GCR-LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN PRODUCT MANIFOLD

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Abstract. We introduce GCR-lightlike submanifold of a semi-Riemannian product manifold and give an example. We study geodesic GCR-lightlike submanifolds of a semi-Riemannian product manifold and obtain some necessary and sufficient conditions for a GCR-lightlike submanifold to be a GCR-lightlike product. Finally, we discuss minimal GCR-lightlike submanifolds of a semi-Riemannian product manifold.

1. Introduction

The significant applications of CR-structures in relativity [3, 4] and growing importance of lightlike submanifolds in mathematical physics and moreover availability of limited information on theory of lightlike submanifolds, motivated Duggal and Bejancu [5] to introduce CR-lightlike submanifolds of indefinite Kaehler manifolds. Similar to CR-lightlike submanifolds, semi-invariant lightlike submanifolds of a semi-Riemannian product manifold were introduced by Atçeken and Kiliç in [1]. Since CR-lightlike submanifold does not include the complex and totally real cases therefore Duggal and Sahin [7] introduced Screen Cauchy-Riemann (SCR)-lightlike submanifold of indefinite Kaehler manifolds, which contains complex and screen real sub-cases. The SCR-lightlike submanifolds, analogously, Screen Semi-Invariant lightlike submanifolds, of semi-Riemannian product manifolds were introduced by Khursheed et al. [9] and Kiliç et al. [10], respectively. Since there is no inclusion relation between SCR and CR cases therefore Duggal and Sahin [8] introduced Generalized Cauchy-Riemann (GCR)-lightlike submanifold of indefinite Kaehler manifolds which acts as an umbrella of real hypersurfaces, invariant, screen real and CR lightlike submanifolds and further developed by [11, 12, 13, 14].
Since the geometry of lightlike submanifolds of semi-Riemannian product manifolds is a topic of chief discussion [16, 17, 18] therefore we introduce GCR-lightlike submanifolds of a semi-Riemannian product manifold. We study geodesic GCR-lightlike submanifolds of a semi-Riemannian product manifold and obtain some necessary and sufficient conditions for a GCR-lightlike submanifold to be a GCR-lightlike product. Finally, we discuss minimal GCR-lightlike submanifolds of a semi-Riemannian product manifold.

2. Lightlike submanifolds

Let \((\bar{M}, \bar{g})\) be a real \((m + n)\)-dimensional semi-Riemannian manifold of constant index \(q\) such that \(m, n \geq 1\), \(1 \leq q \leq m + n - 1\) and \((M, g)\) be an \(m\)-dimensional submanifold of \(\bar{M}\) and \(\bar{g}\) be the induced metric of \(\bar{g}\) on \(M\). If \(\bar{g}\) is degenerate on the tangent bundle \(T M\) of \(M\) then \(M\) is called a lightlike submanifold of \(\bar{M}\), for detail see [5]. For a degenerate metric \(g\) on \(M\), \(TM^\bot\) is a degenerate \(n\)-dimensional subspace of \(T \bar{x} M\). Thus both \(T \bar{x} M\) and \(T \bar{x} M^\bot\) are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace \(RadT \bar{x} M = T \bar{x} M \cap T \bar{x} M^\bot\) which is known as radical (null) subspace. If the mapping \(RadTM : x \in M \rightarrow RadT \bar{x} M\), defines a smooth distribution on \(M\) of rank \(r > 0\) then the submanifold \(M\) of \(\bar{M}\) is called an \(r\)-lightlike submanifold and \(RadTM\) is called the radical distribution on \(M\).

Screen distribution \(S(TM)\) is a semi-Riemannian complementary distribution of \(Rad(TM)\) in \(TM\) therefore

\[
TM = RadTM \perp S(TM)
\]

and \((TM^\bot)\) is a complementary vector subbundle to \(RadTM\) in \((TM^\bot)\). Let \(tr(TM)\) and \(ltr(TM)\) be complementary (but not orthogonal) vector bundles to \(TM\) \(|M\) and to \(RadTM\) in \((SM^\bot)|\), respectively. Then we have

\[
(1) \quad TM |_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^\bot),
\]

\[
(2) \quad tr(TM) = ltr(TM) \perp S(TM^\bot),
\]

\[
(3) \quad TM |_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^\bot).
\]

Let \(u\) be a local coordinate neighborhood of \(M\) and consider the local quasi-orthonormal fields of frames of \(M\) along \(M\), on \(u\) as \(\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\}\), where \(\{\xi_1, \ldots, \xi_r\}, \{N_1, \ldots, N_r\}\) are local lightlike bases of \(\Gamma(RadTM) |_u\), \(\Gamma(ltr(TM)) |_u\) and \(\{W_{r+1}, \ldots, W_n\}, \{X_{r+1}, \ldots, X_m\}\) are local orthonormal bases of \(\Gamma(S(TM^\bot)) |_u\) and \(\Gamma(S(TM)) |_u\) respectively. For this quasi-orthonormal fields of frames, we have:

**Theorem 2.1** ([5]). Let \((M, g, S(TM), S(TM^\bot))\) be an \(r\)-lightlike submanifold of a semi-Riemannian manifold \((M, \bar{g})\). Then there exists a complementary vector bundle \(ltr(TM)\) of \(RadTM\) in \(S(TM^\bot)|_u\) and a basis of \(\Gamma(ltr(TM)) |_u\) consisting of smooth section \(\{N_i\}\) of \(S(TM^\bot)|_u\) where \(u\) is a coordinate neighborhood of \(M\) such that

\[
(4) \quad \bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0 \text{ for any } i, j \in \{1, 2, \ldots, r\},
\]
where \( \{ \xi_1, \ldots, \xi_r \} \) is a lightlike basis of \( \Gamma(\text{Rad}(TM)) \).

Let \( \nabla \) be the Levi-Civita connection on \( M \) then according to the decomposition (3), the Gauss and Weingarten formulas are given by

\[
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \nabla_X U = -A_U X + \nabla^*_X U,
\]

for any \( X, Y \in \Gamma(TM) \) and \( U \in \Gamma(\text{tr}(TM)) \), where \( \{ \nabla_X Y, A_U X \} \) and \( \{ h(X, Y), \nabla^*_X U \} \) belong to \( \Gamma(TM) \) and \( \Gamma(\text{tr}(TM)) \), respectively. Here \( \nabla \) is a torsion-free linear connection on \( M \), \( h \) is a symmetric bilinear form on \( \Gamma(TM) \) which is called second fundamental form, \( A_U \) is a linear operator on \( M \) and known as shape operator.

According to (2) considering the projection morphisms \( L \) and \( S \) of \( \text{tr}(TM) \) on \( \text{ltr}(TM) \) and \( \text{S}(TM^\perp) \) respectively, then (5) become

\[
\nabla_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \nabla_X U = -A_U X + D^l_X U + D^s_X U,
\]

where we put \( h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^*_X U), D^s_X U = S(\nabla^*_X U). \)

As \( h^l \) and \( h^s \) are \( \Gamma(\text{ltr}(TM)) \)-valued and \( \Gamma(\text{S}(TM^\perp)) \)-valued, respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on \( M \). In particular

\[
\nabla_X X = -A_N X + \nabla^l_X N + D^s(X, N), \quad \nabla_X W = -A_W X + \nabla^*_X W + D^l(X, W),
\]

where \( X \in \Gamma(TM) \), \( N \in \Gamma(\text{ltr}(TM)) \) and \( W \in \Gamma(\text{S}(TM^\perp)) \). Using (6) and (7) we obtain

\[
\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y),
\]

\[
\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + \bar{g}(Y, \nabla_X \xi) = 0
\]

for any \( W \in \Gamma(\text{S}(TM^\perp)), \xi \in \Gamma(\text{Rad}(TM)) \). Let \( P \) be the projection morphism of \( TM \) on \( S(TM) \) then using (1), we can induce some new geometric objects on the screen distribution \( S(TM) \) on \( M \) as

\[
\nabla_X PY = \nabla^*_X PY + h^s(X, PY), \quad \nabla_X \xi = -A^*_N X + \nabla^*_X \xi,
\]

for any \( X, Y \in \Gamma(TM) \) and \( \xi \in \Gamma(\text{Rad}(TM)) \), where \( \{ \nabla^*_X PY, A^*_N X \} \) and \( \{ h^s(X, PY), \nabla^*_X \xi \} \) belong to \( \Gamma(S(TM)) \) and \( \Gamma(\text{Rad}(TM)) \), respectively. \( \nabla^*_X \) and \( \nabla^{*l} \) are linear connections on complementary distributions \( S(TM) \) and \( \text{Rad}(TM) \), respectively. \( h^s \) and \( A^* \) are \( \Gamma(\text{Rad}(TM)) \)-valued and \( \Gamma(S(TM)) \)-valued bilinear forms and are called as second fundamental forms of distributions \( S(TM) \) and \( \text{Rad}(TM) \), respectively. Using (6) and (10), we obtain

\[
\bar{g}(h^l(X, PY), \xi) = g(A^*_N X, PY), \quad \bar{g}(h^s(X, PY), N) = g(A_N X, PY)
\]

for any \( X, Y \in \Gamma(TM), \xi \in \Gamma(\text{Rad}(TM)) \) and \( N \in \Gamma(\text{ltr}(TM)) \).
3. Semi-Riemannian product manifolds

Let \((M_1, g_1)\) and \((M_2, g_2)\) be two \(m_1\) and \(m_2\) dimensional semi-Riemannian manifolds with constant indexes \(q_1 > 0\) and \(q_2 > 0\), respectively. Let \(\pi: M_1 \times M_2 \to M_1\) and \(\sigma: M_1 \times M_2 \to M_2\) be the projections which are given by \(\pi(x, y) = x\) and \(\sigma(x, y) = y\) for any \((x, y) \in M_1 \times M_2\). We denote the product manifold by \((\bar{M}, \bar{g}) = (M_1 \times M_2, \bar{g})\), where

\[
\bar{g}(X, Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y),
\]

for any \(X, Y \in \Gamma(T\bar{M})\), where \(*\) means the differential mapping. Then we have

\[
\pi_*^2 = \pi_*, \quad \sigma_*^2 = \sigma_*, \quad \pi_* \sigma_* = \sigma_* \pi_* = 0, \quad \pi_* + \sigma_* = I,
\]

where \(I\) is the identity map of \(T(M_1 \times M_2)\). Thus \((M, \bar{g})\) is a \((m_1 + m_2)\)-dimensional semi-Riemannian manifold with constant index \((q_1 + q_2)\). The semi-Riemannian product manifold \(M = M_1 \times M_2\) is characterized by \(M_1\) and \(M_2\) which are totally geodesic submanifolds of \(\bar{M}\). Now if we put \(F = \pi* - \sigma_*\), then we see that \(F^2 = I\) and

\[
\bar{g}(FX, Y) = \bar{g}(X, FY),
\]

for any \(X, Y \in \Gamma(T\bar{M})\), where \(F\) is called an almost product structure on \(M_1 \times M_2\). If we denote the Levi-Civita connection on \(\bar{M}\) by \(\bar{\nabla}\), then it can be seen that

\[
(\bar{\nabla}_X F)Y = 0,
\]

for any \(X, Y \in \Gamma(T\bar{M})\), that is, \(F\) is parallel with respect to \(\bar{\nabla}\).

4. Generalized Cauchy-Riemann lightlike submanifolds

**Definition 4.1.** Let \((M, g, S(TM))\) be a real lightlike submanifold of a semi-Riemannian product manifold \((\bar{M}, \bar{g})\). Then \(M\) is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles \(D_1\) and \(D_2\) of \(\text{Rad}(TM)\), such that

\[
\text{Rad}(TM) = D_1 \oplus D_2, \quad FD_1 = D_1, \quad FD_2 \subset S(TM).
\]

(B) There exist two subbundles \(D_0\) and \(D'\) of \(S(TM)\), such that

\[
S(TM) = \{FD_2 \oplus D'\} \perp D_0, \quad FD_0 = D_0, \quad FD' = L_1 \perp L_2,
\]

where \(D_0\) is a non degenerate distribution on \(M\), \(L_1\) and \(L_2\) are vector subbundles of \(\text{ltr}(TM)\) and \(S(TM)^\perp\), respectively.

Then the tangent bundle \(TM\) of \(M\) is decomposed as \(TM = D \perp D'\) and

\[
D = \text{Rad}(TM) \oplus D_0 \oplus FD_2.\]

\(M\) is called a proper GCR-lightlike submanifold if \(D_1 \neq \{0\}, D_2 \neq \{0\}, D_0 \neq \{0\}\), which has the following features:

1. The condition (A) implies that \(\dim(\text{Rad}(TM)) \geq 3\).
2. The condition (B) implies that \(\dim(D) = 2s \geq 6\), \(\dim(D') \geq 2\) and \(\dim(D_2) = \dim(L_1)\). Thus \(\dim(M) \geq 8\) and \(\dim(\bar{M}) \geq 12\).
3. Any proper \(8\)-dimensional GCR-lightlike submanifold is \(3\)-lightlike.
Example. Let \( R^2_4 = R^2_5 \times R^2_6 \) be a semi-Riemannian product manifold with the product structure \( F(\partial x^i, \partial y^j) = (\partial y^j, \partial x^i) \), where \((x^i, y^j)\) are cartesian coordinates of \( R^2_4 \). Let \( M \) be a submanifold of \( R^2_4 \) given by:

\[
\begin{align*}
  x_1 &= u_1, & x_2 &= u_5, & x_3 &= u_3, & x_4 &= \sqrt{1 - u_2^2}, & x_5 &= u_6, & x_6 &= u_2,
  y_1 &= u_2, & y_2 &= u_3, & y_3 &= u_8, & y_4 &= u_4, & y_5 &= u_7, & y_6 &= u_1.
\end{align*}
\]

Then \( TM \) is spanned by \( Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8 \), where

\[
Z_1 = \partial x_1 + \partial y_6, \quad Z_2 = \partial y_1 + \partial x_6, \quad Z_3 = \partial x_3 + \partial y_2,
\]

\[
Z_4 = -y_4 \partial x_4 + x_4 \partial y_4, \quad Z_5 = \partial x_2, \quad Z_6 = \partial x_5, \quad Z_7 = \partial y_5, \quad Z_8 = \partial y_3.
\]

Clearly, \( M \) is a 3-lightlike submanifold with \( \text{Rad}(TM) = \text{Span}\{Z_1, Z_2, Z_3\} \) and \( FZ_4 = Z_2 \), therefore \( D_1 = \text{Span}\{Z_1, Z_3\} \). Since \( FZ_3 = \partial y_3 + \partial x_2 = Z_8 + Z_7 \in \Gamma(S(TM)) \), therefore \( D_2 = \text{Span}\{Z_3\} \). Moreover \( FZ_6 = Z_7 \) therefore \( D_0 = \text{Span}\{Z_6, Z_7\} \). The lightlike transversal bundle \( ltr(TM) \) is spanned by

\[
\{N_1 = \frac{1}{2}(-\partial x_1 + \partial y_6), N_2 = \frac{1}{2}(-\partial y_1 + \partial x_6), N_3 = \frac{1}{2}(-\partial x_3 + \partial y_2)\}.
\]

Clearly, \( \text{Span}\{N_1, N_2\} \) is invariant with respect to \( F \) and \( FN_3 = -\frac{1}{2}Z_8 + \frac{1}{2}Z_5 \). Hence \( L_1 = \text{Span}\{N_3\} \). By direct calculations, we obtain \( S(TM^\perp) = \text{Span}\{W = -y_4 \partial y_1 + x_4 \partial x_4\} \). Since \( FZ_4 = W \), thus \( L_2 = S(TM^\perp) \). Hence \( D' = \text{Span}\{FN_3, FW = Z_4\} \). Thus, \( M \) is a proper GCR-lightlike submanifold of semi-Riemannian product manifold \( R^2_4 \).

Let \( Q, P_1 \) and \( P_2 \) be the projections on \( D, FL_1 = M_1 \) and \( FL_2 = M_2 \), respectively. Then for any \( X \in \Gamma(TM) \), we have \( X = QX + P_1X + P_2X \), applying \( F \) to both sides, we obtain

\[
FX = fX + wP_1X + wP_2X,
\]

and we can write the equation (14) as

\[
FX = fX + wX,
\]

where \( fX \) and \( wX \) are the tangential and transversal components of \( FX \), respectively. Similarly

\[
FV = BV + CV,
\]

for any \( V \in \Gamma(tr(TM)) \), where \( BV \) and \( CV \) are the sections of \( TM \) and \( tr(TM) \), respectively. Since \( F \) is parallel on \( M \), using (6), (7), (14) and (16), we obtain

\[
(\nabla_X f)Y = A_{wP_1Y}X + A_{wP_2Y}X + Bh(X, Y).
\]

\[
D^s(X, wP_1Y) = -\nabla_X^s wP_2Y + wP_1\nabla_XY - h^s(X, fY) + Ch^s(X, Y).
\]

\[
D^l(X, wP_2Y) = -\nabla_X^l wP_1Y + wP_1\nabla_XY - h^l(X, fY) + Ch^l(X, Y).
\]
Theorem 4.2. Let $M$ be a GCR-lightlike submanifold of a semi-Riemannian product manifold $N$. Then the induced connection is a metric connection if and only if the following conditions hold

$$\nabla^*_F Y - A^*_FY X \in \Gamma(FD_2 \oplus D_1), \quad \text{when} \quad Y \in \Gamma(D_1),$$

$$\nabla^*_F Y + h^*(X, F Y) \in \Gamma(FD_2 \oplus D_1), \quad \text{when} \quad Y \in \Gamma(D_2),$$

and $Bh(X, F Y) = 0$, when $Y \in \Gamma(\text{Rad}(TM)).$

Proof. Since $F$ is an almost product structure of $M$ therefore we have $\nabla_X Y = \nabla_X F^2 Y$ for any $Y \in \Gamma(\text{Rad}(TM))$ and $X \in \Gamma(TM)$. Then from (13), we obtain $\nabla_X Y = F\nabla_X F^2 Y$ and then using (6) and (16), we obtain

$$\nabla_X Y + h(X, Y) = F(\nabla_X F^2 Y + h(X, F Y)).$$

Since $\text{Rad}(TM) = D_1 \oplus D_2$ therefore using (10), (15) and (16) in (20) and then equating the tangential part for any $Y \in \Gamma(D_1)$, we obtain

$$\nabla_X Y = f(-A^*_FY X + \nabla^*_X F Y) + Bh(X, F Y),$$

and for any $Y \in \Gamma(D_2)$, we obtain

$$\nabla_X Y = f(\nabla^*_X F Y + h^*(X, F Y)) + Bh(X, F Y).$$

Thus from (21), $\nabla_X Y \in \Gamma(\text{Rad}(TM))$, if and only if

$$f(-A^*_FY X + \nabla^*_X F Y) \in \Gamma(FD_2 \oplus D_1) \quad \text{and} \quad Bh(X, F Y) = 0.$$  \hspace{1cm} (23)

From (22), $\nabla_X Y \in \Gamma(\text{Rad}(TM))$, if and only if

$$\nabla^*_X F Y + h^*(X, F Y) \in \Gamma(FD_2 \oplus D_1) \quad \text{and} \quad Bh(X, F Y) = 0.$$  \hspace{1cm} (24)

Thus the assertion follows from (23) and (24). \hspace{1cm} \Box

Theorem 4.3. Let $M$ be a GCR-lightlike submanifold of a semi-Riemannian product manifold $N$. Then

(i) The distribution $D$ is integrable, if and only if,

$$h(X, F Y) = h(FX, Y), \quad \forall \ X, Y \in \Gamma(D).$$

(ii) The distribution $D'$ is integrable, if and only if,

$$A^*_FZ V = A^*_F V Z, \quad \forall \ Z, V \in \Gamma(D').$$

Proof. From (18) and (19), we obtain $w\nabla_X Y = h(X, F Y) - C(h(X, Y))$ for any $X, Y \in \Gamma(D)$, which implies that $w[X, Y] = h(X, F Y) - h(FX, Y)$, which proves (i).

Next, from (17), we have $f\nabla_Z V = -A^*_WZ - Bh(Z, V)$ for any $Z, V \in \Gamma(D')$, therefore $f[Z, V] = A^*_WZ - A^*_W V Z$, which completes the proof. \hspace{1cm} \Box

Theorem 4.4. Let $M$ be a GCR-lightlike submanifold of a semi-Riemannian product manifold $N$. Then $D$ defines a totally geodesic foliation in $M$ if and only if $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. 

Proof. Using the definition of GCR-lightlike submanifolds, \(D\)-defines a totally geodesic foliation in \(M\) if and only if, \(\nabla_X Y \in \Gamma(D)\) for any \(X, Y \in \Gamma(D)\), that is, if and only if
\[
g(\nabla_X Y, F \xi) = g(\nabla_X Y, FW) = 0,
\]
for any \(X, Y \in \Gamma(D), \xi \in \Gamma(D_2)\) and \(W \in \Gamma(L_2)\). From (6) and (13), we obtain
\[
(25) \quad g(\nabla_X Y, F \xi) = \bar{g}(\nabla_X FY, \xi) = \bar{g}(h^l(X, FY), \xi), \quad \forall X, Y \in \Gamma(D), \xi \in \Gamma(D_2).
\]
Similarly, using (6) and (13), we obtain
\[
(26) \quad g(\nabla_X Y, FW) = \bar{g}(\nabla_X FY, W) = \bar{g}(h^s(X, FY), W), \quad \forall X, Y \in \Gamma(D), \quad W \in \Gamma(L_2).
\]
It follows from (25) and (26) that \(D\) defines a totally geodesic foliation in \(M\), if and only if, \(h^s(X, FY)\) has no components in \(L_2\) and \(h^l(X, FY)\) has no components in \(L_1\) for any \(X, Y \in \Gamma(D)\), that is, using (16), \(Bh(X, Y) = 0\) for any \(X, Y \in \Gamma(D)\).

**Theorem 4.5.** Let \(M\) be a GCR-lightlike submanifold of a semi-Riemannian product manifold \(\bar{M}\). Then \(\bar{D}'\)-defines a totally geodesic foliation in \(M\), if and only if, \(A_{wY} X \in \Gamma(D')\) for any \(X, Y \in \Gamma(D')\).

**Proof.** From (17), we obtain that \(f \nabla_X Y = -A_{wY} X - Bh(X, Y)\) for any \(X, Y \in \Gamma(D').\) If \(\bar{D}'\) defines a totally geodesic foliation in \(M\), then \(A_{wY} X = -Bh(X, Y)\), which implies that \(A_{wY} X \in \Gamma(D')\) for any \(X, Y \in \Gamma(D')\). Conversely, let \(A_{wY} X \in \Gamma(D')\) for any \(X, Y \in \Gamma(D')\), therefore \(f \nabla_X Y = 0\), which implies that \(\nabla_X Y \in \Gamma(D')\). Hence the result follows. \(\square\)

**Definition 4.6.** A GCR-lightlike submanifold of a semi-Riemannian product manifold is called \(D\) geodesic (respectively, \(\bar{D}'\) geodesic) GCR-lightlike submanifold if its second fundamental form \(h\) satisfies \(h(X, Y) = 0\) for any \(X, Y \in \Gamma(D)\) (respectively, \(X, Y \in \Gamma(D')\)).

**Theorem 4.7.** Let \(M\) be a GCR-lightlike submanifold of a semi-Riemannian product manifold \(\bar{M}\). Then the distribution \(D\) defines a totally geodesic foliation in \(M\) if and only if \(M\) is \(D\)-geodesic.

**Proof.** Let \(D\) defines a totally geodesic foliation in \(\bar{M}\) then \(\nabla_X Y \in \Gamma(D)\) for any \(X, Y \in \Gamma(D)\). Then using (6) for any \(\xi \in \Gamma(D_2)\) and \(W \in \Gamma(L_2)\), we obtain
\[
\bar{g}(h^l(X, Y), \xi) = \bar{g}(\nabla_X Y, \xi) = 0, \quad \bar{g}(h^s(X, Y), W) = \bar{g}(\nabla_X Y, W) = 0.
\]
Hence \(h^l(X, Y) = h^s(X, Y) = 0\), which implies that \(M\) is \(D\)-geodesic.

Conversely, let us assume that \(M\) is \(D\)-geodesic. Now using (6) and (13), for any \(X, Y \in \Gamma(D), \xi \in \Gamma(D_2)\) and \(W \in \Gamma(L_2)\), we have
\[
\bar{g}(\nabla_X Y, F \xi) = \bar{g}(\nabla_X FY, \xi) = \bar{g}(h^l(X, FY), \xi) = 0.
\]
and
\[ \bar{g}(\nabla_X Y, F W) = \bar{g}(\nabla_X F Y, W) = \bar{g}(h^s(X, F Y), W) = 0. \]
Hence \( \nabla_X Y \in \Gamma(D) \), which completes the proof. \( \square \)

**Theorem 4.8.** Let \( M \) be a GCR-lightlike submanifold of a semi-Riemannian product manifold \( M \). Then \( M \) is \( D \)-geodesic, if and only if,
\[ g(A_W X, Y) = \bar{g}(D(X, W), Y), \]
and
\[ \nabla_X^* F \xi \notin \Gamma(D_0 \perp FL_1), \quad A^*_2 X \notin \Gamma(FL_1), \quad h^l(X, \xi') \notin \Gamma(L_1), \]
for any \( X, Y \in \Gamma(D), \xi \in \Gamma(D_2), \xi' \in \Gamma(Rad(TM)) \) and \( W \in \Gamma(L_2) \).

Proof. Using the definition of GCR-lightlike submanifolds, \( M \) is \( D \)-geodesic, if and only if,
\[ \bar{g}(h^l(X, Y), \xi) = 0, \]
\[ \bar{g}(h^s(X, Y), W) = 0 \]
for any \( X, Y \in \Gamma(D), \xi \in \Gamma(D_2) \) and \( W \in \Gamma(L_2) \). Thus for any \( X, Y \in \Gamma(D) \), first part of assertion follows from (8).

Now, for \( X, Y \in \Gamma(D) \) and \( \xi \in \Gamma(D_2) \), using (6), (10) and (12), we have
\[ \bar{g}(h^l(X, Y), \xi) = \bar{g}(\nabla_X Y, \xi) \]
\[ = -\bar{g}(F Y, \nabla_X F \xi) \]
\[ = -\bar{g}(F Y, \nabla_X F \xi) - \bar{g}(F Y, h^l(X, F \xi)) \]
\[ = -\bar{g}(F Y, \nabla_X F \xi) - \bar{g}(F Y, h^l(X, F \xi)). \]
(27)
Since \( Y \in \Gamma(D) \), this implies that \( Y \in \Gamma(D_0), Y \in \Gamma(D_1), Y \in \Gamma(D_2), \) or \( Y \in \Gamma(FD_2) \). If \( Y \in \Gamma(D_0) \) or \( Y \in \Gamma(D_2) \), then we have
\[ \bar{g}(F Y, h^l(X, F \xi)) = 0, \]
and if \( Y \in \Gamma(D_1) \) or \( Y \in \Gamma(FD_2) \), then we have
\[ \bar{g}(F Y, h^l(X, F \xi)) = g(A^*_2 X, F \xi) + \bar{g}(h^l(X, \xi'), F \xi) \]
for any \( \xi' = F Y \in \Gamma(Rad(TM)) \). Now using (28) and (29) in (27), we obtain
\[ \bar{g}(h^l(X, Y), \xi) = -\bar{g}(F Y, \nabla_X F \xi) - g(A^*_2 X, F \xi) - \bar{g}(h^l(X, \xi'), F \xi), \]
which proves the second part of the assertion. \( \square \)

**Theorem 4.9.** Let \( M \) be a GCR-lightlike submanifold of a semi-Riemannian product manifold \( M \). Then \( M \) is \( D' \)-geodesic, if and only if, \( A_W X \) and \( A^*_2 X \) have no components in \( M_2 \perp FD_2 \), for any \( X \in \Gamma(D'), \xi \in \Gamma(Rad(TM)) \) and \( W \in \Gamma(S(TM^\perp)) \).
Proof. For any $X, Y \in \Gamma(D')$ and $W \in \Gamma(S(TM^\perp))$ using (8), we obtain
\begin{equation}
\bar{g}(h^*(X, Y), W) = g(A_WX, Y),
\end{equation}
and for any $\xi \in \Gamma(Rad(TM))$ using (9) and (10), we obtain
\begin{equation}
\bar{g}(h^l(X, Y), \xi) = g(A^*_\xi X, Y).
\end{equation}
Hence the assertion follows from (30) and (31). \hfill \Box

Definition 4.10. A GCR-lightlike submanifold of a semi-Riemannian product manifold is called mixed-geodesic GCR-lightlike submanifold if its second fundamental form $h$ satisfies $h(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Theorem 4.11. Let $M$ be a GCR-lightlike submanifold of a semi-Riemannian product manifold $M$. Then $M$ is mixed geodesic, if and only if,

\[
A^*_\xi X \in \Gamma(D_0 \perp FL_1), \quad \text{and} \quad A_WX \in \Gamma(D_0 \perp Rad(TM) \perp FL_1)
\]

for any $X \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Proof. Using (9) and (10), for any $X \in \Gamma(D)$, $Y \in \Gamma(D')$ and $\xi \in \Gamma(Rad(TM))$, we obtain
\begin{equation}
\bar{g}(h^l(X, Y), \xi) = g(A^*_\xi X, Y),
\end{equation}
and for any $W \in \Gamma(S(TM^\perp))$ using (8), we obtain
\begin{equation}
\bar{g}(h^l(X, Y), W) = g(A_WX, Y).
\end{equation}
Hence the result follows from (32) and (33). \hfill \Box

Theorem 4.12. Let $M$ be a mixed geodesic GCR-lightlike submanifold of a semi-Riemannian product manifold $M$. Then $A^*_\xi X \in \Gamma(FD_2)$ for any $X \in \Gamma(D')$ and $\xi \in \Gamma(D_2)$.

Proof. Let $X \in \Gamma(D')$ and $\xi \in \Gamma(D_2)$ then we have

\[
h(X, F\xi) = \nabla_X F\xi - \nabla_X F\xi = F\nabla_X \xi + Fh(X, \xi) - \nabla_X F\xi.
\]

Since $M$ is mixed geodesic, therefore $F\nabla_X \xi = \nabla_X F\xi$. Using (10) and (15), we get

\[
-fA^*_\xi X - wA^*_\xi X + F\nabla_X \xi = \nabla_X F\xi + h^*(X, F\xi).
\]
equating the transversal components, we have $wA^*_\xi X = 0$. Thus

\[
A^*_\xi X \in \Gamma(FD_2 \perp D_0).
\]
Now, for any $Z \in \Gamma(D_0)$ and $\xi \in \Gamma(D_2)$, we have

\[
\bar{g}(A^*_\xi X, Z) = \bar{g}(\nabla_X \xi + \nabla^*_\xi, Z) = \bar{g}(\nabla_X \xi, Z) = -g(\xi, \nabla_X Z + h(X, Z)) = 0.
\]
If $A^*_\xi X \in \Gamma(D_0)$, then using the non-degeneracy of $D_0$ for any $Z \in \Gamma(D_0)$, we must have $\bar{g}(A^*_\xi X, Z) \neq 0$. Therefore $A^*_\xi X \notin \Gamma(D_0)$. Hence the assertion is proved. \hfill \Box
Theorem 4.13. Let \( M \) be a mixed geodesic GCR-lightlike submanifold of a semi-Riemannian product manifold \( \bar{M} \). Then the transversal section \( V \in \Gamma(FD') \) is \( D \)-parallel, if and only if, \( \nabla_X FV \in \Gamma(D) \) for any \( X \in \Gamma(D) \).

Proof. Let \( Y \in \Gamma(D') \) such that \( FY = wY = V \in \Gamma(L_1 \perp L_2) \) and \( X \in \Gamma(D) \), then using hypothesis that \( M \) is a mixed geodesic in (17), we have \( f\nabla_X Y = -A_wX = -A_Y X \). Now, \( \nabla_X V = \nabla_X V + A_Y X = \nabla_X FY - f\nabla_X Y \). Since \( \nabla \) is an almost product structure and \( M \) is mixed geodesic therefore we have \( \nabla_X^1 V = w\nabla_X Y \), that is, \( \nabla_X V = w\nabla_X FV \), which proves the theorem. \(\square\)

Theorem 4.14. Let \( M \) be a GCR-lightlike submanifold of a semi-Riemannian product manifold \( M \) such that \( D'(X, V) \in \Gamma(L_2^+) \). Then \( A_{FV} X = F A_V X \) for any \( X \in \Gamma(D) \) and \( V \in \Gamma(L_1^+) \).

Proof. Let \( X \in \Gamma(D) \), \( Y \in \Gamma(D') \) and \( V \in \Gamma(L_1^+) \) then we have

\[
g(A_{FV} X - F A_V X, Y) = g(A_{FV} X, Y) - g(A_V X, FY) = -g(\nabla_X FV, Y) + g(\nabla_X V, FY)
\]

\[
= -g(\nabla_X V, FY) + g(\nabla_X V, FY)
\]

\[
= 0.
\]

(34)

For any \( X \in \Gamma(D) \), \( Z \in \Gamma(D_0) \) and \( V \in \Gamma(L_1^+) \), we have

\[
g(A_{FV} X - F A_V X, Z) = g(A_{FV} X, Z) - g(A_V X, FZ) = -g(\nabla_X FV, Z) + g(\nabla_X V, FZ)
\]

\[
= -g(\nabla_X V, FZ) + g(\nabla_X V, FZ)
\]

\[
= 0.
\]

(35)

For any \( X \in \Gamma(D) \), \( N \in \Gamma(ltr(TM)) \) and \( V \in \Gamma(L_1^+) \), we have

\[
g(A_{FV} X - F A_V X, N) = g(A_{FV} X, N) - g(A_V X, FN) = -g(\nabla_X FV, N) + g(\nabla_X V, FN)
\]

\[
= -g(F\nabla_X V, N) + g(F\nabla_X V, N)
\]

\[
= 0.
\]

(36)

For any \( X \in \Gamma(D) \), \( FN \in \Gamma(FL_1) \) and \( V \in \Gamma(L_1^+) \), we also have

\[
g(A_{FV} X - F A_V X, FN) = g(A_{FV} X, FN) - g(A_V X, N) = -g(\nabla_X FV, FN) + g(\nabla_X V, N)
\]

\[
= -g(F\nabla_X V, FN) + g(\nabla_X V, N)
\]

\[
= -g(\nabla_X V, N) + g(\nabla_X V, N)
\]

\[
= 0.
\]

(37)

Hence the assertion follows from (34)-(37). \(\square\)
5. GCR-Lightlike product

**Definition 5.1** ([15]). A GCR-lightlike submanifold of a semi-Riemannian product manifold $M$ is called a GCR-lightlike product if both the distributions $D$ and $D'$ define totally geodesic foliations in $M$.

**Theorem 5.2.** Let $M$ be a totally geodesic GCR-lightlike submanifold of a semi-Riemannian product manifold $M$. Suppose that there exists a transversal vector bundle of $M$, which is parallel along $D'$ with respect to the Levi-Civita connection on $M$, that is, $\nabla_X V \in \Gamma(tr(TM))$ for any $V \in \Gamma(tr(TM))$ and $X \in \Gamma(D')$. Then $M$ is a GCR-lightlike product.

Proof. Since $M$ is a totally geodesic GCR-lightlike submanifold, therefore $Bh(X,Y) = 0$ for any $X, Y \in \Gamma(D)$. Therefore, the distribution $D$ defines a totally geodesic foliation in $M$. Next, since $\nabla_X V \in \Gamma(tr(TM))$ for any $V \in \Gamma(tr(TM))$ and $X \in \Gamma(D')$, therefore using (7), we obtain $A_V X = 0$, then from (17), we get $f\nabla_X Y = 0$ for any $X, Y \in \Gamma(D')$, which implies that $\nabla_X Y \in \Gamma(D')$. Hence the distribution $D'$ defines a totally geodesic foliation in $M$. Thus $M$ is a GCR-lightlike product.

**Definition 5.3.** A lightlike submanifold of a semi-Riemannian manifold is said to be an irrotational submanifold if $\nabla_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Thus $M$ is an irrotational lightlike submanifold, if and only if, $h^s(X, \xi) = 0, h^a(X, \xi) = 0$.

**Theorem 5.4.** Let $M$ be an irrotational GCR-lightlike submanifold of a semi-Riemannian product manifold $M$. Then $M$ is a GCR-lightlike product if the following conditions are satisfied:

(A) $\nabla_X U \in \Gamma(S(TM^\bot))$ for any $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$.

(B) $A_V^s Y \in \Gamma(FL_2)$ for any $Y \in \Gamma(D)$.

Proof. Using (7) with (A), we get $A_W X = 0, D^i(X, W) = 0$ and $\nabla_X Y = 0$ for any $X \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\bot))$. Therefore for any $X, Y \in \Gamma(D)$ and $W \in \Gamma(S(TM^\bot))$ and using (8), we obtain $g(h^s(X,Y), W) = 0$, then non-degeneracy of $S(TM^\bot)$ implies that $h^s(X,Y) = 0$. Hence, $Bh^s(X,Y) = 0$. Now, let $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TM))$, then using (B), we have $g(h^i(X,Y), \xi) = -g(\nabla_X Y, \xi) = g(A^s_V X, Y) = 0$. Then using (4), we get $h^i(X,Y) = 0$. Hence $Bh^i(X,Y) = 0$. Thus the distribution $D$ defines a totally geodesic foliation in $M$.

Next, let $X, Y \in \Gamma(D')$, then $FY = wy \in \Gamma(L_1 \perp L_2) \subset tr(TM)$. Using (17), we obtain $f\nabla_X Y = -Bh(X,Y)$, comparing the components along $D$, we get $f\nabla_X Y = 0$, which implies that $\nabla_X Y \in \Gamma(D')$. Thus the distribution $D'$ defines a totally geodesic foliation in $M$. Hence $M$ is a GCR-lightlike product.

**Theorem 5.5.** Let $M$ be a GCR-lightlike submanifold of a semi-Riemannian product manifold $M$. Then $M$ is a GCR-lightlike product if and only if $(\nabla_X f)Y = 0$ for any $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D')$. 


Proof. Let \((\nabla_X f)Y = 0\) for any \(X, Y \in \Gamma(D)\) or \(X, Y \in \Gamma(D')\). Let \(X, Y \in \Gamma(D)\), then \(wY = 0\) and (17) gives that \(Bh(X, Y) = 0\). Hence using the Theorem (4.4), the distribution \(D\) defines a totally geodesic foliation in \(M\). Next, let \(X, Y \in \Gamma(D')\). Since \(BV \in \Gamma(D')\) for any \(V \in \Gamma(tr(TM))\), then (17) implies that \(A_{wY}X \in \Gamma(D')\). Hence using Theorem 4.5, the distribution \(D'\) defines a totally geodesic foliation in \(M\). Since both the distributions \(D\) and \(D'\) define totally geodesic foliations in \(M\), hence \(M\) is a GCR-lightlike product.

Conversely, let \(M\) be a GCR-lightlike product, therefore the distributions \(D\) and \(D'\) define totally geodesic foliations in \(M\). Using (13), for any \(X, Y \in \Gamma(D)\), we have \(\nabla_X FY = F\nabla_X Y\), then comparing the transversal components, we obtain \(h(X, FY) = Fh(X, Y)\) and then \((\nabla_X f)Y = \nabla_X fY = f\nabla_X Y = \nabla_X FY - h(X, FY) = F\nabla_X Y + h(X, FY) = 0\), that is \((\nabla_X f)Y = 0\) for any \(X, Y \in \Gamma(D)\). Let \(D'\) defines a totally geodesic foliation in \(M\) and using (13), we have \(\nabla_X FY = F\nabla_X Y\), then comparing the tangential components on both sides, we obtain \(-A_{wY}X = Bh(X, Y)\), then (17) implies that \((\nabla_X f)Y = 0\), which completes the proof. \(\square\)

**Definition 5.6 ([6]).** A lightlike submanifold \((M, g)\) of a semi-Riemannian manifold \((\bar{M}, g)\) is said to be totally umbilical in \(M\) if there is a smooth transversal vector field \(H \in \Gamma(tr(TM))\) on \(M\), called the transversal curvature vector field of \(M\), such that, for any \(X, Y \in \Gamma(TM)\),

\[
h(X, Y) = Hg(X, Y).
\]

Using (7), it is clear that \(M\) is a totally umbilical, if and only if, on each coordinate neighborhood \(U\) there exist smooth vector fields \(H^1 \in \Gamma(ltr(TM))\) and \(H^\alpha \in \Gamma(S(TM^\perp))\) such that

\[
h^1(X, Y) = H^1g(X, Y), \quad h^\alpha(X, Y) = H^\alpha g(X, Y), \quad D^\alpha(X, W) = 0
\]

for any \(X, Y \in \Gamma(TM)\) and \(W \in \Gamma(S(TM^\perp))\). \(M\) is called totally geodesic if \(H = 0\), that is, if \(h(X, Y) = 0\).

**Lemma 5.7.** Let \(M\) be a totally umbilical GCR-lightlike submanifold of semi-Riemannian product manifold \(M\). Then the distribution \(D'\) defines a totally geodesic foliation in \(M\).

Proof. Let \(X, Y \in \Gamma(D')\) then (17) implies that \(f\nabla_X Y = -A_{wY}X = Bh(X, Y)\), then for any \(Z \in \Gamma(D_0)\), we have

\[
g(f\nabla_X Y, Z) = -g(A_{wY}X, Z) - g(Bh(X, Y), Z) = \bar{g}(\nabla_X wY, Z) = \bar{g}(\nabla_X FY, Z) = \bar{g}(\nabla_X Y, FZ) = \bar{g}(\nabla_X Y, Z') = -g(Y, \nabla_X Z'),
\]

where \(Z' = FZ \in \Gamma(D_0)\). Since \(X \in \Gamma(D')\) and \(Z \in \Gamma(D_0)\), then from (18) and (19), we have \(wP\nabla_X Z = h(X, fZ) - Ch(X, Z) = Hg(X, fZ) - Chg(X, Z) = 0\), therefore \(wP\nabla_X Z = 0\), which implies that \(\nabla_X Z \in \Gamma(D)\). Thus (40) implies
that $g(f \nabla_X Y, Z) = 0$, then the non-degeneracy of $D_0$ implies that $f \nabla_X Y = 0$. Hence $\nabla_X Y \in \Gamma(D')$ for any $X, Y \in \Gamma(D')$. Thus the result follows. □

**Theorem 5.8.** Let $M$ be a totally umbilical GCR-lightlike submanifold of semi-Riemannian product manifold $M$. Then $M$ is a GCR-lightlike product if and only if $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

**Proof.** Let $M$ be a GCR-lightlike product therefore the distributions $D$ and $D'$ define totally geodesic foliations in $M$. Therefore using Theorem 4.4, we have $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. Now using the hypothesis for $X \in \Gamma(D')$ and $Y \in \Gamma(D)$, we have $Bh(X, Y) = g(X, Y)BH = 0$. Thus $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

Conversely, let $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Now for any $X, Y \in \Gamma(D)$, we have $Bh(X, Y) = 0$, which implies that $D$ defines a totally geodesic foliation in $M$. Let $X, Y \in \Gamma(D')$, then (17) implies that $A_{wY}X = -f \nabla_X Y - Bh(X, Y)$ and using Lemma 5.7, we obtain $f A_{wY}X + wA_{wY}X = -h(X, Y)$, comparing the tangential components on both sides, we have $f A_{wY}X = 0$, which implies that $A_{wY}X \in \Gamma(D')$. Hence using Theorem 4.5, the distribution $D'$ defines a totally geodesic foliation in $M$. Hence the result follows. □

**Theorem 5.9.** Let $M$ be a GCR-lightlike submanifold of a semi-Riemannian product manifold $M$. Then $M$ is totally geodesic manifold, if and only if, $Rad(TM)$ and $S(TM^\perp)$ are Killing distributions on $M$.

**Proof.** For any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, consider

$$
\bar{g}(h(X, Y), \xi) = \bar{g}(\nabla_X Y, \xi) = X \bar{g}(Y, \xi) - \bar{g}(\nabla_X \xi, Y) \\
= \bar{g}([\xi, X], Y) - \bar{g}(\nabla_\xi X, Y) \\
= \bar{g}([\xi, X], Y) - \xi \bar{g}(X, Y) + \bar{g}(\nabla_\xi Y, X) \\
= -\xi \bar{g}(X, Y) + \bar{g}([\xi, X], Y) + \bar{g}([\xi, Y], X) - \bar{g}(\nabla_\xi Y, X, \xi) \\
= -(L_\xi \bar{g})(X, Y) - \bar{g}(h(X, Y), \xi),
$$

which implies that

$$
2\bar{g}(h(X, Y), \xi) = -(L_\xi \bar{g})(X, Y)
$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.

Similarly, for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$, we have

$$
\bar{g}(h(X, Y), W) = \bar{g}(\nabla_X Y, W) = X \bar{g}(Y, W) - \bar{g}(\nabla_X W, Y) \\
= \bar{g}([W, X], Y) - \bar{g}(\nabla_W X, Y) \\
= \bar{g}([W, X], Y) - W \bar{g}(X, Y) + \bar{g}(\nabla_W Y, X) \\
= -W \bar{g}(X, Y) + \bar{g}([W, X], Y) + \bar{g}([W, Y], X) - \bar{g}(\nabla_Y X, W) \\
= -(L_W \bar{g})(X, Y) - \bar{g}(h(X, Y), W),
$$

(43)
which implies that
\[(44)\quad 2\bar{g}(h(X, Y), W) = -(L_W\bar{g})(X, Y)\]
for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$. Thus from (42) and (44), we have $h(X, Y) = 0$, if and only if, $(L_\xi\bar{g})(X, Y) = 0$ and $(L_W\bar{g})(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$. Thus the result follows. \hfill \Box

**Theorem 5.10.** Let $M$ be a totally umbilical GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. If the induced connection is a metric connection, then $h^*(X, Y) = 0$ for any $X, Y \in \Gamma(D_0$).

*Proof.* Let the induced connection $\nabla$ be a metric connection, then from Theorem 2.2 on page 159 of [5], we have $h^l = 0$. Hence using hypothesis in (19), we get $WP_1\nabla X Y = 0$, therefore, $\nabla X Y \in \Gamma(S(TM))$, which implies that $h^*(X, Y) = 0$ for any $X, Y \in \Gamma(D_0)$. Thus the result follows. \hfill \Box

6. Minimal GCR-lightlike submanifolds

**Definition 6.1** ([2]). A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be minimal if $h^s = 0$ on $\text{Rad}(TM)$ and $\text{trace } h = 0$, where trace is written with respect to $g$ restricted to $S(TM)$.

**Theorem 6.2.** Let $M$ be a totally umbilical GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then $M$ is minimal, if and only if, $M$ is totally geodesic.

*Proof.* Suppose $M$ is minimal then $h^s(X, Y) = 0$ for any $X, Y \in \Gamma(\text{Rad}(TM))$. Since $M$ is totally umbilical therefore $h^l(X, Y) = H^l g(X, Y) = 0$ for any $X, Y \in \Gamma(\text{Rad}(TM))$. Now, choose an orthonormal basis $\{e_1, e_2, \ldots, e_{m-r}\}$ of $S(TM)$ then from (39), we obtain

$$\text{trace } h(e_i, e_i) = \sum_{i=1}^{m-r} \epsilon_i g(e_i, e_i) H^l + \epsilon_i g(e_i, e_i) H^s = (m - r) H^l + (m - r) H^s.$$ 

Since $M$ is minimal and $\text{ltr}(TM) \cap S(TM^\perp) = \{0\}$, we get $H^l = 0$ and $H^s = 0$. Hence $M$ is totally geodesic. Converse follows directly. \hfill \Box

**Theorem 6.3.** A totally umbilical proper GCR-lightlike submanifold of a semi-Riemannian product manifold $M$ is minimal, if and only if,

$$\text{trace } A_{W_p} = 0 \quad \text{and} \quad \text{trace } A^*_{\xi_k} = 0 \quad \text{on} \quad D_0 \perp FL_2$$

for $W_p \in \Gamma(S(TM^\perp))$, where $k \in \{1, 2, \ldots, r\}$ and $p \in \{1, 2, \ldots, n - r\}$. 
Proof. Using (38), it is clear that $h^s(X,Y) = 0$ on $Rad(TM)$. Using the definition of a GCR-lightlike submanifold, we have

$$
\text{trace } h|_{S(TM)} = \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{j=1}^{b} h(F\xi_j, F\xi_j) + \sum_{j=1}^{b} h(FN_j, N_j) + \sum_{l=1}^{c} h(FW_l, FW_l),
$$

where $a = \dim(D_0)$, $b = \dim(D_2)$ and $c = \dim(L_2)$. Since $M$ is totally umbilical therefore from (38), we have $h(F\xi_j, F\xi_j) = h(FN_j, N_j) = 0$. Thus above equation becomes

$$
\text{trace } h|_{S(TM)} = \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{l=1}^{c} h(FW_l, FW_l)
= \sum_{i=1}^{a} \frac{1}{r} \sum_{k=1}^{r} g(h^l(Z_i, Z_i), \xi_k)N_k
+ \sum_{i=1}^{a} \frac{1}{n-r} \sum_{p=1}^{n-r} g(h^s(Z_i, Z_i), W_p)W_p
+ \sum_{l=1}^{c} \frac{1}{r} \sum_{k=1}^{r} g(h^l(FW_l, FW_l), \xi_k)N_k
+ \sum_{l=1}^{c} \frac{1}{n-r} \sum_{p=1}^{n-r} g(h^s(FW_l, FW_l), W_p)W_p,
$$

(45)
where $\{W_1, W_2, \ldots, W_{n-r}\}$ is an orthonormal basis of $S(TM^\perp)$. Using (8) and (11) in (45), we obtain

$$
\text{trace } h|_{S(TM)} = \sum_{i=1}^{a} \frac{1}{r} \sum_{k=1}^{r} g(A^*_k Z_i, Z_i)N_k
+ \sum_{i=1}^{a} \frac{1}{n-r} \sum_{p=1}^{n-r} g(AW_p Z_i, Z_i)W_p
+ \sum_{l=1}^{c} \frac{1}{r} \sum_{k=1}^{r} g(A^*_k FW_l, FW_l)N_k
+ \sum_{l=1}^{c} \frac{1}{n-r} \sum_{p=1}^{n-r} g(AW_p FW_l, FW_l)W_p.
$$

Thus $\text{trace } h|_{S(TM)} = 0$, if and only if, $\text{trace } AW_p = 0$ and $\text{trace } A\xi_k = 0$ on $D_0 \perp FL_2$. Hence the result follows. □
Theorem 6.4. Let $M$ be an irrotational lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then $M$ is minimal, if and only if,

$\text{trace } A^*_{\xi_k}|_{S(TM)} = 0$ and $\text{trace } A_{W_j}|_{S(TM)} = 0$, where $W_j \in \Gamma(S(TM^\perp))$, $k \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, n-r\}$.

Proof. $M$ is irrotational implies that $h^s(X, \xi) = 0$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad}(TM))$, therefore $h^s = 0$ on $\text{Rad}(TM)$. Also

$$\text{trace } h|_{S(TM)} = \sum_{i=1}^{m-r} \epsilon_i (h^l(e_i, e_j) + h^s(e_i, e_j))$$

$$= \sum_{i=1}^{m-r} \epsilon_i \left(\frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(e_i, e_i), \xi_k)N_k + \frac{1}{n-r} \sum_{j=1}^{n-r} \bar{g}(h^s(e_i, e_i), W_j)W_j\right)$$

$$= \sum_{i=1}^{m-r} \epsilon_i \left(\frac{1}{r} \sum_{k=1}^{r} \bar{g}(A^*_{\xi_k}e_i, e_i)N_k + \frac{1}{n-r} \sum_{j=1}^{n-r} \bar{g}(A_{W_j}e_i, e_i)W_j\right).$$

Hence theorem follows. $\square$

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