 Nil\textsuperscript{*}\textit{-}COHERENT RINGS

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Abstract. Let $R$ be a ring and $\text{Nil}\textsuperscript{*}(R)$ be the prime radical of $R$. In this paper, we say that a ring $R$ is left $\text{Nil}\textsuperscript{*}$-coherent if $\text{Nil}\textsuperscript{*}(R)$ is coherent as a left $R$-module. The concept is introduced as the generalization of left $J$-coherent rings and semiprime rings. Some properties of $\text{Nil}\textsuperscript{*}$-coherent rings are also studied in terms of $N$-injective modules and $N$-flat modules.

1. Introduction

Throughout $R$ is an associative ring with identity and all modules are unitary. $R\text{Mod}(M_R)$ stands for the category of all left (right) $R$-modules. $\text{Hom}(M, N)$ (resp. $\text{Ext}^n(M, N)$) means $\text{Hom}_R(M, N)$ (resp. $\text{Ext}^n_R(M, N)$), and similarly $M \otimes N$ (resp. $\text{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\text{Tor}_n^R(M, N)$). The character module $M^+$ is defined by $M^+ = \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$. The Jacobson radical of $R$ is denoted by $J(R)$. If $X$ is a subset of $R$, the right (left) annihilator of $X$ in $R$ is denoted by $r(X)$ ($l(X)$). We will use the usual notations from [1, 8, 9, 13, 14, 22].

We first recall some known notions needed in the sequel.

Let $C$ be the class of $R$-modules. For an $R$-module $M$, $C \in C$ is called a $C$-cover [8] of $M$ if there is a homomorphism $g : C \to M$ such that the following hold: (1) For any homomorphism $g' : C' \to M$ with $C' \in C$, there exists a homomorphism $f : C' \to C$ with $g' = gf$. (2) If $f$ is an endomorphism of $C$ with $gf = g$, then $f$ must be an automorphism. If (1) holds but (2) may not, $g : C \to M$ is called a $C$-precover. Dually we have the definition of a $C$-(pre)envelope. $C$-covers and $C$-envelopes may not exist in general, but if they exist, they are unique up to isomorphism. A homomorphism $g : M \to C$ with $C \in C$ is said to a $C$-envelope with the unique mapping property (see [9]) if for any homomorphism $g' : M \to C'$ with $C' \in C$, there is a unique homomorphism $f : C \to C'$ such that $fg = g'$. Dually, we have the definition of $C$-cover with the unique mapping property.

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Let $M$ be a left $R$-module. A right $C$-resolution of $M$ is a complex (need not be exact) $0 \to M \to F^0 \to F^1 \to \cdots$ with each $F^i \in C$. Write

$$L^0 = M, \ L^1 = \text{Coker}(M \to F^0), \ L^i = \text{Coker}(F^{i-2} \to F^{i-1}) \text{ for } i \geq 2.$$ 

Here $M \to F^0, L^1 \to F^1, L^i \to F^i$ for $i \geq 2$ are $C$-preenvelopes. The $n$th cokernel $L^n(n \geq 0)$ is called the $n$th $C$-cosyzygy of $M$.

A left $C$-resolution of $M$ is a complex $\cdots \to I_1 \to I_0 \to M \to 0$ with each $I_i \in C$. Write

$$K_0 = M, \ K_1 = \text{Ker}(I_0 \to M), \ K_i = \text{Ker}(I_{i-1} \to I_{i-2}) \text{ for } i \geq 2.$$ 

Here $I_0 \to M, I_1 \to K_1, I_i \to K_i$ for $i \geq 2$ are $C$-precovers. The $n$th kernel $K_n(n \geq 0)$ is called the $n$th $C$-syzygy of $M$.

A left $C$-resolution $\cdots \to I_1 \to I_0 \to M \to 0$ is called minimal if every $I_i \to K_i$ is $C$-cover for any $i \geq 0$.

Let $R$ be a ring. A left $R$-module $M$ is coherent if every finitely generated submodule of $M$ is finitely presented. The ring $R$ is said to be left coherent if $R$ is a coherent as a left $R$-module. Since coherence of rings and modules first appeared in [2], their generalizations have been studied extensively by many authors (see, [3, 4, 7, 9, 11, 15, 17]). A ring $R$ is called left $J$-coherent [7] if the Jacobson radical $J(R)$ of $R$ is a coherent left $R$-module. $R$ is said to be left $P$-coherent [17] (resp. left min-coherent [15]) if every principal (resp. minimal) left ideal of $R$ is finitely presented.

Recall that the prime radical $Nil_*(R)$ [14] $(N(R)$ for short) of $R$ is the intersection of all prime ideals of $R$. $N(R)$ contains all nilpotent one-side ideal of $R$. A ring $R$ is semiprime if $N(R) = 0$. We say that a ring $R$ is left $Nil_*$-coherent if the prime radical $N(R)$ of $R$ is a coherent left $R$-module, or equivalently, any finitely generated left ideal in $N(R)$ is finitely presented. $Nil_*$-coherent rings are introduced, in this paper, as the generalization of $J$-coherent rings and semiprime rings. Some examples of left $Nil_*$-coherent rings are given, and some properties of left $Nil_*$-coherent rings are studied. We prove that if $R$ is right perfect, then $R$ is left $Nil_*$-coherent if and only if $R$ is left coherent. To characterize left $Nil_*$-coherent rings, we introduce left $N$-injective modules and right $N$-flat modules. The class of left $N$-injective (resp. right $N$-flat) $R$-modules is denoted $\mathcal{N}I$ (resp. $\mathcal{N}F$). We also show that if $R$ is left $Nil_*$-coherent, then every right $R$-module has an $\mathcal{N}F$-preenvelope and every left $R$-module has an $\mathcal{NI}$-cover.

In [8], Enochs and Jenda investigated the global dimension of a left Noetherian ring using the left injective resolutions of left $R$-modules. Mao recently generalized their results to left coherent rings (see [16]). In the third section of this paper, left strongly $Nil_*$-coherent rings and the $N$-injective dimensions are defined. We study the $N$-injective dimensions of modules and rings in terms of left $\mathcal{NI}$-resolutions and right $\mathcal{N}F$-resolutions of modules.
2. \textit{Nil}_*-coherent rings

\textbf{Definition 2.1.} A ring \( R \) is said to be left \textit{Nil}_*-coherent if the prime radical \( N(R) \) of \( R \) is coherent left \( R \)-module, or equivalently, every finitely generated left ideal in \( N(R) \) is finitely presented. Similarly, we have the concept of right \textit{Nil}_*-coherent rings.

\textbf{Remark 2.2.} Here give some examples of \textit{Nil}_*-coherent rings.

(1) Obviously, left \( J \)-coherent rings are left \textit{Nil}_*-coherent because \( N(R) \subseteq J(R) \).

(2) A semiprime ring is right and left \textit{Nil}_*-coherent. Moreover, a domain is right and left \textit{Nil}_*-coherent.

The following examples show that \textit{Nil}_*-coherent rings are non-trivial generalizations of \( J \)-coherent rings and semiprime rings.

\textbf{Example 1.} Let \( R \) be a valuation ring of rank \( R > 1 \). Then \( [[x]] \), the ring of power series in one variable \( x \), is a commutative domain, and so it is \textit{Nil}_*-coherent. But \( [[x]] \) is not a \( J \)-coherent ring by [7, Example 3.16].

\textbf{Example 2.} Let \( R = \left( \frac{\mathbb{Z}}{2} \right) \). Then \( R \) is a coherent ring, and hence it is a \textit{Nil}_*-coherent ring. However, \( R \) is not semiprime because there is a nilpotent ideal \((0 \ 2) \neq 0\).

From the next example, we can see that the definition of \textit{Nil}_*-coherent rings is not left-right symmetric.

\textbf{Example 3.} Let \( L = \mathbb{Q}(x_2, x_3, \ldots) \) be a subfield of \( K = \mathbb{Q}(x_1, x_2, \ldots) \) with \( \mathbb{Q} \) the field of rational numbers, and there exists a field isomorphism \( \varphi : K \to L \). We define a ring by taking \( R = K \times K \) with multiplication

\[
(x, y)(x', y') = (xx', \varphi(x)y' + yy'), \text{ where } x, y, x', y' \in K.
\]

It is easy to see that \( R \) has exactly three right ideals, \((0), R, \text{ and } (0, K) = (0, 1)R \). So \( R \) is right \textit{Nil}_*-coherent. Let \( a = (0, 1) \). Note that \( Ra \subseteq N(R) \) and \( l(a) \) is not finitely generated. Then \( R \) is not left \textit{Nil}_*-coherent.

Similar to [7, Proposition 2.10, Corollary 2.11 and Corollary 2.12], we have the following results.

\textbf{Proposition 2.3.} Let \( \varphi : R \to S \) be a ring homomorphism such that \( S \) is a finitely generated left \( R \)-module and \( N(S) \) is a coherent left \( R \)-module. If \( R \) is a left \textit{Nil}_*-coherent ring, then so is \( S \).

\textit{Proof.} Let \( M \) be a finitely generated submodule of the left \( S \)-module \( N(S) \). By assumption, \( M \) is a finitely generated submodule of the left \( R \)-module \( N(S) \), and hence \( M \) is a finitely presented left \( R \)-module. So \( M \) is a finitely presented left \( S \)-module by [11, Theorem 1]. Therefore, \( S \) is a left \textit{Nil}_*-coherent ring.

\textbf{Corollary 2.4.} Let \( R \) be a left \textit{Nil}_*-coherent ring. Then \( M_n(R) \), the ring of \( n \times n \) matrices over \( R \), is also a left \textit{Nil}_*-coherent ring for every positive integer \( n \).
Proof. By [14, Theorem 10.21], \( N(M_n(R)) = M_n(N(R)) \cong N(R)^{n^2} \). \( N(R)^{n^2} \) is a coherent left \( R \)-module by assumption, so is \( N(M_n(R)) \). Then the result comes from Proposition 2.3.

**Corollary 2.5.** If \( R \) is a left \( \text{Nil}^*_* \)-coherent ring and a finitely generated left ideal \( I \subseteq N(R) \), then the quotient ring \( R/I \) is also left \( \text{Nil}^*_* \)-coherent.

**Proof.** We have \( N(R/I) = N(R)/I \) in terms of [14, Exercise 10.20]. Now let \( X \) be a finitely generated submodule of the left \( R \)-module \( N(R/I) \). Then there is a finitely generated left \( R \)-module \( J \) with \( I \subseteq J \subseteq N(R) \) and \( X = J/I \). Since \( R \) is left \( \text{Nil}^*_* \)-coherent, \( J \) is a finitely presented left \( R \)-module, so is \( X \) by [13, Lemma 4.54]. Thus \( N(R/I) \) is a coherent left \( R \)-module. Therefore, \( R/I \) is a left \( \text{Nil}^*_* \)-coherent ring by Proposition 2.3.

**Proposition 2.6.** A direct product of rings \( R = R_1 \times R_2 \times \cdots \times R_n \) is left \( \text{Nil}^*_* \)-coherent if and only if \( R_i \) is left \( \text{Nil}^*_* \)-coherent for \( i = 1, \ldots, n \).

**Proof.** Note that \( N(R) = N(R_1) \times N(R_2) \times \cdots \times N(R_n) \). If \( R \) is left \( \text{Nil}^*_* \)-coherent, then \( N(R) \) is coherent left \( R \)-module, so is \( N(R_i) \) for all \( i \). By Proposition 2.3, \( R_i \) is left \( \text{Nil}^*_* \)-coherent.

Conversely, it is enough to prove the assertion for \( n = 2 \). There exists an exact sequence \( 0 \to N(R_1) \to N(R) \to N(R_2) \to 0 \). Hence \( N(R_2) \cong N(R)/N(R_1) \) is a coherent \( R_2 \)-module, thus, a coherent \( R \)-module by [9, Theorem 2.4.1]. Similarly, \( N(R_1) \) is a coherent \( R \)-module. By [9, Theorem 2.2.1(2)], \( N(R) \) is a coherent \( R \)-module, and hence \( R \) is left \( \text{Nil}^*_* \)-coherent.

If \( R \) is the direct product of \( R_1 \) and \( R_2 \), where \( R_1 \) is a left \( J \)-coherent ring that is not semiprime and \( R_2 \) is a semiprime ring that is not left \( J \)-coherent, then \( R \) is a left \( \text{Nil}^*_* \)-coherent ring that is neither left \( J \)-coherent nor semiprime.

Let \( M \) be a bimodule over \( R \). The trivial extension of \( R \) and \( M \) is \( R \rtimes M = \{ (a, x) | a \in R, x \in M \} \) with addition defined componentwise and multiplication defined by \( (a, x)(b, y) = (ab, ay + xb) \). For convenience, we write \( I \rtimes X = \{ (a, x) | a \in I, x \in X \} \), where \( I \) is a subset of \( R \) and \( X \) is a subset of \( M \). The below result is a generalization of [4, Theorem 12].

**Proposition 2.7.** A ring \( R \) is left coherent if and only if \( R \rtimes R \) is left \( \text{Nil}^*_* \)-coherent.

**Proof.** \((\Rightarrow)\). It follows from [4, Theorem 12] and Remark 2.2(1).

\((\Leftarrow)\). Set \( S = R \rtimes R \). We first prove that \( R \) is left \( P \)-coherent. For any \( a \in R, S(0, a) \subseteq N(S) \) and \( l_S(0, a) = l_R(a) \rtimes R \). Since \( S \) is left \( \text{Nil}^*_* \)-coherent, \( l_R(a) \rtimes R \) is a finitely generated left ideal of \( S \). Write \( l_R(a) \rtimes R = S(a_1, b_1) + \cdots + S(a_n, b_n) \) with all \( (a_i, b_i) \in S \). It follows that \( l_R(a) = Ra_1 + \cdots + Ra_n \). So \( R \) is left \( P \)-coherent.

Now since \( R \rtimes R \) is left \( \text{Nil}^*_* \)-coherent, \( M_n(R) \rtimes M_n(R) \cong M_n(R \rtimes R) \) is left \( \text{Nil}^*_* \)-coherent (for all \( n > 0 \)) by Corollary 2.4. Thus, \( M_n(R) \) is left \( P \)-coherent, and so \( R \) is left coherent by [17, Proposition 2.4].
Left $\text{Nil}_*\text{-coherent}$ rings are left min-coherent. In fact, if $Ra$ is a minimal left ideal of $R$, then we have either $(Ra)^2 = 0$, or $Ra = Re$ for some idempotent $e^2 = e \in R$ (see [14, Lemma 10.22]). The following example is constructed to show that min-coherent rings need not be $\text{Nil}_*\text{-coherent}$.

**Example 4.** Let $R$ be a countable direct product of the polynomial ring $\mathbb{Q}[y, z]$ (see [13, Example 4.61(a)]). Then $R[x]$ is not a coherent ring. Note that $R[x] \cong R[x]$ is not $\text{Nil}_*\text{-coherent}$ by Proposition 2.10. But $(R \times R)[x]$ is min-coherent because both socles are zero.

In order to characterize $\text{Nil}_*\text{-coherent}$ rings, we introduce $N$-injective modules and $N$-flat modules as the following.

**Definition 2.8.** A left $R$-module $M$ is said to be $N$-injective if $\text{Ext}^1(R/I, M) = 0$ for every finitely generated left ideal $I$ in $N(R)$. A right $R$-module $F$ is called $N$-flat if $\text{Tor}_1(F, R/I) = 0$ for every finitely generated left ideal $I$ in $N(R)$. Usually, we can define right $N$-injective modules and left $N$-flat modules.

**Remark 2.9.** (1) In what follows, $\mathcal{N}T$ (resp. $\mathcal{N}F$) stands for the class of all $N$-injective left $R$-modules (resp. $N$-flat right $R$-modules). By the definition, it is clear that $\mathcal{N}T$ (resp. $\mathcal{N}F$) is closed under direct sums, direct summands, direct products (resp. direct limits) and extensions.

(2) A right $R$-module $F$ is $N$-flat if and only if $F^+$ is $N$-injective by the standard isomorphism $\text{Ext}^1(N, F^+) \cong \text{Tor}_1(F, N)^+$ for every finitely generated left ideal $I$ in $N(R)$.

(3) Recall that a left $R$-module $M$ (resp. right $R$-module $F$) is $J$-injective (resp. $J$-flat) if $\text{Ext}^1(R/I, M) = 0$ (resp. $\text{Tor}_1(F, R/I) = 0$) for any finitely generated ideal $I$ in $J(R)$ (see [7]). It is easy to see that left $J$-injective (resp. right $J$-flat) $R$-modules are left $N$-injective (resp. right $N$-flat). If $R$ is left Artinian, then left $J$-injective (resp. right $J$-flat $R$-modules coincide with left $N$-injective (resp. right $N$-flat).

**Proposition 2.10.** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is a semiprime ring.
2. Every left (or right) $R$-module is $N$-injective.
3. Every left (or right) simple $R$-module is $N$-injective.
4. Every principle left (or right) ideal in $N(R)$ is $N$-injective.
5. Every right (or left) $R$-module is $N$-flat.
6. Every finitely generated left (or right) ideal in $N(R)$ is a pure submodule of $R$.

**Proof.** (1)$\Rightarrow$(2) is trivial since $N(R) = 0$. (2)$\Rightarrow$(3) and (2)$\Rightarrow$(4) are clear.

(2)$\Rightarrow$(5) holds by Remark 2.9(2).

(3)$\Rightarrow$(1). Let $a \in N(R)$. If $N(R) + l(a) \neq R$, then we take a maximal left ideal $M$ of $R$ such that $N(R) + l(a) \subseteq M$. Then $R/M$ is $N$-injective by (3). Note that the homomorphism $f : Ra \rightarrow R/M$ given by $f(xa) = x + M$, $x \in R$ is well-defined. So there exists $c \in R$ such that $f = (c + M)$. Then $1 + M =...
\[ f(a) = a(c + M) = ac + M, \] which implies that \( 1 - ac \in M \), which yields \( 1 \in M \), a contradiction. Therefore \( N(R) + l(a) = R \) and so \( l(a) = R \) because \( N(R) \) is a small ideal of \( R \). So \( a = 0 \). Hence \( N(R) = 0 \).

(5) \( \Rightarrow \) (6). For any finitely generated left ideal \( I \) in \( N(R) \) and any right \( R \)-module \( M \), \( \text{Tor}_1(M, R/I) = 0 \) since \( M \) is \( N \)-flat. Then \( R/I \) is flat, and hence \( I \) is a pure submodule of \( R \).

(6) \( \Rightarrow \) (2). Let \( I \) be a finitely generated left ideal in \( N(R) \). Then \( R/I \) is flat by (6), and so it is projective. Thus every left \( R \)-module is \( N \)-injective.

(4) \( \Rightarrow \) (1). Suppose that \( N(R) \neq 0 \), then there exists an non-zero superfluous submodule \( Ra \) in \( N(R) \). Thus \( \text{Ext}^1(R/Ra, Ra) = 0 \) by (3), and so the exact sequence \( 0 \to Ra \to R \to R/Ra \to 0 \) splits. Therefore \( Ra \) is a direct summand of \( R \). Since \( Ra \) is superfluous, \( Ra = 0 \), a contradiction. Hence \( R \) is a semiprime ring.

\[ \Box \]

Let \( R = \mathbb{Z} \), the integer ring. By the proposition above, any \( R \)-module is \( N \)-injective and \( N \)-flat. However, \( \mathbb{Z} \) is not injective and \( \mathbb{Z}/2\mathbb{Z} \) is not flat as \( R \)-module.

Similar to [7, Theorem 2.13], [15, Theorem 4.5] and [17, Theorem 2.7], we have the following theorem which characterize \( \text{Nil}^*_N \)-coherent rings in terms of, among others, \( N \)-injective modules, \( N \)-flat modules and \( N \)-flat preenvelope.

Theorem 2.11. Let \( R \) be a ring. Then the following are equivalent:

1. \( R \) is a left \( \text{Nil}^*_N \)-coherent ring.
2. Any direct product of copies of \( R \) is \( N \)-flat.
3. Any direct product of \( N \)-flat right \( R \)-modules is \( N \)-flat.
4. Any direct limit of \( N \)-injective left \( R \)-modules is \( N \)-injective.
5. For any finitely generated left ideal \( I \) in \( N(R) \) and any family \( \{M_i\} \) of right \( R \)-modules, \( \text{Tor}_1(\prod M_i, R/I) \cong \prod \text{Tor}_1(M_i, R/I) \).
6. A left \( R \)-module \( M \) is \( N \)-injective if and only if \( M^+ \) is \( N \)-flat.
7. A right \( R \)-module \( M \) is \( N \)-injective if and only if \( M^{++} \) is \( N \)-injective.
8. A right \( R \)-module \( P \) is \( N \)-flat if and only if \( P^{++} \) is \( N \)-flat.
9. Every right \( R \)-module has an \( NF \)-preenvelope.

Corollary 2.12. The following statements hold for any ring \( R \):

1. \( NI \) and \( NF \) are closed under pure submodules.
2. If \( R \) is left \( \text{Nil}^*_N \)-coherent, then \( NI \) and \( NF \) are closed under pure quotient modules.

Proof. (1). The proof is similar to that of [7, Lemma 2.4].

(2). For a pure exact sequence \( 0 \to A \to B \to C \to 0 \) of left \( R \)-modules with \( B \) \( N \)-injective, there is a split exact sequence \( 0 \to C^+ \to B^+ \to A^+ \to 0 \). By Theorem 2.11, \( B^+ \) is \( N \)-flat, so is \( C^+ \). Thus \( C \) is \( N \)-injective by Theorem 2.11 again. The \( NF \) case is similar.

The following result will consider the existence of \( NI \)-covers over a left \( \text{Nil}^*_N \)-coherent ring.
Proposition 2.13. Let $R$ be a left $\text{Nil}_{*}$-coherent ring. Then every left $R$-module has an $N\mathcal{I}$-cover.

Proof. By Corollary 2.12(2), $N\mathcal{I}$ is closed under pure quotient modules. By Remark 2.9(1), $N\mathcal{I}$ is closed under direct sums. Then, in view of [12, Theorem 2.5], every left $R$-module has an $N\mathcal{I}$-cover. □

Remark 2.14. If $R$ is a left $\text{Nil}_{*}$-coherent ring, then every right $R$-module has a right $N\mathcal{F}$-resolution by Theorem 2.11, and every right $R$-module has a left $N\mathcal{I}$-resolution by Proposition 2.13.

In general, an $N\mathcal{I}$-cover need not be an epimorphism and an $N\mathcal{F}$-preenvelope need not be a monomorphism. Now we consider when every left $R$-module has an epic $N\mathcal{I}$-cover and when every right $R$-module has a monic $N\mathcal{F}$-preenvelope.

Proposition 2.15. Let $R$ be left $\text{Nil}_{*}$-coherent. Then the following are equivalent:

1. $R$ is $N$-injective as left $R$-module.
2. For any left $R$-module, there is an epimorphic $N\mathcal{I}$-cover.
3. For any right $R$-module, there is a monomorphic $N\mathcal{F}$-preenvelope.
4. Every $(FP)$-injective right $R$-module is $N$-flat.
5. Every flat left $R$-module is $N$-injective.

Proof. (1) ⇒ (3). Let $M$ be any right $R$-module. Then $M$ has an $N\mathcal{F}$-preenvelope $f: M \rightarrow F$ by Theorem 2.11. Since $(rR)^+ = (rR)^+$ is a cogenerator in the category of right $R$-modules, there is an exact sequence $0 \rightarrow M \rightarrow \prod (rR)^+$. By Theorem 2.11, $\prod (rR)^+$ is $N$-flat. So there exists a homomorphism $g: F \rightarrow \prod (rR)^+$ such that $gf = i$. Since $i$ is a monomorphism, so is $f$.

(3) ⇒ (4). Note that the $FP$-injective right $R$-module $E$ embeds in a $N$-flat right $R$-module by (3). Thus $E$ is $N$-flat by Corollary 2.12.

(4) ⇒ (5). For any flat left $R$-module $F$, $F^+$ is injective. Then $F^+$ is $N$-flat by assumption, and hence $F$ is $N$-injective by Theorem 2.11.

(5) ⇒ (2). For any left $R$-module $M$, in view of Proposition 2.13, there is an $N\mathcal{I}$-cover $f: C \rightarrow M$. Note that $R$ is also $N$-injective by hypothesis, so $f$ is an epimorphism.

(2) ⇒ (1). By assumption, $R$ has an epimorphic $N\mathcal{I}$-cover $\varphi: D \rightarrow R$, then we have an exact sequence $0 \rightarrow K \rightarrow D \xrightarrow{\varphi} R \rightarrow 0$ with $K = \text{Ker}\varphi$ and $D$ $N$-injective. Note that $R$ is projective, so the sequence is split, then $R$ is $N$-injective as left $R$-module by Remark 2.9 (1). □

Corollary 2.16. The following are equivalent for a ring $R$.

1. $R$ is semiprime.
2. $R$ is left $N$-injective and every finitely generated left ideal in $N(R)$ is projective.

Proof. (1) ⇒ (2) is clear.
(2) ⇒ (1). We firstly prove that every quotient module of a $N$-injective left $R$-module is $N$-injective. Let $B$ be any $N$-injective left $R$-module and $A \subseteq B$. For any finitely generated left ideal $I$ in $N(R)$ and a homomorphism $f : I \to B/A$, $I$ is projective, so there is a homomorphism $g : I \to B$ such that $\pi g = f$, where $\pi : B \to B/A$ is the canonical epimorphism. Then there is a homomorphism $h : R \to B$ such that $hi = g$ since $B$ is $N$-injective, where $i : I \to R$ is an inclusion. Thus, $f = \pi hi$, and hence $B/A$ is $N$-injective.

Thus, for any left $R$-module $M$, there is a monomorphic $N$-injective cover $\alpha : E \to M$ by [20, Proposition 4]. Since $R$ is left $N$-injective, then $\alpha$ is epimorphic by Proposition 2.15, whence $M$ is left $N$-injective. By Proposition 2.10, $R$ is semiprime. □

Remark 2.17. The ring $R$ in Example 2 is left hereditary, and hence every finitely generated left ideal in $N(R)$ is projective. But it is not semiprime, so $R$ is not left $N$-injective by Corollary 2.16. Thus, there exists a ring whose every left $R$-module has an $N\mathcal{I}$-cover but need not be an epimorphism and every right $R$-module has an $N\mathcal{F}$-preenvelope but need not be a monomorphism.

Recall that a ring $R$ is right perfect [18] if $R/J(R)$ is semisimple and $J(R)$ is right $T$-nilpotent. It was shown that if $R$ is right perfect, then $R$ is left $J$-coherent if and only if $R$ is left coherent (see [7]). At the end of this section, we extend this result onto left $\text{Nil}_*$-coherent rings.

**Proposition 2.18.** If $R$ is right perfect, then $R$ is left $\text{Nil}_*$-coherent if and only if $R$ is left coherent.

**Proof.** (⇒) is clear.

(⇐). We first prove that every $N$-flat right $R$-module is flat. Let $F$ be right $N$-flat. Note that $N(R) \cong \lim\limits_{\to} I_i$, where $I_i$ range over all finitely generated submodules of $N(R)$. Then

$$\text{Tor}_1(F, R/N(R)) = \text{Tor}_1(F, \lim\limits_{\to} R/I_i) = \lim\limits_{\to} \text{Tor}_1(F, R/I_i) = 0.$$ Since $N(R) \subseteq J(R)$ is also right $T$-nilpotent, $F$ is right flat by [23, Theorem 5.2].

Now let $M$ be any $N$-injective left $R$-module. Then $M^+$ is $N$-flat by Theorem 2.11, and hence $M^+$ is flat by the preceding result. Thus $M^{++}$ is $FP$-injective, whence $M$ is $FP$-injective because $M$ is a pure submodule of $M^{++}$. By Theorem 2.11 again, any direct limit of $FP$-injective left $R$-modules is $FP$-injective, which implies $R$ is left coherent. □

### 3. Strongly $\text{Nil}_*$-coherent rings

A class $C$ of left $R$-modules is said to be coresolving [19] if $E \in C$ for all injective left $R$-modules $E$, if $C$ is closed under extensions, and if given an exact sequence of left $R$-modules $0 \to A \to B \to C \to 0$, $E \in C$ whenever $A, B \in C$. Dually, we have the definition of resolving.

In the present section, we study the ring that $N\mathcal{I}$ is coresolving.
Lemma 3.1. Let \( R \) be a ring. Then the following are equivalent:

1. \( \mathcal{I} \) is coresolving.
2. \( \text{Ext}^k(R/I, M) = 0 \) for any \( N \)-injective left \( R \)-module \( M \) and any finitely generated left ideal \( I \) in \( N(R) \), \( k \geq 1 \).
3. \( R \) is left \( \text{Nil}_* \)-coherent and \( NF \) is resolving.
4. \( R \) is left \( \text{Nil}_* \)-coherent and \( \text{Tor}_k(N, R/I) = 0 \) for any \( N \)-flat right \( R \)-module \( N \) and any finitely generated left ideal \( I \) in \( N(R) \), \( k \geq 1 \).

Proof. The proof is similar to that of [7, Lemma 3.4]. □

Definition 3.2. We call the ring satisfying the equivalent conditions in Lemma 3.1 left strongly \( \text{Nil}_* \)-coherent. Dually, the notion of right strongly \( \text{Nil}_* \)-coherent rings can be defined.

Example 5. (1) By Proposition 2.10, a semiprime ring is left and right strongly \( \text{Nil}_* \)-coherent.
(2) If a ring \( R \) satisfies the condition that every finitely generated left ideal in \( N(R) \) is projective, then \( R \) is left strongly \( \text{Nil}_* \)-coherent by the proof of Corollary 2.16.
(3) A right perfect and left \( \text{Nil}_* \)-coherent ring is left strongly \( \text{Nil}_* \)-coherent by Proposition 2.18 and Lemma 3.1.

Remark 3.3. We claim that the definition of strongly \( \text{Nil}_* \)-coherent rings is also not left-right symmetric. Indeed, the ring \( R \) in Example 3 is right \( \text{Nil}_* \)-coherent but not left \( \text{Nil}_* \)-coherent. Note that it has only three right ideals, 0, \((0, K) = (0, 1)R\) and \( R \). Thus \( R \) is left prefect by [18, Theorem B.39], and hence \( R \) is right strongly \( \text{Nil}_* \)-coherent ring but not left strongly \( \text{Nil}_* \)-coherent.

Definition 3.4. The left \( N \)-injective dimension of a left \( R \)-module \( M \), denoted by \( l.N - \text{Id}(M) \), is defined as the least nonnegative integer \( n \) such that \( \text{Ext}^{n+1}(R/I, M) = 0 \) for any finitely generated left ideal \( I \) in \( N(R) \). If no such \( n \) exists, then \( l.N - \text{Id}(M) = \infty \). Set \( l.N - \text{Id}(R) = \sup \{ l.N - \text{Id}(M) : M \in R \text{-mod} \} \) and call \( l.N - \text{Id}(R) \) the left \( N \)-injective dimension of \( R \).

By Proposition 2.10, \( l.N - \text{Id}(R) = 0 \) if and only if \( R \) is a semiprime ring. Then the \( N \)-injective dimension of \( R \) can measure how far away a ring is from being a semiprime ring.

Proposition 3.5. Let \( R \) be a left strongly \( \text{Nil}_* \)-coherent ring. Then the following are equivalent for a left \( R \)-module \( M \):

1. \( l.N - \text{Id}(M) \leq n \).
2. \( \text{Ext}^{n+1}(R/I, M) = 0 \) for any finitely generated left ideal \( I \) in \( N(R) \).
3. \( \text{Ext}^{n+k}(R/I, M) = 0 \) for every finitely generated left ideal \( I \) in \( N(R) \), and \( k \geq 1 \).
4. For every exact sequence \( 0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow L_n \rightarrow 0 \) with each \( E_i \) \( N \)-injective, \( L_n \) is \( N \)-injective.

Proof. The proof is similar to that of [7, Lemma 3.6]. □
Lemma 3.8. Let \( N \) be an exact sequence of left \( R \)-modules. Then:

1. \( l.N - \text{Id}(B) \leq \sup\{l.N - \text{Id}(A), l.N - \text{Id}(C)\} \).
2. \( l.N - \text{Id}(A) \leq \sup\{l.N - \text{Id}(B), l.N - \text{Id}(C) + 1\} \).
3. \( l.N - \text{Id}(C) \leq \sup\{l.N - \text{Id}(B), l.N - \text{Id}(A) - 1\} \).

Proof. (1). For any finitely generated left ideal \( I \) in \( N(R) \), we have the following exact sequence

\[
\text{Ext}^n(R/I, A) \rightarrow \text{Ext}^n(R/I, B) \rightarrow \text{Ext}^n(R/I, C)
\]

\[
\rightarrow \text{Ext}^{n+1}(R/I, A) \rightarrow \text{Ext}^{n+1}(R/I, B).
\]

Let \( l.N - \text{Id}(B) = n \). If \( l.N - \text{Id}(C) \leq n - 1 \), by Proposition 3.5, \( \text{Ext}^n(R/I, C) = \text{Ext}^{n+1}(R/I, B) = 0 \). Then \( \text{Ext}^{n+1}(R/I, A) = 0 \), and hence \( l.N - \text{Id}(A) \leq n \) by Proposition 3.5 again. If \( l.N - \text{Id}(A) < n \), then \( \text{Ext}^n(R/I, A) = 0 \), so \( \text{Ext}^n(R/I, B) = 0 \), and hence \( l.N - \text{Id}(B) < n \), contradicting with assumption. Thus \( l.N - \text{Id}(A) = n \), and (1) follows. If \( l.N - \text{Id}(C) \geq n \), it is clear that (1) hold.

Similarly, we can prove (2) and (3). \( \square \)

By Proposition 3.6, we immediately deduce the following corollary.

Corollary 3.7. Let \( R \) be a strongly \( \text{Nil}_n \)-coherent ring and \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) be an exact sequence of left \( R \)-modules with \( B \) \( N \)-injective. If \( 0 < l.N - \text{Id}(A) < \infty \), then \( l.N - \text{Id}(A) = l.N - \text{Id}(C) + 1 \).

Lemma 3.8. Let \( R \) be a ring and \( M \) a left \( R \)-module. There is an exact sequence \( 0 \rightarrow M \rightarrow I \rightarrow N \rightarrow 0 \) with \( I \) \( N \)-injective and \( \text{Ext}^1(N, I') = 0 \) for all \( N \)-injective left \( R \)-modules \( I' \). Moreover, \( \text{Tor}_1(F, N) = 0 \) for all \( N \)-flat right \( R \)-modules \( F \).

Proof. In view of [10, Theorem 4.1.6] and [21, Corollary 3.5], left \( R \)-module \( M \) has a special \( \mathcal{N} \)-preenvelope \( f : M \rightarrow I \), that is, there is an exact sequence \( 0 \rightarrow M \rightarrow I \rightarrow N \rightarrow 0 \), where \( I \) is \( N \)-injective and \( \text{Ext}^1(N, I') = 0 \) for all \( N \)-injective left \( R \)-modules \( I' \).

For any \( N \)-flat right \( R \)-module \( F \), \( F^+ \) is \( N \)-injective by Remark 2.9(2). Thus \( \text{Tor}_1(F, N) \simeq \text{Ext}^1(N, F^+) = 0 \), and hence \( \text{Tor}_1(F, N) = 0 \). \( \square \)

Proposition 3.9. Let \( R \) be a left strongly \( \text{Nil}_n \)-coherent ring and \( M \) a left \( R \)-module. Then \( l.N - \text{Id}(M) \leq n(n \geq 0) \) if and only if for every left \( \mathcal{N} \)-resolution \( \cdots \rightarrow I_n \rightarrow I_{n-1} \rightarrow \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow N \rightarrow 0 \) of any right \( R \)-module \( N \), \( \text{Hom}(M, I_n) \rightarrow \text{Hom}(M, K_n) \) is an epimorphism, where \( K_n \) is the \( n \)th \( \mathcal{N} \)-syzygy of \( N \).

Proof. We proceed by induction on \( n \). For \( n \geq 1 \), by Lemma 3.8, there is an exact sequence \( 0 \rightarrow M \rightarrow I \rightarrow N \rightarrow 0 \), where \( I \) is \( N \)-injective and \( \text{Ext}^1(N, I') = 0 \) for all \( N \)-injective left \( R \)-modules \( I' \). Then we have the following commutative diagram
\[ \begin{align*}
\text{Hom}(I, I_n) & \to \text{Hom}(I, K_n) \to 0 \\
\downarrow & \downarrow \\
\text{Hom}(M, I_n) & \to \text{Hom}(M, K_n) \\
\downarrow & \downarrow \\
0 & \to 0.
\end{align*} \]

Since \( I_n \to K_n \) is an \( \mathcal{NZ} \)-precover of \( K_n \), the first arrow is exact. In addition, the first column is exact since \( \text{Ext}^1(N, I_n) = 0 \). Furthermore, there is a commutative diagram

\[ 
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \text{Hom}(N, K_n) & \text{Hom}(N, I_{n-1}) & \text{Hom}(N, K_{n-1}) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \text{Hom}(I, K_n) & \text{Hom}(I, I_{n-1}) & \text{Hom}(I, K_{n-1}) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \text{Hom}(M, K_n) & \text{Hom}(M, I_{n-1}) & \text{Hom}(M, K_{n-1}) & 0.
\end{array} \]

\( l \cdot N - Id(M) \leq n \) if and only if \( l \cdot N - Id(N) \leq n - 1 \) by Corollary 3.7 if and only if \( \text{Hom}(N, I_{n-1}) \to \text{Hom}(N, K_{n-1}) \) is an epimorphism by induction if and only if \( \text{Hom}(I, K_n) \to \text{Hom}(M, K_n) \) is an epimorphism by the second diagram if and only if \( \text{Hom}(M, I_n) \to \text{Hom}(M, K_n) \) is an epimorphism by the first diagram.

For \( n = 0 \), let \( K_0 = M \). Then \( \text{Hom}(M, I_0) \to \text{Hom}(M, M) \) is an epimorphism means \( \text{Hom}(I, M) \to \text{Hom}(M, M) \) is an epimorphism. Thus \( 0 \to M \to I \to N \to 0 \) splits, and hence \( M \) is \( N \)-injective. Conversely, if \( M \) is \( N \)-injective, then it is clear that \( \text{Hom}(M, I_0) \to \text{Hom}(M, K_0) \) is an epimorphism. \( \square \)

Let \( \mathcal{C}, \mathcal{D} \) and \( \mathcal{E} \) be categories of modules and \( T : \mathcal{C} \times \mathcal{D} \to \mathcal{E} \) be an additive functor contravariant in the first variable and covariant in the second. Let \( \mathcal{I} \) and \( \mathcal{F} \) be the classes of modules of \( \mathcal{C} \) and \( \mathcal{D} \) respectively. Then \( T \) is said to be right balanced by \( \mathcal{I} \times \mathcal{F} \) if for each module \( M \) of \( \mathcal{C} \), there is a \( T(-, \mathcal{F}) \) exact complex \( \cdots \to I_1 \to I_0 \to M \to 0 \) with each \( I_i \in \mathcal{I} \), and for each module \( N \) of \( \mathcal{D} \), there is a \( T(\mathcal{I}, -) \) exact complex \( 0 \to N \to F^0 \to F^1 \to \cdots \) with \( F^i \in \mathcal{F} \). Similarly, we have the definition of left balance. If \( T \) is covariant in both variables, then we would postulate the existence of complexes \( \cdots \to I_1 \to I_0 \to M \to 0 \) and \( \cdots \to F_1 \to F_0 \to N \to 0 \) or \( 0 \to M \to F^0 \to F^1 \to \cdots \) and \( 0 \to N \to F^0 \to F^1 \to \cdots \) to define the left or right balance functors relative to \( \mathcal{I} \times \mathcal{F} \), respectively.

**Lemma 3.10.** If \( R \) is left strongly \( \text{Nil}_* \)-coherent, then \( - \otimes - \) on \( \mathcal{M}_R \times_R \mathcal{M} \) is right balanced by \( \mathcal{N} \mathcal{F} \times \mathcal{N} \mathcal{I} \).

**Proof.** Let \( M \) be any right \( R \)-module. By Remark 2.14, there is a right \( \mathcal{N} \mathcal{F} \)-resolution \( 0 \to M \to F^0 \to F^1 \to \cdots \). For any \( N \)-injective left \( R \)-module \( N \),
\( N^+ \) is \( N \)-flat by Theorem 2.11. Thus we have an exact sequence
\[
\cdots \to \text{Hom}(F^1, N^+) \to \text{Hom}(F^0, N^+) \to \text{Hom}(M, N^+) \to 0.
\]
Hence
\[
\cdots \to (N \otimes F^1)^+ \to (N \otimes F^0)^+ \to (N \otimes M)^+ \to 0
\]
is exact. Then \( 0 \to N \otimes M \to N \otimes F^0 \to N \otimes F^1 \to \cdots \) is exact. In addition, by Lemma 3.8, the right \( NF \times NI \)-resolution \( 0 \to G \to I^0 \to I^1 \to \cdots \) of any left \( R \)-module \( G \) is exact, so the sequence \( 0 \to G \otimes F \to I^0 \otimes F \to I^1 \otimes F \to \cdots \) is exact for any \( F \in NF \) by Lemma 3.8 again, as desired. \( \square \)

**Remark 3.11.** (1) Tor\(^n\)(\( - , - \)) denotes the \( n \)th right derived functor of \( - \otimes - \) with respect to the pair \( NF \times NI \). If \( R \) is a left strongly \( Nil^* \)-coherent ring, for any right \( R \)-module \( M \) and left \( R \)-module \( N \), Tor\(^n\)(\( M, N \)) can be computed using either the right \( NF \)-resolution of \( M \) or the right \( NI \)-resolution of \( N \) by Lemma 3.10.

(2) If \( R \) is a left strongly \( Nil^* \)-coherent ring, by the proof of Lemma 3.8, every left \( R \)-module has a right \( NI \)-resolution. So Hom\(( - , - \)) is left balanced on \( R \mathcal{M} \times \mathcal{M} \) by \( NI \times NI \). Let \( \text{Ext}_n(-, -) \) be the \( n \)th left derived functor of Hom\(( - , - \)) with respect to the pair \( NI \times NI \). Then, for two left \( R \)-modules \( M \) and \( N \), \( \text{Ext}_n(M, N) \) can be computed using the right \( NI \)-resolution of \( M \) or the left \( NI \)-resolution of \( N \).

We are now in a position to prove the following theorem.

**Theorem 3.12.** If \( R \) is left strongly \( Nil^* \)-coherent and \( n \geq 0 \), then the following are equivalent:

1. \( l \mathcal{N} - \text{Id}(R) \leq n \).
2. If \( 0 \to M \to F^0 \to F^1 \to \cdots \) is a right \( NF \)-resolution of right \( R \)-module \( M \), then the sequence is exact at \( F^k \) for \( k \geq n - 1 \), where \( F^{-1} = M \).
3. For every flat left \( R \)-module \( F \), there is an exact sequence \( 0 \to F \to A^0 \to A^1 \to \cdots \to A^n \to 0 \) with each \( A^i \in NI \).
4. For every injective right \( R \)-module \( A \), there is an exact sequence \( 0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0 \) with each \( F_i \in NF \).
5. If \( \cdots \to I_1 \to I_0 \to M \to 0 \) is a left \( NI \)-resolution of a left \( R \)-module \( M \), then the sequence is exact at \( I_k \) for \( k \geq n - 1 \), where \( I_{-1} = M \).

**Proof.** (3)\( \Rightarrow \) (1) is trivial.

(1)\( \Rightarrow \) (2). By Remark 3.11 (1), the right derived functor Tor\(^n\)(\( R, M \)) can be computed using either a right \( NF \)-resolution of \( M \) or a right \( NI \)-resolution of \( R \).

If \( n \geq 2 \), we have the exact sequence \( 0 \to R \to A^0 \to \cdots \to A^n \to 0 \) with \( A^i \in NI \), so Tor\(^k\)(\( R, M \)) = 0 for \( k \geq n - 1 \). Computing using \( 0 \to M \to F^0 \to F^1 \to \cdots \) in (2), we see that the sequence \( \cdots \to R \otimes F^{n-2} \to R \otimes F^{n-1} \to R \otimes F^n \to \cdots \) is exact at \( R \otimes F^k \) for \( k \geq n - 1 \), so \( 0 \to M \to F^0 \to F^1 \to \cdots \) is exact at \( F^k \) for \( k \geq n - 1 \).
If \( n = 1 \), \( 0 \rightarrow R \rightarrow A^0 \rightarrow A^1 \rightarrow 0 \) is exact, where \( A^i \) is \( N \)-injective. So \( \text{Tor}^1(R, M) = 0 \) as above, \( F^0 \rightarrow F^1 \rightarrow F^2 \) is exact and \( R \otimes M \rightarrow \text{Tor}^0(R, M) \) is epic. Computing the latter morphism using \( 0 \rightarrow M \rightarrow F^0 \rightarrow F^1, \) we have \( M \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots \) is exact.

If \( n = 0 \), then \( R \) is \( N \)-injective as a right \( R \)-module. But the balance of \(- \otimes -\) then gives \( 0 \rightarrow R \otimes M \rightarrow R \otimes F^0 \rightarrow R \otimes F^1 \rightarrow \cdots \) is exact. Thus \( 0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots \) is exact.

(2) \(\Rightarrow\) (3). Let \( 0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots \) be a right \( \mathcal{NF} \)-resolution of a finitely presented left \( R \)-module \( M \). By assumption, the sequence is exact at \( F^k \) for \( k \geq n - 1 \). Let \( 0 \rightarrow F \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots \) be exact with \( F \) flat and each \( A^i \) \( N \)-injective. If \( n \geq 2 \), we get \( \text{Tor}^k(F, M) = 0 \) for \( k \geq n - 1 \) since \( F \) is flat. Computing using \( 0 \rightarrow F \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots \), then \( A^{n-2} \otimes M \rightarrow A^{n-1} \otimes M \rightarrow A^n \otimes M \rightarrow A^{n+1} \otimes M \) is exact. By [8, Lemma 8.4.23], \( K = \text{Ker}(A^n \rightarrow A^{n+1}) \) is a pure submodule of \( A^n \), hence \( K \) is also \( N \)-injective by Corollary 2.12. Then \( 0 \rightarrow F \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots \rightarrow A^{n-1} \rightarrow K \rightarrow 0 \) gives the desired exact sequence.

If \( n = 1 \), then \( M \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots \) is exact. Thus \( \text{Tor}^k(F, M) = 0 \) for \( k \geq 1 \) and \( F \otimes M \rightarrow \text{Tor}^0(F, M) \) is epic. So \( F \otimes M \rightarrow A^0 \otimes M \rightarrow A^1 \otimes M \rightarrow A^2 \otimes M \) is exact. By [8, Lemma 8.4.23] again, we get the exact sequence \( 0 \rightarrow F \rightarrow A^0 \rightarrow K \rightarrow 0 \) with \( K = \text{Ker}(A^1 \rightarrow A^2) \) \( N \)-injective.

If \( n = 0 \), then \( 0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots \) is exact, so \( \text{Tor}^k(F, M) = 0 \) for \( k \geq 0 \) and \( F \otimes M \rightarrow \text{Tor}^0(F, M) \) is an isomorphism. This gives that \( 0 \rightarrow F \otimes M \rightarrow A^0 \otimes M \rightarrow A^1 \otimes M \) is exact, which implies \( F \) is a pure submodule of \( A^n \), hence \( F \) is \( N \)-injective.

(5) \(\Rightarrow\) (1). By assumption, \( I_n \rightarrow I_{n-1} \rightarrow I_{n-2} \) is exact at \( I_{n-1} \). Thus \( I_n \rightarrow K_n \) is epic, where \( K_n = \text{Ker}(I_{n-1} \rightarrow I_{n-2}) \). Hence \( \text{Hom}(R, I_n) \rightarrow \text{Hom}(R, K_n) \) is epic. By Proposition 3.9, \( 1.N - \text{Id}(R) \leq n \).

(1) \(\Rightarrow\) (5). If \( n \geq 2 \). Let \( 0 \rightarrow R \rightarrow A^0 \rightarrow \cdots \rightarrow A^n \rightarrow 0 \) be a right \( \mathcal{NF} \)-resolution of a right \( R \)-module \( M \), then \( \text{Ext}_k(R, M) = 0 \) for \( k \geq n - 1 \). By Remark 3.11 (2), we can compute \( \text{Ext}_k(R, M) = 0 \) using a left \( \mathcal{NF} \)-resolution of \( M \).

(4) \(\Rightarrow\) (5). If \( n = 1 \), then there is an exact sequence \( 0 \rightarrow R \rightarrow A^0 \rightarrow A^1 \rightarrow 0 \) with \( A^i \in \mathcal{NF} \). So \( 0 \rightarrow \text{Hom}(A^1, M) \rightarrow \text{Hom}(A^0, M) \rightarrow \text{Hom}(R, M) \) is exact. Thus \( \text{Ext}_k(R, M) = 0 \) for \( k \geq 1 \) and \( \text{Ext}_0(R, M) \rightarrow \text{Hom}(R, M) \) is a monomorphism. But computing \( \text{Ext}_0(R, M) \) using a left \( \mathcal{NF} \)-resolution of \( M \), we see that \( I_1 \rightarrow I_0 \rightarrow M \) is exact at \( I_0 \), so \( \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow M \rightarrow 0 \) is exact at \( I_k \) for \( k \geq 0 \).

If \( n = 0 \), then \( R \) is \( N \)-injective as a left \( R \)-module. So every \( \mathcal{NF} \)-precovar is epic, and hence \( \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow M \rightarrow 0 \) is exact.

The proof of (4) \(\Rightarrow\) (5) is dual to that of (2) \(\Rightarrow\) (3).}

**Proposition 3.13.** Let \( R \) be a left strongly \( Nil_* \)-coherent ring and \( wD(R) < \infty \), where \( wD(R) \) is the weak global dimension of \( R \). Then \( 1.N - \text{Id}(R) = 1.N - \text{Id}(R) \leq wD(R) \).
Proof. We first prove the right inequality. By the definitions of left \( N \)-injective dimensions of modules and rings, we have

\[ l.N - \text{Id}(R) = \sup \{ l.pd(R/I) \mid I \text{ is finitely generated left ideal in } N(R) \} , \]

where \( l.pd(R/I) \) is the left projective dimension of \( R/I \). Then \( l.N - \text{Id}(R) \leq wD(R) \). We suppose that \( l.N - \text{Id}(R) = n < \infty \).

For the left equality, it suffices to prove \( l.N - \text{Id}(R) \leq l.N - \text{Id}(R) \). Hence assume that \( l.N - \text{Id}(R) = m < \infty \). By the similar proof of [7, Proposition 3.10], it can be proven that \( l.N - \text{Id}(F) \leq m \) for any free left \( R \)-module \( F \). Note that, for any left \( R \)-module \( M \), there exists an exact sequence \( 0 \to K_n \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0 \) with each \( F_i \) free. Then \( l.N - \text{Id}(K_n) = n \) and \( l.N - \text{Id}(F_i) \leq m \). By Proposition 3.5, \( \text{Ext}^{m+1}_R(R/I, M) \cong \text{Ext}^{n+n+1}_R(R/I, K_n) = 0 \) for every finitely generated left ideal \( I \) in \( N(R) \), and hence \( l.N - \text{Id}(M) \leq m \). Therefore, \( l.N - \text{Id}(R) = l.N - \text{Id}(R) \).

Example 6. Let \( \mathbb{F}[x] \) be a polynomial ring over a field \( \mathbb{F} \). Then \( \mathbb{F}[x] \) is semiprime, and hence \( l.N - \text{Id}(R) = l.N - \text{Id}(R) = 0 \). It is easy to verify that \( wD(R) = 1 \).

Lemma 3.14. Let \( R \) be a left strongly \( \text{Nil}_\ast \)-coherent ring and \( M \) a left \( R \)-module. If \( \text{Ext}_R^i(E, M) = 0 \) for all \( N \)-injective left \( R \)-modules \( E \), then \( M \) has an \( \mathcal{N}_I \)-cover \( L \to M \) with \( L \) injective.

Proof. In view of Proposition 2.13, \( M \) has an \( \mathcal{N}_I \)-cover \( f : L \to M \). For the exact sequence \( 0 \to L \to E \to L' \to 0 \) with \( E \) injective, \( L' \) is \( N \)-injective. Thus \( \text{Hom}(E, M) \to \text{Hom}(L, M) \to 0 \) is exact since \( \text{Ext}_R^1(L', M) = 0 \), and hence there is \( g \in \text{Hom}(E, M) \) such that \( f = gi \). Then there exists \( h : E \to L \) such that \( g = fh \) since \( f : L \to M \) is an \( \mathcal{N}_I \)-cover of \( M \). So \( f = fhi \), implies \( hi \) is isomorphism. Therefore, \( L \) is injective.

Theorem 3.15. If \( R \) is left strongly \( \text{Nil}_\ast \)-coherent and \( n \geq 1 \), then the following are equivalent:

1. \( l.N - \text{Id}(R) \leq n \).
2. Every \( n \)th \( \mathcal{N}_I \)-syzygy of a left \( R \)-module is \( N \)-injective.
3. Every \( (n-1) \)th \( \mathcal{N}_I \)-syzygy of a right \( R \)-module has an \( \mathcal{N}_I \)-cover which is a monomorphism.
   
   Moreover, if \( n \geq 2 \), then the above conditions are equivalent to:
4. Every \( (n-2) \)th \( \mathcal{N}_I \)-syzygy in a minimal left \( \mathcal{N}_I \)-resolution of a left \( R \)-module has an \( \mathcal{N}_I \)-cover with the unique mapping property.

Proof. (1)\( \Rightarrow \) (2). Let \( K_n \) be \( n \)th \( \mathcal{N}_I \)-syzygy of a left \( R \)-module. Then \( l.N - \text{Id}(K_n) \leq n \). So \( \text{Hom}(K_n, K_n) \to \text{Hom}(K_n, K_n) \) is an epimorphism by Proposition 3.9, whence \( K_n \) is \( N \)-injective.

(2)\( \Rightarrow \) (3). Let \( f : I_{n-1} \to K_{n-1} \) be an \( \mathcal{N}_I \)-precovers of the \((n-1)\)th \( \mathcal{N}_I \)-syzygy \( K_{n-1} \), and \( K_n = \text{Ker}(f) \). Then we have the exact sequence \( 0 \to K_n \to \).
$I_{n-1} \to \text{im}(f) \to 0$. By assumption, $K_n$ is $N$-injective, so is im$(f)$. Thus the inclusion im$(f) \to K_{n-1}$ is an $\mathcal{N}$-$\mathcal{T}$-cover which is a monomorphism.

(3)$\Rightarrow$(2). Let $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to N \to 0$ be any left $\mathcal{N}$-$\mathcal{T}$-resolution of a left $R$-module $N$ and \( K_n = \text{Ker}(I_{n-1} \to I_{n-2}) \). Let \( K_{n-1} = \text{Ker}(I_{n-2} \to I_{n-3}) \). Hence \( K_{n-1} \) has a monomorphic $\mathcal{N}$-$\mathcal{T}$-cover \( I \to K_{n-1} \) by assumption. Thus \( K_n \oplus I \cong I_{n-1} \) in terms of [8, Lemma 8.6.3]. So $K_n$ is $N$-injective by Remark 2.9(1).

(2)$\Rightarrow$(1). Let $M$ be a left $R$-module. For a left $\mathcal{N}$-$\mathcal{T}$-resolution $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to N \to 0$ of a left $R$-module $N$, $I_n \to K_n$ is a split epimorphism since $K_n$ is $N$-injective. Thus $\text{Hom}(M, I_n) \to \text{Hom}(M, K_n)$ is epimorphic, hence $l_N - Id(M) \leq n$ by Proposition 3.9. Then $l_N - l.dim(R) \leq n$.

(3)$\Rightarrow$(4). Let $\cdots \to I_{n-3} \to I_{n-4} \to \cdots \to I_1 \to I_0 \to M \to 0$ be a minimal $\mathcal{N}$-$\mathcal{T}$-resolution of a left $R$-module $M$ with $K_{n-2} = \text{Ker}(I_{n-3} \to I_{n-4})$. By assumption, $K_{n-1} = \text{Ker}(I_{n-2} \to I_{n-3})$ has a monomorphic $\mathcal{N}$-$\mathcal{T}$-cover $i : I_{n-1} \to K_{n-1}$. Note $\text{Ext}^1(I, K_{n-1}) = 0$ for all $N$-injective right $R$-modules $I$ by Wakamatsu’s Lemma. Thus $I_{n-1}$ is injective by Lemma 3.14. But $K_{n-1}$ has no nonzero injective submodule by [15, Corollary 1.2.8]. Thus $I_{n-1} = 0$, and hence $\text{Hom}(I, K_{n-1}) = \text{Hom}(I, I_{n-1}) = 0$ for any $N$-injective left $R$-module $I$. So we have the exact sequence $0 \to \text{Hom}(I, I_{n-2}) \to \text{Hom}(I, K_{n-2}) \to 0$ for any $N$-injective left $R$-module $I$, as desired.

(4)$\Rightarrow$(2). Let $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to M \to 0$ be an $\mathcal{N}$-$\mathcal{T}$-resolution of a left $R$-module $M$ with $K_n = \text{Ker}(I_{n-1} \to I_{n-2})$. By assumption, $M$ has a minimal $\mathcal{N}$-$\mathcal{T}$-resolution of the form $0 \to I_{n-2} \to I_{n-3} \to \cdots \to I_1' \to I_0' \to M \to 0$. In view of [8, Corollary 8.6.4], $K_n \oplus I_{n-2} \oplus I_{n-3}' \oplus \cdots \cong I_{n-1} \oplus I_{n-2}' \oplus I_{n-3}' \oplus \cdots$. Thus $K_n$ is $N$-injective. \( \square \)

Corollary 3.16. If $R$ is left strongly Nil$_+$-coherent, then the following are equivalent:

1. $l.N - l.dim(R) \leq 2$.
2. Every left $R$-module has an $\mathcal{N}$-$\mathcal{T}$-cover with the unique mapping property.

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