MERIDIAN SURFACES IN $\mathbb{E}^4$ WITH POINTWISE 1-TYPE GAUSS MAP

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Abstract. In the present article we study a special class of surfaces in the four-dimensional Euclidean space, which are one-parameter systems of meridians of the standard rotational hypersurface. They are called meridian surfaces. We show that a meridian surface has a harmonic Gauss map if and only if it is part of a plane. Further, we give necessary and sufficient conditions for a meridian surface to have pointwise 1-type Gauss map and find all meridian surfaces with pointwise 1-type Gauss map.

1. Introduction

The study of submanifolds of Euclidean space or pseudo-Euclidean space via the notion of finite type immersions began in the late 1970’s with the papers [6, 7] of B.-Y. Chen and has been extensively carried out since then. An isometric immersion $x : M \to \mathbb{E}^m$ of a submanifold $M$ in Euclidean $m$-space $\mathbb{E}^m$ is said to be of finite type [6] if $x$ identified with the position vector field of $M$ in $\mathbb{E}^m$ can be expressed as a finite sum of eigenvectors of the Laplacian $\Delta$ of $M$, i.e.,

$$x = x_0 + \sum_{i=1}^k x_i,$$

where $x_0$ is a constant map, $x_1, x_2, \ldots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different, then $M$ is said to be of $k$-type. Many results on finite type immersions have been collected in the survey paper [8]. Similarly, a smooth map $\phi$ of an $n$-dimensional Riemannian manifold $M$ of $\mathbb{E}^m$ is said to be of finite type if $\phi$ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of $\Delta$. The notion of finite type immersion is naturally extended to the Gauss map $G$ on $M$ in Euclidean space [10]. Thus, a submanifold $M$ of Euclidean space has 1-type Gauss map $G$, if $G$ satisfies $\Delta G = \mu (G + C)$ for some $\mu \in \mathbb{R}$ and some constant vector $C$ (of [2], [3], [4], [13]). However, the Laplacian

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of the Gauss map of some typical well-known surfaces such as the helicoid, the catenoid and the right cone in the Euclidean 3-space $E^3$ takes a somewhat different form, namely, $\Delta G = \lambda(G + C)$ for some non-constant function $\lambda$ and some constant vector $C$. Therefore, it is worth studying the class of surfaces satisfying such an equation. A submanifold $M$ of the Euclidean space $E^m$ is said to have pointwise 1-type Gauss map if its Gauss map $G$ satisfies

$$\Delta G = \lambda(G + C)$$

for some non-zero smooth function $\lambda$ on $M$ and some constant vector $C$ [11]. A pointwise 1-type Gauss map is called proper if the function $\lambda$ defined by (1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1) is zero. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind ([9], [11], [14], [15]). In [11] M. Choi and Y. Kim characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. Also, together with B. Y. Chen, they proved that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature [9]. Moreover, they characterized the rational surfaces of revolution with pointwise 1-type Gauss map. In [17] D. Yoon studied Vranceanu rotation surfaces in Euclidean 4-space $E^4$. He obtained classification theorems for the flat Vranceanu rotation surfaces with 1-type Gauss map and an equation in terms of the mean curvature vector [16]. For the general case see [1].

The study of meridian surfaces in the Euclidean 4-space $E^4$ was first introduced by G. Ganchev and the third author in [12]. The meridian surfaces are one-parameter systems of meridians of the standard rotational hypersurface in $E^4$. In this paper we investigate the meridian surfaces with pointwise 1-type Gauss map. We give necessary and sufficient conditions for a meridian surface to have pointwise 1-type Gauss map and find all meridian surfaces with pointwise 1-type Gauss map of first and second kind.

2. Preliminaries

In the present section we recall definitions and results of [5]. Let $x : M \to E^m$ be an immersion from an $n$-dimensional connected Riemannian manifold $M$ into an $m$-dimensional Euclidean space $E^m$. We denote by $\langle \cdot, \cdot \rangle$ the metric tensor of $E^m$ as well as the induced metric on $M$. Let $\nabla'$ be the Levi-Civita connection of $E^m$ and $\nabla$ the induced connection on $M$. Then the Gauss and Weingarten formulas are given, respectively, by

$$\nabla'_X Y = \nabla_X Y + h(X, Y),$$
$$\nabla'_X \xi = - A_\xi X + D_X \xi,$$

where $X, Y$ are vector fields tangent to $M$ and $\xi$ is a vector field normal to $M$. Moreover, $h$ is the second fundamental form, $D$ is the linear connection induced in the normal bundle $T^\perp M$, called normal connection, and $A_\xi$ is the
shape operator in the direction of $\xi$ that is related with $h$ by

$$\langle h(X, Y), \xi \rangle = \langle A_{\xi}X, Y \rangle.$$  

The covariant differentiation $\nabla h$ of the second fundamental form $h$ on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\perp M$ of $M$ is defined by

$$(\nabla_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields $X, Y$ and $Z$ tangent to $M$. The Codazzi equation is given by

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z).$$

We denote by $R$ the curvature tensor associated with $\nabla$, i.e.,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$  

The equations of Gauss and Ricci are given, respectively, by

$$\langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle,$$

$$\langle R^\perp(X, Y)\xi, \eta \rangle = \langle [A_{\xi}, A_{\eta}]X, Y \rangle,$$

for vector fields $X, Y, Z, W$ tangent to $M$ and $\xi, \eta$ normal to $M$.

The mean curvature vector field $H$ of an $n$-dimensional submanifold $M$ in $E^m$ is given by

$$H = \frac{1}{n} \text{trace } h.$$  

A submanifold $M$ is said to be minimal (respectively, totally geodesic) if $H \equiv 0$ (respectively, $h \equiv 0$).

We shall recall the definition of Gauss map $G$ of a submanifold $M$. Let $G(n, m)$ denotes the Grassmannian manifold consisting of all oriented $n$-planes through the origin of $E^m$ and $\wedge^n E^m$ be the vector space obtained by the exterior product of $n$ vectors in $E^m$. In a natural way, we can identify $\wedge^n E^m$ with some Euclidean space $E^N$ where $N = \binom{m}{n}$. Let $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ be an adapted local orthonormal frame field in $E^m$ such that $e_1, e_2, \ldots, e_n$ are tangent to $M$ and $e_{n+1}, e_{n+2}, \ldots, e_m$ are normal to $M$. The map $G : M \to G(n, m)$ defined by $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$ is called the Gauss map of $M$. It is a smooth map which carries a point $p$ in $M$ into the oriented $n$-plane in $E^m$ obtained by the parallel translation of the tangent space of $M$ at $p$ in $E^m$.

For any real function $\phi$ on $M$ the Laplacian of $\phi$ is defined by

$$\Delta \phi = -\sum_i (\nabla'_{e_i} \nabla'_{e_i} \phi - \nabla_{e_i} \nabla_{e_i} \phi).$$
3. Classification of meridian surfaces with pointwise 1-type Gauss map

Let \( \{e_1, e_2, e_3, e_4\} \) be the standard orthonormal frame in \( \mathbb{E}^4 \), and \( S^2(1) \) be the 2-dimensional sphere in \( \mathbb{E}^3 = \text{span}\{e_1, e_2, e_3\} \), centered at the origin \( O \). We consider a smooth curve \( c : r = r(v), v \in J, J \subset \mathbb{R} \) on \( S^2(1) \), parameterized by the arc-length \( (r'^2)(v) = 1 \). Let \( t(v) = r'(v) \) be the tangent vector field of \( c \). We consider the moving frame field \( \{t(v), n(v), r(v)\} \) of the curve \( c \) on \( S^2(1) \). With respect to this orthonormal frame field the following Frenet formulas hold:

\[
\begin{align*}
r' &= t; \\
t' &= \kappa n - r; \\
n' &= -\kappa t,
\end{align*}
\]

where \( \kappa(v) = \langle t'(v), n(v) \rangle \) is the spherical curvature of \( c \).

Let \( f = f(u), \ g = g(u) \) be non-zero smooth functions, defined in an interval \( I \subset \mathbb{R} \), such that \( (f'(u))^2 + (g'(u))^2 = 1, \ u \in I \). We consider the surface \( M^2 \) in \( \mathbb{E}^4 \) constructed in the following way:

\[ M^2 : z(u, v) = f(u) r(v) + g(u) e_4, \ u \in I, v \in J \]

(see [12]).

The surface \( M^2 \) lies on the rotational hypersurface \( M^3 \) in \( \mathbb{E}^4 \) obtained by the rotation of the meridian curve \( \alpha : u \to (f(u), g(u)) \) about the \( Oe_4 \)-axis in \( \mathbb{E}^4 \). \( M^2 \) is called a meridian surface on \( M^3 \) since it is a one-parameter system of meridians of \( M^3 \).

The tangent space of \( M^2 \) is spanned by the vector fields:

\[
\begin{align*}
z_u &= f' r + g' e_4; \\
z_v &= f t,
\end{align*}
\]

and hence, the coefficients of the first fundamental form of \( M^2 \) are \( E = 1; \ F = 0; \ G = f'^2(u) \). Taking into account (3) and (5), we calculate the second partial derivatives of \( z(u, v) \):

\[
\begin{align*}
z_{uu} &= f'' r + g'' e_4; \\
z_{uv} &= f' t; \\
z_{vv} &= f \kappa n - f r.
\end{align*}
\]

Let us denote \( x = z_u, \ y = \frac{z_v}{f} = t \) and consider the following orthonormal normal frame field of \( M^2 \):

\[
\begin{align*}
n_1 &= n(v); \\
n_2 &= -g'(u) r(v) + f'(u) e_4.
\end{align*}
\]

Thus we obtain a positive orthonormal frame field \( \{x, y, n_1, n_2\} \) of \( M^2 \). We denote by \( \kappa_\alpha \) the curvature of the meridian curve \( \alpha \), i.e.,

\[ \kappa_\alpha(u) = f'(u) g''(u) - g'(u) f'''(u). \]
By covariant differentiation with respect to $x$ and $y$, and a straightforward calculation we obtain

\[ \nabla'_x x = \kappa \alpha n_2; \]
\[ \nabla'_x y = 0; \]
\[ \nabla'_y x = \frac{f'}{f} y; \]
\[ \nabla'_y y = -\frac{f'}{f} x + \frac{\kappa}{f} n_1 + \frac{g'}{f} n_2; \]

and

\[ \nabla'_y n_1 = 0; \]
\[ \nabla'_y n_1 = -\frac{\kappa}{f} y; \]
\[ \nabla'_x n_2 = -\kappa \alpha x; \]
\[ \nabla'_y n_2 = -\frac{g'}{f} y, \]

where $\kappa(v)$ and $\kappa_\alpha(u)$ are the curvatures of the spherical $c$ and the meridian curve $\alpha$, respectively (see [12]).

Equalities (7) imply the following result.

**Lemma 3.1.** Let $M^2$ be a meridian surface given with the surface patch (4). Then

\[ A_{n_1} = \begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & \frac{f}{\kappa} \end{bmatrix}, \quad A_{n_2} = \begin{bmatrix} \kappa \alpha & 0 \\ 0 & \frac{g'}{f} \end{bmatrix}. \]

So, the Gauss curvature is given by

\[ K = \frac{\kappa_\alpha g'}{f} \]

and the mean curvature vector field $H$ of $M^2$ is

\[ H = \frac{\kappa}{2f} n_1 + \frac{\kappa_\alpha f + g'}{2f} n_2. \]

The Gauss map $G$ of $M^2$ is defined by $G = x \wedge y$. Using (2), (6), and (7) we calculate that the Laplacian of the Gauss map is expressed as

\[ \Delta G = \frac{(f\kappa_\alpha)^2 + \kappa^2 + g'^2}{f^2} x \wedge y - \frac{\kappa'}{f^2} x \wedge n_1 \]
\[ - \frac{\kappa f'}{f^2} y \wedge n_1 - \frac{f'g'}{f^2} y \wedge n_2, \]

where $\kappa' = \frac{d}{dv}(\kappa)$.
First, we suppose that the Gauss map of $M^2$ is harmonic, i.e., $\Delta G = 0$. Then from (8) we get

\begin{align}
\kappa_\alpha &= 0; \\
\kappa &= 0; \\
g' &= 0.
\end{align}

(9)

So, (6) and (9) imply that $M^2$ is a totally geodesic surface in $\mathbb{E}^4$. Conversely, if $M^2$ is totally geodesic, then $\Delta G = 0$.

Thus we obtain the following result.

**Theorem 3.2.** Let $M^2$ be a meridian surfaces in the Euclidean space $\mathbb{E}^4$. The Gauss map of $M^2$ is harmonic if and only if $M^2$ is part of a plane.

Now, we suppose that the meridian surface $M^2$ is of pointwise 1-type Gauss map, i.e., $G$ satisfies (1), where $\lambda \neq 0$. Then, from equalities (1) and (8) we get

\begin{align}
\lambda + \lambda \langle C, x \wedge y \rangle &= \frac{(f \kappa_\alpha)^2 + \kappa^2 + g'^2}{f^2}; \\
\lambda \langle C, x \wedge n_1 \rangle &= -\frac{\kappa'}{f^2}; \\
\lambda \langle C, y \wedge n_1 \rangle &= -\frac{\kappa f'}{f^2}; \\
\lambda \langle C, y \wedge n_2 \rangle &= -\frac{f'g' - f(f \kappa_\alpha)'}{f^2}.
\end{align}

(10)

Using (8) we obtain

\begin{align}
\lambda \langle C, x \wedge n_2 \rangle &= 0; \\
\lambda \langle C, n_1 \wedge n_2 \rangle &= 0.
\end{align}

(11)

Differentiating (11) with respect to $u$ and $v$ we get

\begin{align}
\kappa_\alpha \langle C, x \wedge n_1 \rangle &= 0; \\
\frac{f'}{f} \langle C, y \wedge n_2 \rangle - \frac{g'}{f} \langle C, x \wedge y \rangle &= 0; \\
-\frac{\kappa}{f} \langle C, y \wedge n_2 \rangle + \frac{g'}{f} \langle C, y \wedge n_1 \rangle &= 0.
\end{align}

(12)

Since $\lambda \neq 0$ equalities (10) and (12) imply

\begin{align}
\kappa_\alpha \kappa' &= 0; \\
\kappa (f \kappa_\alpha)' &= 0; \\
\lambda f'^2 g' &= g' \left(1 + (f \kappa_\alpha)^2 + \kappa^2\right) - ff'(f \kappa_\alpha)'.
\end{align}

(13)

We distinguish the following cases.
**Case I:** $g' = 0$. In such case $\kappa_\alpha = 0$. Then equality (8) implies that

$$\Delta G = \frac{\kappa^2}{f^2} x \wedge y - \frac{\kappa'}{f^2} x \wedge n_1 - \frac{\kappa f'}{f^2} y \wedge n_1.$$  

If we assume that $M^2$ has pointwise 1-type Gauss map of the first kind, i.e., $C = 0$, then from (14) we get $\kappa' = 0$ and $\kappa f' = 0$, which imply $\kappa = 0$ since $f' \neq 0$. Hence $\Delta G = 0$, which contradicts the assumption that $\lambda \neq 0$. Consequently, in the case $g' = 0$ there are no meridian surfaces of pointwise 1-type Gauss map of the first kind.

Now we consider meridian surfaces of pointwise 1-type Gauss map of the second kind, i.e., $C \neq 0$. So we suppose that $\kappa \neq 0$. From equalities (1) and (14) we obtain

$$C = \left( \frac{\kappa^2}{\lambda f^2} - 1 \right) x \wedge y - \frac{\kappa'}{\lambda f^2} x \wedge n_1 - \frac{\kappa f'}{\lambda f^2} y \wedge n_1.$$  

Using (6), (7) and (15) we obtain

$$\nabla'_x C = \frac{\kappa}{\lambda f^3} \left( 3\kappa' \lambda - \kappa \lambda \right) x \wedge y$$

$$+ \frac{1}{\lambda f^3} \left( -\kappa'' \lambda + k' \lambda' + \kappa^3 \lambda + \kappa \lambda - \kappa \lambda^2 f^2 \right) x \wedge n_1$$

$$+ \frac{f'}{\lambda f^3} \left( -2 \kappa' \lambda + \kappa \lambda' \right) y \wedge n_1.$$  

The last formulas imply that $C = \text{const}$ if and only if $\kappa = \text{const}$ and $\lambda = \frac{\kappa^2 + 1}{f^2}$.

The condition $\kappa = \text{const} \neq 0$ implies that the curve $c$ on $S^2(1)$ is a circle with non-zero constant spherical curvature. Since $g' = 0$ and $(f'^2 + g'^2) = 1$ we get $f(u) = \pm u + a$, $g(u) = b$, where $a = \text{const}$, $b = \text{const}$. In this case $M^2$ is a developable ruled surface. Moreover, from (7) it follows that $\nabla'_x n_2 = 0$; $\nabla'_x n_2 = 0$, which implies that $M^2$ lies in the 3-dimensional space spanned by $\{x, y, n_1\}$.

Conversely, if $g' = 0$ and $\kappa = \text{const}$, by direct computation we get

$$\Delta G = \frac{\kappa^2 + 1}{f^2} (G + C),$$  

where $C = -\frac{1}{\kappa^2 + 1} x \wedge y - \frac{\kappa f'}{\kappa^2 + 1} y \wedge n_1$. Hence, $M^2$ is a surface with pointwise 1-type Gauss map of the second kind.

Summing up we obtain the following result.

**Theorem 3.3.** Let $M^2$ be a meridian surface given with parametrization (4) and $g' = 0$. Then $M^2$ has pointwise 1-type Gauss map of the second kind if and only if the curve $c$ is a circle with non-zero constant spherical curvature.
and the meridian curve $\alpha$ is determined by $f(u) = \pm u + a$; $g(u) = b$, where $a = \text{const}$, $b = \text{const}$. In this case $M^2$ is a developable ruled surface lying in 3-dimensional space.

**Case II:** $g' \neq 0$. In such case from the third equality of (13) we obtain

$$\lambda = \frac{g' \left(1 + (f \kappa)'^2 + \kappa^2\right) - f f'(f \kappa)'}{f^2 g'}.$$  

First we shall consider the case of pointwise 1-type Gauss map surfaces of the first kind. From (8) it follows that $M^2$ is of the first kind ($C = 0$) if and only if

$$\kappa = \text{const};$$
$$f'g' - f(f \kappa)' = 0.$$ 

The first equality of (17) implies that $\kappa = \text{const}$. There are two subcases:

1. $\kappa = 0$. Then the meridian curve $\alpha$ is determined by the equation

$$f'g' - f(f \kappa)' = 0.$$ 

The equalities $\kappa = f'' - f f'' \sqrt{1 - f'^2}$, $f'^2 + g'^2 = 1$ imply that $\kappa = -\frac{f''}{f'}$. Hence equation (18) can be rewritten in the form

$$f'\sqrt{1 - f'^2} + f \left(\frac{f f''}{\sqrt{1 - f'^2}}\right)' = 0.$$ 

Since $\kappa = 0$, $M^2$ lies in the 3-dimensional space spanned by $\{x, y, n_2\}$.

Conversely, if $\kappa = 0$ and the meridian curve $\alpha$ is determined by a solution $f(u)$ of differential equation (19), the function $g(u)$ is defined by $g' = \sqrt{1 - f'^2}$, then the surface $M^2$, parameterized by (4), is a surface of pointwise 1-type Gauss map of the first kind.

2. $\kappa \neq 0$. Then the second equality of (17) implies that $f' = 0$. In this case $f(u) = a$; $g(u) = \pm u + b$, where $a = \text{const}$, $b = \text{const}$. By a result of [12], $M^2$ is a developable ruled surface in a 3-dimensional space, since $\kappa = 0$ and $\kappa = \text{const}$. It follows from (16) that $\lambda = \frac{1 + \kappa^2}{a^2} = \text{const}$, which implies that $M^2$ has 1-type Gauss map, i.e., $M^2$ is non-proper. The converse is also true.

Thus we obtain the following result.

**Theorem 3.4.** Let $M^2$ be a meridian surface given with parametrization (4) and $g' \neq 0$. Then $M^2$ has pointwise 1-type Gauss map of the first kind if and only if one of the following holds:

(i) the curve $c$ is a great circle on $S^2(1)$ and the meridian curve $\alpha$ is determined by the solutions of the following differential equation

$$f'\sqrt{1 - f'^2} + f \left(\frac{f f''}{\sqrt{1 - f'^2}}\right)' = 0;$$
(ii) the curve $c$ is a circle on $S^2(1)$ with non-zero constant spherical curvature and the meridian curve $\alpha$ is determined by $f(u) = a; g(u) = \pm u + b$, where $a = \text{const}, b = \text{const}$. In this case $M^2$ is a developable ruled surface in a 3-dimensional space. Moreover, $M^2$ is non-proper.

Now we shall consider the case of pointwise 1-type Gauss map surfaces of the second kind. It follows from equalities (13) that there are three subcases.

1. $\kappa_n = 0$. In this subcase

\begin{equation}
\Delta G = \frac{\kappa^2 + g'^2}{f^2} x \wedge y - \frac{\kappa'}{f^2} x \wedge n_1 - \frac{\kappa f'}{f^2} y \wedge n_1 - \frac{f'g'}{f^2} y \wedge n_2.
\end{equation}

From equalities (1) and (20) we obtain

\[ C = \left( \frac{\kappa^2 + g'^2}{\lambda f^2} - 1 \right) x \wedge y - \frac{\kappa'}{\lambda f^2} x \wedge n_1 - \frac{\kappa f'}{\lambda f^2} y \wedge n_1 - \frac{f'g'}{\lambda f^2} y \wedge n_2. \]

The third equality in (13) implies that in this case $\lambda = \frac{1 + \kappa^2}{f^2}$ and hence, $C$ is expressed as follows:

\begin{equation}
C = -\frac{1}{1 + \kappa^2} \left( f'^2 x \wedge y + \kappa' x \wedge n_1 + \kappa f' y \wedge n_1 + f'g' y \wedge n_2 \right).
\end{equation}

Using (6), (7) and (21) we obtain

\[
\nabla_x C = -\frac{1}{1 + \kappa^2} \left( 2f'' f' x \wedge y + \kappa f'' y \wedge n_1 + (f'g'' + f''g') y \wedge n_2 \right);
\]

\[
\nabla_y C = \frac{1}{f(1 + \kappa^2)^2} \left( (2\kappa \kappa' f'^2 + \kappa \kappa'(1 + \kappa^2)) x \wedge y \\
+ (2\kappa \kappa' - (1 + \kappa^2) \kappa'') x \wedge n_1 \right) \\
+ \frac{1}{f(1 + \kappa^2)^2} (-2\kappa' f' y \wedge n_1 + 2\kappa \kappa' f' g' y \wedge n_2).
\]

The last formulas imply that $C = \text{const}$ if and only if $\kappa = \text{const}, f' = a = \text{const}, g' = b = \text{const}, a^2 + b^2 = 1$.

The condition $\kappa = \text{const}$ implies that the curve $c$ is a circle on $S^2(1)$. The meridian curve $\alpha$ is given by $f(u) = au + a_1; g(u) = bu + b_1$, where $a_1 = \text{const}, b_1 = \text{const}$. In this case $M^2$ is a developable ruled surface lying in a 3-dimensional space.

Conversely, if $f(u) = au + a_1; g(u) = bu + b_1$ and $\kappa = \text{const}$, then

\[ \Delta G = \frac{\kappa^2 + b'^2}{f^2} x \wedge y - \frac{\kappa a}{f^2} y \wedge n_1 - \frac{ab}{f^2} y \wedge n_2. \]

Hence, by direct computation we get

\[ \Delta G = \frac{1 + \kappa^2}{f^2} (G + C), \]

where $C = -\frac{a}{1 + \kappa^2} (a x \wedge y + \kappa y \wedge n_1 + b y \wedge n_2)$. Consequently, $M^2$ is a surface of pointwise 1-type Gauss map of the second kind.
2. $\kappa = 0$. In this subcase
\begin{equation}
\Delta G = \frac{(f\kappa_\alpha)^2 + g^2}{f^2} x \wedge y - \frac{f'g' - f(f\kappa_\alpha)'}{f^2} y \wedge n_2.
\end{equation}

From equalities (1) and (22) we obtain
\begin{equation}
C = \left(\frac{(f\kappa_\alpha)^2 + g^2}{f^2} - 1\right) x \wedge y - \frac{f'g' - f(f\kappa_\alpha)'}{\lambda f^2} y \wedge n_2.
\end{equation}

Using the third equality of (13) we obtain that $C$ is expressed as follows:
\begin{equation}
C = -\frac{f'g' - f(f\kappa_\alpha)'}{\lambda f^2} \left(\frac{f'}{g'} x \wedge y + y \wedge n_2\right),
\end{equation}
where $\lambda = \frac{1}{f'} \left(1 + (f\kappa_\alpha)^2 - \frac{ff''}{g'} (f\kappa_\alpha)'ight)$. We denote
\begin{equation}
\varphi = -\frac{f'g' - f(f\kappa_\alpha)'}{\lambda f^2}.
\end{equation}

Then equalities (6), (7) and (23) imply
\begin{equation}
\nabla_\perp C = \left(\left(\frac{f'}{g'}\right)' + \varphi \kappa_\alpha\right) x \wedge y + \left(\varphi' - \varphi \frac{f'}{g'} \kappa_\alpha\right) y \wedge n_2;
\end{equation}
\begin{equation}
\nabla_\parallel C = 0.
\end{equation}

It follows from (25) that $C = \text{const}$ if and only if $\varphi' = \varphi \frac{f'}{g'} \kappa_\alpha$, or equivalently
\begin{equation}
(\ln \varphi)' = \frac{f'}{g'} \kappa_\alpha.
\end{equation}

Using that $f\kappa_\alpha = -\frac{ff''}{\sqrt{1-f'^2}}$ from (24) we get
\begin{equation}
\varphi = \frac{-\sqrt{1-f'^2} (f(1-f'^2)(ff'')^2f'f'' + f'(1-f'^2)^2)}{ff'(ff'')(1-f'^2) + f^2f'' + (1-f'^2)^2}.
\end{equation}

Now, formulas (26) and (27) imply that $C = \text{const}$ if and only if the function $f(u)$ is a solution of the following differential equation
\begin{equation}
\left(\ln \frac{-\sqrt{1-f'^2} (f(1-f'^2)(ff'')^2f'f'' + f'(1-f'^2)^2)}{ff'(ff'')(1-f'^2) + f^2f'' + (1-f'^2)^2}\right)' = -\frac{f'f''}{1-f'^2}.
\end{equation}

Conversely, if $\kappa = 0$ and the meridian curve $\alpha$ is determined by a solution $f(u)$ of differential equation (28), $g(u)$ is defined by $g' = \sqrt{1-f'^2}$, then the surface $M^2$, parameterized by (4), is a surface of pointwise 1-type Gauss map of the second kind.

3. $\kappa = \text{const} \neq 0$ and $f\kappa_\alpha = a = \text{const}$, $a \neq 0$. In this subcase
\begin{equation}
\Delta G = \frac{a^2 + \kappa^2 + g^2}{f^2} x \wedge y - \frac{\kappa f'}{f^2} y \wedge n_1 - \frac{f'g'}{f^2} y \wedge n_2.
\end{equation}
From equalities (1), (16) and (29) we obtain
\[ C = -\frac{1}{1 + a^2 + \kappa^2} \left( f'^2 x \wedge y + \kappa f' y \wedge n_1 + f' g' y \wedge n_2 \right). \]

Then equalities (6), (7) and (30) imply
\[ \nabla_x C = -\frac{1}{1 + a^2 + \kappa^2} \left( f'' f' x \wedge y + \kappa f'' y \wedge n_1 + g' f'' y \wedge n_2 \right); \]
\[ \nabla_y C = 0. \]

Formulas (31) imply that \( C = \text{const} \) if and only if \( f'' = 0 \). But, if \( f'' = 0 \), then \( \kappa = 0 \), which contradicts the assumption that \( f \kappa \neq 0 \).

Consequently, if \( \kappa = \text{const} \neq 0 \) and \( f \kappa = a = \text{const} \), \( a \neq 0 \), then there are no meridian surfaces of pointwise 1-type Gauss map of the second kind.

Summing up we obtain the following result.

**Theorem 3.5.** Let \( M^2 \) be a meridian surface given with parametrization (4) and \( g' \neq 0 \). Then \( M^2 \) has pointwise 1-type Gauss map of the second kind if and only if one of the following holds:

(i) the curve \( c \) is a circle on \( S^2(1) \) and the meridian curve \( \alpha \) is determined by \( f(u) = au + a_1; g(u) = bu + b_1 \), where \( a, a_1, b, b_1 \) are constants. In this case \( M^2 \) is a developable ruled surface lying in a 3-dimensional space;

(ii) the curve \( c \) is a great circle on \( S^2(1) \) and the meridian curve \( \alpha \) is determined by the solutions of the following differential equation
\[ \left( \ln \frac{f'^2 f''}{f' f''(f''/f^2 + f' f''/(1 - f'^2)^2 + f''(1 - f'^2)^2)} \right)' = -\frac{f' f''}{1 - f'^2}. \]

Theorem 3.3, Theorem 3.4, and Theorem 3.5 describe all meridian surfaces with pointwise 1-type Gauss map.

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**References**


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