GENERATING SETS OF STRICTLY ORDER-PRESERVING TRANSFORMATION SEMIGROUPS ON A FINITE SET

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Abstract. Let \( O_n \) and \( PO_n \) denote the order-preserving transformation and the partial order-preserving transformation semigroups on the set \( X_n = \{1, \ldots, n\} \), respectively. Then the strictly partial order-preserving transformation semigroup \( SPO_n \) on the set \( X_n \), under its natural order, is defined by \( SPO_n = PO_n \setminus O_n \). In this paper we find necessary and sufficient conditions for any subset of \( SPO(n, r) \) to be a (minimal) generating set of \( SPO(n, r) \) for \( 2 \leq r \leq n - 1 \).

1. Introduction

The partial transformation semigroup \( P_X \) and the full transformation semigroup \( T_X \) on a set \( X \), the semigroups analogue of the symmetric group \( S_X \), have been much studied over the last fifty years, for both finite and infinite \( X \). Here we are concerned solely with the case where \( X = X_n = \{1, \ldots, n\} \), and we write respectively \( P_n, T_n \) and \( S_n \) rather than \( P_{X_n}, T_{X_n} \) and \( S_{X_n} \). Among recent contributions are [1, 2, 6, 10, 11]. The domain, image, height and kernel of \( \alpha \in P_n \) are defined by

\[
\text{dom} (\alpha) = \{ x \in X_n : \text{there exists } y \in X_n \text{ such that } x\alpha = y \},
\]

\[
\text{im} (\alpha) = \{ y \in X_n : \text{there exists } x \in X_n \text{ such that } x\alpha = y \},
\]

\[
h (\alpha) = |\text{im} (\alpha)|,
\]

\[
\ker (\alpha) = \{ (x, y) \in X_n \times X_n : (x, y) \in \text{dom} (\alpha) \text{ and } x\alpha = y\alpha \text{ or } x, y \notin \text{dom} (\alpha) \}.
\]

respectively. Notice that \( \ker (\alpha) \) is an equivalence relation on \( X_n \) and the equivalence classes of \( \ker (\alpha) \) are all of the pre-image sets of elements in \( \text{im} (\alpha) \) together with \( X_n \setminus \text{dom} (\alpha) \). Then, the set \( \text{kp} (\alpha) = \{ y\alpha^{-1} : y \in \text{im} (\alpha) \} \) is called the kernel partition of \( \alpha \) and the ordered pair \( \text{ks} (\alpha) = (\text{kp} (\alpha), X_n \setminus \text{dom} (\alpha)) \)

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is called the kernel structure of $\alpha$. For any $\alpha, \beta \in P_n$ notice that

$$ks(\alpha) = ks(\beta) \iff kp(\alpha) = kp(\beta) \iff \ker(\alpha) = \ker(\beta) \text{ and } \dom(\alpha) = \dom(\beta).$$

Moreover, for any $\alpha, \beta$ in $P_n$ it is well known that $\dom(\alpha \beta) \subseteq \dom(\alpha)$, $\ker(\alpha) \subseteq \ker(\alpha \beta)$, and that $\alpha \in P_n$ is an idempotent if and only if $x\alpha = x$ for all $x \in \im(\alpha)$. We denote the set of all idempotents in any subset $U$ of any semigroup by $E(U)$. (See [3, 7] for other terms in semigroup theory which are not explained here.)

The order-preserving transformation semigroup $O_n$ and the partial order-preserving transformation semigroup $PO_n$ on $X_n$, under its natural order, are defined by

$$O_n = \{ \alpha \in T_n \setminus S_n : x \leq y \Rightarrow x\alpha \leq y\alpha \ (\forall x, y \in X_n) \},$$

$$PO_n = O_n \cup \{ \alpha \in T_n \setminus T_n : x \leq y \Rightarrow x\alpha \leq y\alpha \ (\forall x, y \in \dom(\alpha)) \},$$

respectively. Since $\dom(\alpha \beta) \subseteq \dom(\alpha)$, for all $\alpha, \beta \in PO_n$, $SPO_n = PO_n \setminus O_n$ is a subsemigroup of $PO_n$, which is called the strongly partial order-preserving transformation semigroup on $X_n$, and

$$SPO(n, r) = \{ \alpha \in SPO_n : |\im(\alpha)| \leq r \}$$

is (under usual composition) a subsemigroup of $SPO_n$ for $1 \leq r \leq n - 1$. Notice that $SPO(n, n - 1) = SPO_n$.

Let $A = \{ A_1, \ldots, A_k \}$ be a partition of a set $Y \subseteq X_n$. Then $A$ is called an ordered partition, and we write $A = (A_1, \ldots, A_k)$, if $x < y$ for all $x \in A_i$ and $y \in A_{i+1}$ (1 ≤ $i$ ≤ $k - 1$) (the idea of ordering a family of sets appeared on p335 of [8]). Moreover, a set $\{a_1, \ldots, a_k\}$, such that $|\{a_1, \ldots, a_k\} \cap A_i| = 1$ for each 1 ≤ $i$ ≤ $k$, is called a transversal (or a cross-section) of $A$. For $\alpha \in PO_n$ with height $k$, we have the order on the kernel classes $A_1, \ldots, A_k$ of $kp(\alpha)$ as defined above and $\ker(\alpha) = \bigcup_{i=1}^{k+1} (A_i \times A_i)$, where $\emptyset \neq A_{k+1} = X_n \setminus \dom(\alpha)$. Without loss of generality, if $(A_1, \ldots, A_k)$ is an ordered partition of $\dom(\alpha)$, then $A_1 \alpha < \cdots < A_k \alpha$, and moreover, $\alpha$ can be written in the following tabular forms:

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_k & A_{k+1} \\ A_1 \alpha & A_2 \alpha & \cdots & A_k \alpha & - \end{pmatrix} \quad \text{or} \quad \alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ A_1 \alpha & A_2 \alpha & \cdots & A_k \alpha \end{pmatrix}.$$

For any $\alpha, \beta$ in $SPO(n, r)$, it is easy to show by using the definitions of the Green's equivalences that

$$(\alpha, \beta) \in \mathcal{L} \iff \im(\alpha) = \im(\beta), \quad (\alpha, \beta) \in \mathcal{R} \iff ks(\alpha) = ks(\beta),$$

$$(\alpha, \beta) \in \mathcal{D} \iff h(\alpha) = h(\beta) \quad \text{and} \quad (\alpha, \beta) \in \mathcal{H} \iff \alpha = \beta$$

(see for the definitions of the Green's equivalences [7, pages 45–47]). For each $r$ such that $1 \leq r \leq n - 1$, we denote Green's $D$-class of all elements in $SPO(n, r)$ of height $k$ by $D_k$ for $1 \leq k \leq r$. 

Let \( II = (V(II), \overrightarrow{E}(II)) \) be a digraph. For two vertices \( u, v \in V(II) \) we say \( u \) is connected to \( v \) in \( II \) if there exists a directed path from \( u \) to \( v \), that is, either \((u, v) \in \overrightarrow{E}(II)\) or \((u, w_1), \ldots, (w_i, w_{i+1}), \ldots, (w_n, v) \in \overrightarrow{E}(II)\) for some \( w_1, \ldots, w_n \in V(II) \). Moreover, we say \( II \) is strongly connected if, for any two vertices \( u, v \in V(II) \), \( u \) is connected to \( v \) in \( II \). Let \( X \) be a non-empty subset of Green’s \( D \)-class \( D_r \) of \( SPO(n, r) \) for \( 2 \leq r \leq n - 1 \). Then we define a digraph \( \Gamma_X \) as follows:

- the vertex set of \( \Gamma_X \), denoted by \( V = V(\Gamma_X) \), is \( X \); and
- the directed edge set of \( \Gamma_X \), denoted by \( \overrightarrow{E} = \overrightarrow{E}(\Gamma_X) \), is

\[
\overrightarrow{E} = \{(\alpha, \beta) \in V \times V : \alpha \beta \in D_r\}.
\]

Let \( S \) be any semigroup, and let \( A \) be any non-empty subset of \( S \). Then the subsemigroup generated by \( A \), that is, the smallest subsemigroup of \( S \) containing \( A \), is denoted by \( \langle A \rangle \). The rank of a finitely generated semigroup \( S \), a semigroup generated by its a finite subset, is defined by

\[
\text{rank}(S) = \min\{ |A| : \langle A \rangle = S \}.
\]

Gomes and Howie pointed out that the semigroup \( SPO_n \) is not idempotent generated and the rank of \( SPO_n \) is \( 2n - 2 \) in [5]. Moreover, for \( 2 \leq r \leq n - 2 \), Garba proved in [4] that the subsemigroup \( SPO(n, r) \) is generated by idempotents of height \( r \), and the rank of \( SPO(n, r) \) is \( \sum_{k=r}^{n-1} \binom{n}{k}(k-1) \).

The main goal of this paper is to find necessary and sufficient conditions for any subset of \( SPO(n, r) \) to be a (minimal) generating set of \( SPO(n, r) \) for \( 2 \leq r \leq n - 1 \).

### 2. Generating sets of \( SPO_n \)

For convenience we state and prove probably a well known proposition.

**Proposition 1.** For \( n, k \geq 2 \) and \( 1 \leq r \leq n - 1 \), let \( \alpha, \beta, \alpha_1, \ldots, \alpha_k \in D_r \) in \( SPO(n, r) \). Then,

(i) \( \alpha \beta \in D_r \) if and only if \( \text{im}(\alpha \beta) = \text{im}(\beta) \). In other words, \( \text{im}(\alpha) \) is a transversal of the kernel partition \( \text{kp}(\beta) \) of \( \beta \).

(ii) \( \alpha_1 \cdots \alpha_k \in D_r \) if and only if \( \alpha_1 \alpha_{i+1} \in D_r \) for each \( 1 \leq i \leq k - 1 \).

**Proof.** The proof of (i) is clear. We prove (ii):

\( \Rightarrow \) This part of the proof is also clear.

\( \Leftarrow \) Suppose that \( \alpha_1 \alpha_{i+1} \in D_r \) for each \( 1 \leq i \leq k - 1 \). We use the inductive hypothesis on \( k \) to complete the proof.

For \( k = 2 \) the claim is obviously true. Suppose that the claim holds for \( k - 1 \geq 2 \). If \( \text{im}(\alpha_{k-1}) = \{y_1, \ldots, y_r\} \), then \( \text{im}(\alpha_1 \cdots \alpha_{k-1}) = \{y_1, \ldots, y_r\} \) since \( \text{im}(\alpha_1 \cdots \alpha_{k-1}) \subseteq \text{im}(\alpha_{k-1}) \) and \( \alpha_1 \cdots \alpha_{k-1}, \alpha_k \in D_r \). Thus it follows from (i) that

\[
\text{im}(\alpha_1 \cdots \alpha_{k-1} \alpha_k) = \{y_1 \alpha_k, \ldots, y_r \alpha_k\} = \text{im}(\alpha_{k-1} \alpha_k) = \text{im}(\alpha_k),
\]

and so \( \alpha_1 \cdots \alpha_{k-1} \alpha_k \in D_r \), as required. \( \square \)
For $0 \leq s \leq r \leq n$ recall the set
\[ [r, s] = \{ \alpha \in PO_n : |\text{dom}(\alpha)| = r \text{ and } |\text{im}(\alpha)| = s \} \]
(defined in [5, p. 276]). It is indicated in [5] that the top Green’s $D$-class $D_{n-1} = [n-1, n-1]$ in $SPO_n$ does not generate $SPO_n$, and that $SPO_n$ is not idempotent generated.

Suppose that $A$ is a (minimal) generating set of $SPO_n$. Since there exist $n$ different Green’s $L$-classes and $n$ different Green’s $R$-classes in $D_{n-1}$, $A$ must contain at least $n$ elements from $D_{n-1}$. Next notice that a typical element $\alpha \in [n-1, n-2] \subseteq D_{n-2}$ has the form:
\[ \alpha = \left( \begin{array}{ccc}
  a_1 & \cdots & a_{n-1} \\
  b_1 & \cdots & b_{n-1}
\end{array} \right) \in [n-1, n-2], \]
where $a_1 < \cdots < a_{n-1}$, $b_1 \leq \cdots \leq b_{n-1}$ and all but one of the inequalities between the $b$’s are strict. Then $\alpha$ is called of kernel type $i$ if $b_i = b_{i+1}$, and we write $K(\alpha) = i$, and so the possible values for $K(\alpha)$ are $1, 2, \ldots, n-2$. It is shown in [5] that $A \cap [n-1, n-2]$ must contain at least one element of each of the $n-2$ possible kernel types in $[n-1, n-2]$. For each $i = 2, \ldots, n$ let
\[(1) \quad \alpha_i : X_n \setminus \{i\} \to X_n \setminus \{i-1\} \quad \text{and} \quad \alpha_1 : X_n \setminus \{1\} \to X_n \setminus \{n\}\]
be the unique order-preserving (bijective) transformations. That is, let $\alpha_1$ denote the unique order-preserving (bijective) transformation from $X_n \setminus \{1\}$ onto $X_n \setminus \{n\}$ and, for each $i \in \{2, \ldots, n\}$, let $\alpha_i$ denote the unique order-preserving (bijective) transformation from $X_n \setminus \{i\}$ onto $X_n \setminus \{i-1\}$. For each $i = 1, \ldots, n-2$, let
\[(2) \quad \beta_i : X_n \setminus \{n\} \to X_n\]
be the order-preserving transformation defined by
\[ j\beta_i = \begin{cases} 
  i+1 & \text{if } j = i \\
  j & \text{otherwise.}
\end{cases} \]
Then it is clear that $\{\alpha_1, \ldots, \alpha_n\} \subseteq [n-1, n-1] = D_{n-1}$ and $\{\beta_1, \ldots, \beta_{n-2}\} \subseteq [n-1, n-2]$ and it is showed in [5] that
\[(3) \quad Z = \{\alpha_1, \ldots, \alpha_n\} \cup \{\beta_1, \ldots, \beta_{n-2}\}\]
is a minimal generating set of $SPO_n$. Thus $\text{rank}(SPO_n) = 2n - 2$. Although $SPO_n = SPO(n, n-1)$ is not idempotent generated, it is shown in [4] that $SPO(n, r)$ is generated by the idempotents in $D_r$, and that
\[ \text{rank}(SPO(n, r)) = \sum_{k=r}^{n-1} \binom{n}{k} \binom{k - 1}{r - 1} \]
for $2 \leq r \leq n - 2$.

Notice that if $\alpha, \beta \in [r, r] \subseteq SPO(n, r)$ for $1 \leq r \leq n - 1$, then it follows from Proposition 1(i) that $\alpha \beta \in D_r$ if and only if $\text{im}(\alpha) = \text{dom}(\beta)$. 
Theorem 2. Let A be a subset of SPO_n, and let B = A \cap [n-1, n-1] and C = A \cap [n-1, n-2]. Then A is a generating set of SPO_n if and only if

(i) there exists \lambda_i \in B such that dom(\lambda_i) = X_n \setminus \{i\} for each i = 1, 2, \ldots, n,

(ii) there exists \gamma_i \in B such that im(\gamma_i) = X_n \setminus \{n\} and \gamma_i \in B such that im(\gamma_i) = X_n \setminus \{i-1\} for each i = 2, 3, \ldots, n,

(iii) \lambda_i is connected to \gamma_i in the digraph \Gamma_B for each i = 1, 2, \ldots, n; and

(iv) there exists at least one element of each of the n - 2 possible kernel types in C.

Proof. (⇒) Suppose that A is a generating set of SPO_n. Let \alpha_i, with i = 1, 2, \ldots, n, be the elements defined in (1). Since A is a generating set, there exist \lambda_{i,1}, \ldots, \lambda_{i,k} \in A such that

\alpha_i = \lambda_{i,1} \cdot \cdots \lambda_{i,k}

for each i \in \{1, \ldots, n\}. Since ker(\lambda_{i,1}) \subseteq ker(\alpha_i), dom(\alpha_i) \subseteq dom(\lambda_{i,1}), im(\alpha_i) \subseteq im(\lambda_{i,1}) and \alpha_i \in [n-1, n-1], it follows that dom(\alpha_i) = X_n \setminus \{i\} = dom(\lambda_{i,1}) for each i = 1, 2, \ldots, n and that im(\alpha_i) = X_n \setminus \{n\} = im(\lambda_{i,1}) and im(\alpha_i) = X_n \setminus \{i-1\} = im(\lambda_{i,k}) for each i = 2, 3, \ldots, n. Then it is clear that \lambda_{i,1}, \ldots, \lambda_{i,k} \in B. Let \lambda_i = \lambda_{i,1} and \gamma_i = \lambda_{i,k}. Hence, the first two conditions hold. Moreover, it follows from Proposition 1(ii) that \lambda_{i,j} \lambda_{i,j+1} \in [n-1, n-1] for each 1 \leq j \leq k-1, and so there exists a directed edge from \lambda_{i,j} to \lambda_{i,j+1} in \Gamma_B for each 1 \leq j \leq k-1. Thus, \lambda_i is connected to \gamma_i in the digraph \Gamma_B. Therefore, the third condition holds as well. Since the last condition follows from the result in [5, p. 280], the first part of the proof is complete.

(⇐) For this part of the proof it is enough to show that the generating set \mathcal{Z} given in (3) is a subset of \langle A \rangle.

For any element \alpha_i defined in (1) it follows from the conditions that there exist \lambda_i, \gamma_i \in B such that dom(\lambda_i) = dom(\alpha_i), im(\gamma_i) = im(\alpha_i) and \lambda_i is connected to \gamma_i in the digraph \Gamma_B. Thus there exists a directed path from \lambda_i to \gamma_i, say

\lambda_i = \sigma_1 \rightarrow \sigma_2 \rightarrow \cdots \rightarrow \sigma_{k-1} \rightarrow \sigma_k = \gamma_i,

where \sigma_1, \ldots, \sigma_k \in B, and hence, \sigma_j \sigma_{j+1} \in [n-1, n-1] for each 1 \leq j \leq k-1. It follows from Proposition 1(ii) that \delta = \sigma_1 \cdot \cdots \sigma_k \in [n-1, n-1]. Thus we have im(\delta) = im(\sigma_k) = im(\gamma_i) = im(\alpha_i) and dom(\delta) = dom(\sigma_1) = dom(\lambda_i) = dom(\alpha_i), and so \alpha_i = \delta \in \langle B \rangle \subseteq \langle A \rangle since \alpha_i, \delta \in [n-1, n-1] are order-preserving bijections.

It follows from the last condition that, for each i = 1, 2, \ldots, n - 2, we may choose and fix an element with the kernel type i in C and say \theta_i. Thus, we have

\theta_i = \begin{pmatrix} a_1 & \cdots & a_{i-1} & a_i & a_{i+1} & a_{i+2} & \cdots & a_{n-1} \\ b_1 & \cdots & b_{i-1} & b_i & b_{i+1} & b_{i+1} & \cdots & b_{n-2} \end{pmatrix} \in C
for some $a_1 < \cdots < a_{n-1}$ and $b_1 < \cdots < b_{n-2}$ in $X_n$. For any element $\beta_i$ ($1 \leq i \leq n-2$) defined in (2), consider two strictly partial order-preserving transformations

$$
\varphi_i = \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix} \in [n-1,n-1] \text{ and } \\
\psi_i = \begin{pmatrix} b_1 & \cdots & b_{i-1} & b_i & \cdots & b_{n-2} \\ 1 & \cdots & i-1 & i+1 & \cdots & n-1 \end{pmatrix} \in [n-2,n-2].
$$

From [5, Lemmas 3.4 and 3.12] we have that $[r,r] \subseteq \langle B \rangle$ for all $r = 1, \ldots, n-1$, and so $\varphi_i, \psi_i \in \langle B \rangle$ ($1 \leq i \leq n-2$). Thus it follows from the fact

$$
\beta_i = \varphi_i \theta_i \psi_i
$$

that $\beta_i \in \langle A \rangle$ for all $i = 1, \ldots, n-2$. Therefore, $Z$ defined in (3) is a subset of $\langle A \rangle$, and so $\text{SPO}_n = \langle A \rangle$.

Since a generating set of $\text{SPO}_n$ must contain at least $n$ elements from $[n-1,n-1]$ and at least $n-2$ elements from $[n-1,n-2]$, we have the following corollary from Theorem 2:

**Corollary 3.** Let $B$ be a subset of $[n-1,n-1]$ with cardinality $n$ and let $C$ be a subset of $[n-1,n-2]$ with cardinality $n-2$. Then $B \cup C$ is a minimal generating set of $\text{SPO}_n$ if and only if

(i) there exists exactly one element $\lambda_i \in B$ such that $\text{dom}(\lambda_i) = X_n \setminus \{i\}$ for each $i = 1, \ldots, n$,

(ii) there exist exactly one element $\gamma_i \in B$ such that $\text{im}(\gamma_i) = X_n \setminus \{n\}$ and exactly one element $\gamma_i \in B$ such that $\text{im}(\gamma_i) = X_n \setminus \{i-1\}$ for each $i = 2, 3, \ldots, n$,

(iii) $\lambda_i$ is connected to $\gamma_i$ in the digraph $\Gamma_B$ for $i = 1, 2, \ldots, n$; and

(iv) there exists exactly one element of each of the $n-2$ possible kernel types in $C$.

For example, consider $\text{SPO}_3$, $B = \{\sigma_1, \sigma_2, \sigma_3\} \subseteq [n-1,n-1]$ with cardinality 3, and $C = \{\emptyset\} \subseteq [n-1,n-2]$ with cardinality 1 where

$$
\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \\
\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \emptyset = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}.
$$

First of all it is easy to see that $B$ satisfies the first two conditions and $C$ satisfies the last condition of Corollary 3. Moreover, since $\text{im}(\sigma_i)$ is a transversal of the kernel partition $\text{kp}(\sigma_{i+1})$, there exists a directed edge from $\sigma_i$ to $\sigma_{i+1}$ for each $1 \leq i \leq 3$ (where $\sigma_4 = \sigma_1$) in the digraph $\Gamma_B$. Thus $\Gamma_B$ is a Hamiltonian digraph. It follows from their definitions that Hamiltonian digraphs are strongly connected (see [9, pages 88 and 148]) the third condition of Corollary 3 is satisfied as well, and so $A = B \cup C$ is a minimal generating set of $\text{SPO}_3$. 


Notice that if \( A \subseteq SPO_n \) is a (minimal) generating set of \( SPO_n \), then we may use the paths in \( \Gamma_B \) to write each \( \alpha_i \) (for \( 1 \leq i \leq n \)) defined in (1) as a product of elements from \( B \), where \( B = A \cap [n-1, n-1] \). For example, consider the minimal generating set \( A = \{ \sigma_1, \sigma_2, \sigma_3, \theta \} \) of \( SPO_3 \) given above and consider the transformation \( \alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix} \) defined in (1). Since \( B = \{ \sigma_1, \sigma_2, \sigma_3 \} \), \( \text{dom}(\alpha_1) = \text{dom}(\sigma_1) \), \( \text{im}(\alpha_1) = \text{im}(\sigma_2) \) and \( \sigma_1 \rightarrow \sigma_2 \) is a path in \( \Gamma_B \), it follows that \( \text{dom}(\sigma_1 \sigma_2) = \text{dom}(\alpha_1) \) and \( \text{im}(\sigma_1 \sigma_2) = \text{im}(\alpha_1) \), that is, \( \sigma_1 \sigma_2 = \alpha_1 \).

3. Generating sets of \( SPO(n, r) \)

Now we consider the subsemigroups \( SPO(n, r) \) for all \( 2 \leq r \leq n-2 \). Since \( \alpha \in D_k \) (\( 1 \leq k \leq r \)) can not be written as a product of elements with height smaller than \( k \), and since \( SPO(n, r) \) is generated by its idempotents of height \( r \), it is enough to consider only the subsets of \( D_r \) as a generating set of \( SPO(n, r) \). Moreover, a subset \( X \) of \( D_r \) is a generating set of \( SPO(n, r) \) if and only if, for each idempotent \( \xi \) in \( E(D_r) \), there exist \( \alpha, \beta \in X \) such that

(i) \( \text{ks}(\alpha) = \text{ks}(\xi) \),

(ii) \( \text{im}(\beta) = \text{im}(\xi) \), and

(iii) \( \alpha \) is connected to \( \beta \) in the digraph \( \Gamma_X \).

Theorem 4. Let \( X \) be a subset of Green’s \( \mathcal{D} \)-class \( D_r \) in \( SPO(n, r) \) for \( 2 \leq r \leq n-2 \). Then \( X \) is a generating set of \( SPO(n, r) \) if and only if, for each idempotent \( \xi \) in \( E(D_r) \), there exist \( \alpha, \beta \in X \) such that

(1) \( \text{ks}(\alpha) = \text{ks}(\xi) \),

(2) \( \text{im}(\beta) = \text{im}(\xi) \), and

(3) \( \alpha \) is connected to \( \beta \) in the digraph \( \Gamma_X \).

Proof. The proof is similar to the proof of Theorem 2. But it is much easier since each \( SPO(n, r) \) (for \( 2 \leq r \leq n-2 \)) is idempotent generated. \( \square \)

Similarly, since \( \text{rank}(SPO(n, r)) = \sum_{k=r}^{n-1} \binom{n}{k} \binom{k-1}{r-1} \) for \( 2 \leq r \leq n-2 \), we have the following corollary:

Corollary 5. For \( 2 \leq r \leq n-2 \) let \( X \) be a subset of Green’s \( \mathcal{D} \)-class \( D_r \) with cardinality \( \sum_{k=r}^{n-1} \binom{n}{k} \binom{k-1}{r-1} \). Then \( X \) is a minimal generating set of \( SPO(n, r) \) if and only if, for each idempotent \( \xi \in E(D_r) \), there exist \( \alpha, \beta \in X \) such that \( \text{ks}(\alpha) = \text{ks}(\xi) \), \( \text{im}(\beta) = \text{im}(\xi) \) and \( \alpha \) is connected to \( \beta \) in \( \Gamma_X \). \( \square \)

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