Lemma 2.3.** Suppose \( A \) is an \( n \times m \) Boolean matrix and \( A = XY \) is a Boolean rank factorization of \( A \), where \( b(A) = b \). If \( A \) has no zero lines, neither \( X \) nor \( Y \) has a zero line.

**Proof.** Since \( A \) has no zero lines, \( X \) has no zero rows and \( Y \) has no zero columns. Suppose that \( X \) has a zero column, and without loss of generality, let it be the \( i \)th column. Let \( X' \) be the matrix obtained from \( X \) by deleting its \( i \)th column, and let \( Y' \) be the matrix obtained from \( Y \) by deleting its \( i \)th row. Then, \( X' \) is an \( n \times (b - 1) \) matrix, \( Y' \) is a \( (b - 1) \times m \) matrix, and \( X'Y' = A \). Therefore, the Boolean rank of \( A \) is at most \( b - 1 \). This is a contradiction. Hence, \( X \) has no zero columns. Similarly, \( Y \) has no zero rows. This establishes the result. \( \square \)

**Lemma 2.4.** Let \( A \) be an \( n \times n \) Boolean irreducible matrix, with \( p(A) = p \), and let \( A = XY \) be a Boolean rank factorization of \( A \), with \( b(A) = b \). Then,

(i) \( XY \) is irreducible, with \( p(YX) \geq p \).

(ii) \( \text{cindex}(XY) \leq \text{cindex}(YX) + 1 \).

**Proof.** If \( A \) is primitive, we have the result by Proposition 2.2. Suppose that \( p(A) \geq 2 \). Then, we may suppose that

\[
A = \begin{bmatrix}
O & A_0 & O & \cdots & O \\
O & O & A_1 & \cdots & O \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
O & O & O & \cdots & A_{p-2} \\
A_{p-1} & O & O & \cdots & O
\end{bmatrix},
\]

in which the zero matrices on the diagonal are square matrices of orders \( n_0, n_1, \ldots, n_{p-1} \), respectively (see [3]). Further, there exists a permutation matrix \( P \) such that

\[
XP = \begin{bmatrix}
X_0 & O & O & \cdots & O \\
O & X_1 & O & \cdots & O \\
O & O & X_2 & \cdots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & O & O & \cdots & X_{p-1}
\end{bmatrix}, \quad P^TY = \begin{bmatrix}
O & Y_0 & O & \cdots & O \\
O & O & Y_1 & \cdots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & O & O & \cdots & Y_{p-2} \\
Y_{p-1} & O & O & \cdots & O
\end{bmatrix}.
\]
where $A_i = X_i Y_i$ is a Boolean rank factorization of $A_i$, with $b(A_i) = b_i$. Moreover, $Y X$ is permutationally similar to

\[
\begin{bmatrix}
O & Y_0 X_1 & O & \cdots & O \\
O & O & Y_1 X_2 & \cdots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & O & O & \cdots & Y_{p-2} X_{p-1} \\
Y_{p-1} X_0 & O & O & \cdots & O
\end{bmatrix},
\]

in which the zero matrices on the diagonal are square matrices of orders $b_0, b_1, \ldots, b_{p-1}$, respectively. Let

\[
\begin{align*}
\bar{A}_0 &= A_0 A_1 \cdots A_{p-2} A_{p-1} = X_0 Y_0 X_1 Y_1 \cdots X_{p-2} Y_{p-2} X_{p-1} Y_{p-1}, \\
\bar{A}_1 &= A_1 A_2 \cdots A_{p-1} A_0 = X_1 Y_1 X_2 Y_2 \cdots X_{p-1} Y_{p-1} X_0 Y_0, \\
& \quad \vdots \\
\bar{A}_{p-1} &= A_{p-1} A_0 \cdots A_{p-3} A_{p-2} = X_{p-1} Y_{p-1} X_0 Y_0 \cdots X_{p-3} Y_{p-3} X_{p-2} Y_{p-2}.
\end{align*}
\]

For each $i$, there exists a positive integer $l$ such that $\bar{A}_i^l = J_{n_i}$ since $\bar{A}_i$ is primitive. For each $i$, neither $X_i$ nor $Y_i$ has a zero line by Lemma 2.3. Then, we have

\[
\begin{align*}
(Y_0 X_1 Y_2 \cdots Y_{p-2} X_{p-1} Y_{p-1} X_0)^{l+1} &= Y_0 \bar{A}_i^l (X_1 Y_1 X_2 \cdots Y_{p-2} X_{p-1} Y_{p-1} X_0) \\
&= Y_0 J_{n_i} (X_1 Y_1 X_2 \cdots Y_{p-2} X_{p-1} Y_{p-1} X_0) = J_{b_0}, \\
(Y_1 X_2 Y_3 \cdots Y_{p-1} X_0 Y_1)^{l+1} &= Y_1 \bar{A}_i^l (X_2 Y_2 X_3 \cdots Y_{p-1} X_0 Y_1) \\
&= Y_1 J_{n_i} (X_2 Y_2 X_3 \cdots Y_{p-1} X_0 Y_1) = J_{b_1}, \\
& \quad \vdots \\
(Y_{p-1} X_0 Y_1 \cdots Y_{p-3} X_{p-2} Y_{p-2} X_{p-1})^{l+1} &= Y_{p-1} \bar{A}_0^l (X_0 Y_0 X_1 \cdots Y_{p-3} X_{p-2} Y_{p-2} X_{p-1}) \\
&= Y_{p-1} J_{n_0} (X_0 Y_0 X_1 \cdots Y_{p-3} X_{p-2} Y_{p-2} X_{p-1}) = J_{b_{p-1}}.
\end{align*}
\]

Therefore, $Y X$ is irreducible and $p(Y X) \geq p = p(XY)$ by (1).

Suppose that $\text{cindex}(XY) = k$. By the definition of the competition index of an irreducible Boolean matrix,

\[
A^k (A^T)^k = (XY)^k ((XY)^T)^k = \begin{bmatrix}
J_{n_0} & O & O & \cdots & O \\
O & J_{n_1} & O & \cdots & O \\
O & O & J_{n_2} & \cdots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & O & O & \cdots & J_{n_{p-1}}
\end{bmatrix}.
\]
For each $i$, neither $X_i$ nor $Y_i$ has a zero line by Lemma 2.3. Then, we have

$$X_iX_i^T \geq I_n, \quad \text{and} \quad Y_iY_i^T \geq I_n.$$  

If we suppose that all subscripts are taken by modulo $p$, we have

$$(A_{ij}A_{ij+1} \cdots A_{ij+k})(A_{ij}A_{ij+1} \cdots A_{ij+k})^T = J_{n_i},$$

$$(X_iY_iX_{i+1}Y_{i+1} \cdots X_{i+k}Y_{i+k})(X_iY_iX_{i+1}Y_{i+1} \cdots X_{i+k}Y_{i+k})^T = J_{n_i}.$$  

by (2). Therefore, we have

$$(Y_{i-1}X_iY_{i+1}X_{i+1} \cdots Y_{i+k}X_{i+k+1})(Y_{i-1}X_iY_{i+1}X_{i+1} \cdots Y_{i+k}X_{i+k+1})^T$$

$$= (Y_{i-1}X_iY_{i+1}X_{i+1} \cdots Y_{i+k})(X_{i+k+1}X_{i+k+1}^T)(Y_{i-1}X_iY_{i+1}X_{i+1} \cdots Y_{i+k})^T$$

$$\geq (Y_{i-1}X_iY_{i+1}X_{i+1} \cdots Y_{i+k})J_{n_{i+k+1}}(Y_{i-1}X_iY_{i+1}X_{i+1} \cdots Y_{i+k})^T$$

$$= Y_{i-1}(X_iY_iX_{i+1}Y_{i+1} \cdots Y_{i+k})J_{n_{i+k+1}}(X_iY_iX_{i+1}Y_{i+1} \cdots Y_{i+k})^TY_{i-1}^T$$

$$= Y_{i-1}J_{n_i}Y_{i-1}^T = J_{b_{i-1}}.$$  

Then, we have

$$(YX)^{k+1}(YX)^{k+1} = \begin{bmatrix}
J_{b_0} & O & O & \cdots & O \\
O & J_{b_1} & O & \cdots & O \\
O & O & J_{b_2} & \cdots & O \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
O & O & O & \cdots & J_{b_{n-1}}
\end{bmatrix}.$$  

Thus, we have $\text{cindex}(YX) \leq k + 1 = \text{cindex}(XY) + 1$. This establishes the result. $\square$

In Lemma 2.4, the condition that $A$ is irreducible is required. See Example 2.5.

**Example 2.5.** Consider the Boolean reducible matrix $A$ such that

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 
\end{bmatrix}.$$  

Then, we have $b(A) = 4$ and a Boolean rank factorization $A = XY$ for $X$ and $Y$ such that

$$X = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 
\end{bmatrix}, \quad Y = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 
\end{bmatrix}. $$
Then, we have
\[
YX = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

We have \(\text{cindex}(A) = 3\) and \(\text{cindex}(YX) = 1\). Therefore, we have
\[
\text{cindex}(XY) > \omega + 1.
\]

**Proposition 2.6** (Akelbek, Fital, and Shen [1]). Suppose that \(A\) is an \(n \times n\) (\(n \geq 2\)) primitive Boolean matrix with Boolean rank \(b(A) = b\). Then,
\[
\text{cindex}(A) \leq \omega + 1.
\]

If \(3 \leq b \leq n - 1\), the equality holds if and only if \(A\) is permutationally similar to one of the forms \(M_1, M_3,\) and \(M_5\) in Table 1.

In Table 1, the rows and columns of \(M_1, M_3,\) and \(M_5\) are partitioned conformally, so that each diagonal block is square, and the top left-hand side submatrix common to each has \(b\) blocks in its partition.

**Theorem 2.7.** Suppose that \(A\) is an \(n \times n\) irreducible Boolean matrix with Boolean rank \(b(A) = b\), where \(3 \leq b \leq n - 1\). Then, we have
\[
\text{cindex}(A) \leq \omega + 1.
\]

The equality holds if and only if \(A\) is permutationally similar to one of the forms \(M_1, M_3,\) and \(M_5\) in Table 1.

**Proof.** If \(p(A) = 1\), we have the result by Proposition 2.6. Suppose that \(p(A) \geq 2\). We claim that \(\text{cindex}(A) < \omega + 1\). Let \(A = XY\) be a Boolean rank factorization of \(A\). Then, \(YX\) is a \(b \times b\) irreducible matrix, with \(p = p(YX) \geq p(XY) \geq 2\) by Lemma 2.4. By Lemma 2.4 and Proposition 1.2, we have
\[
\text{cindex}(A) \leq \text{cindex}(YX) + 1 \leq \begin{cases} 
p \cdot \omega_r + p, & \text{when } r \geq 2, \\
p, & \text{when } r < 2,
\end{cases}
\]
where \(r = \left\lfloor \frac{b}{p} \right\rfloor\). If \(r < 2\), we obtain the result. Suppose that \(2 \leq p \leq \left\lfloor \frac{b}{2} \right\rfloor\).

Then, we have
\[
\text{cindex}(A) \leq \text{cindex}(YX) + 1 \leq p \cdot \omega_r + p \leq \frac{b^2}{2p} + \frac{5}{2}p - b.
\]

Let \(g(p) = \frac{b^2}{2p} + \frac{5}{2}p - b (2 \leq p \leq \left\lfloor \frac{b}{2} \right\rfloor)\). Then, \(g(p)\) attains the maximum value when \(p = 2\). \(g(2) = \frac{b^2 - 4b + 5}{4} < \left\lfloor \frac{b^2 - 8b + 2}{2} \right\rfloor + 1 = \omega + 1\) since \(b \geq 2p \geq 4\). Then,
\[
\text{cindex}(A) < \omega + 1.
\]
This establishes the result. \(\Box\)
3. A bound on the competition index of a nearly reducible matrix

The irreducible Boolean matrix $A$ is called nearly reducible if each matrix obtained from $A$ by the replacement of a 1 with a 0 is a reducible Boolean matrix. Thus, the digraph $D$ is minimally strong if and only if its adjacency matrix $A$ is nearly reducible.

The term rank of a Boolean matrix $A$, denoted by $t(A)$, is defined to be the largest number of 1s in $A$, with at most one 1 in each column and at most one 1 in each row. Then, we have $b(A) \leq t(A)$.

Proposition 3.1 (Cho and Kim [4]). Let $D$ be a strongly connected digraph of order $n(\geq 3)$. If $p(D) > \frac{n}{2}$, we have

$$\text{cindex}(D) \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$  

Theorem 3.2. Let $A$ be a nearly reducible $n \times n$ Boolean matrix, where $n \geq 8$. Then, we have

$$\text{cindex}(A) \leq \left\lceil \frac{(n-2)^2+1}{2} \right\rceil + 1.$$  

The equality holds if and only if $A$ is permutationally similar to

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$  

Proof. Denote $p = p(A)$.

Case 1. $p > \frac{n}{2}$.

By Proposition 3.1, we have $\text{cindex}(D) \leq \left\lfloor \frac{n-1}{2} \right\rfloor < \left\lceil \frac{(n-2)^2+1}{2} \right\rceil + 1$.

Case 2. $2 \leq p \leq \frac{n}{2}$.

By Theorem 2.1, we have

$$\text{cindex}(A) \leq p \cdot \omega_{n/p} + p - 1 \leq \frac{n^2}{2p} + \frac{5}{2} p - n - 1.$$  

Let $g(p) = \frac{n^2}{2p} + \frac{5}{2} p - n - 1 (2 \leq p \leq \frac{n}{2})$. Then, $g(p)$ attains the maximum value when $p = 2$. $g(2) = \frac{n^2 - 4n + 16}{4} \leq \left\lceil \frac{(n-2)^2+1}{2} \right\rceil + 1$, where $n \geq 8$.

Case 3. $p = 1$.

If $b(A) \leq n - 2$, we have $\text{cindex}(A) \leq \omega_{n-2} + 1 < \left\lceil \frac{(n-2)^2+1}{2} \right\rceil + 1$ by Theorem 2.7.
If \( b(A) = n - 1 \), we have 
\[
cindex(A) \leq \omega_{n-1} + 1 = \left\lfloor \frac{(n-2)^2 + 1}{2} \right\rfloor + 1,
\]
and the equality holds if and only if \( A \) is permutationally similar to
\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 & 0 \\
\end{bmatrix},
\]
by Theorem 2.7.

Suppose that \( b(A) = n \). Then we have \( t(A) = n \) since \( t(A) \geq b(A) \). Thus we have an \( n \times n \) permutation submatrix in \( A \). If there is no \( n \)-cycle in \( D = D(A) \), \( s(D) = s \leq \left\lfloor \frac{n}{2} \right\rfloor \). By Proposition 1.1 we have 
\[
cindex(A) < \left\lfloor \frac{(n-2)^2 + 1}{2} \right\rfloor + 1 \] since \( n \geq 8 \). If there is an \( n \)-cycle in \( D \), \( D \) is isomorphic to an \( n \)-cycle since \( A \) is a nearly reducible Boolean matrix. However, \( p(C_n) = n \) is not primitive. This establishes the result. \( \square \)

References

THE COMPETITION INDEX OF A NEARLY REDUCIBLE BOOLEAN MATRIX 2011

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