AUTOCOMMUTATORS AND AUTO-BELL GROUPS

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Abstract. Let \( x \) be an element of a group \( G \) and \( \alpha \) be an automorphism of \( G \). Then for a positive integer \( n \), the autocommutator \( [x, \alpha]_n \) is defined inductively by \( [x, \alpha] = x^{-1}x\alpha = x^{-1}\alpha(x) \) and \( [x, \alpha]_n = [[x, \alpha], \alpha] \). We call the group \( G \) to be \( n \)-auto-Engel if \( [x, \alpha]_n = [\alpha, x]_n = 1 \) for all \( x \in G \) and every \( \alpha \in \text{Aut}(G) \), where \( [\alpha, x] = [x, \alpha]^{-1} \). Also, for any integer \( n \neq 0, 1 \), a group \( G \) is called an \( n \)-auto-Bell group when \( [x, \alpha]_n = [x, \alpha]^n \) for every \( x \in G \) and each \( \alpha \in \text{Aut}(G) \). In this paper, we investigate the properties of such groups and show that if \( G \) is an \( n \)-auto-Bell group, then the factor group \( G/L_3(G) \) has finite exponent dividing \( 2n(n-1) \), where \( L_3(G) \) is the third term of the upper autocentral series of \( G \). Also, we give some examples and results about \( n \)-auto-Bell abelian groups.

1. Introduction

Let \( G \) be a group and let \( \text{Aut}(G) \) denote the automorphism group of \( G \). For \( \alpha \in \text{Aut}(G) \) and \( x \in G \), the autocommutator of \( x \) and \( \alpha \) is defined to be \( [x, \alpha] = x^{-1}x\alpha = x^{-1}\alpha(x) \). The absolute centre and the autocommutator subgroup of \( G \) are the subgroups \( L(G) = \{ x \in G : [x, \alpha] = 1 \text{ for all } \alpha \in \text{Aut}(G) \} \) and \( K(G) = \langle [x, \alpha] : x \in G, \alpha \in \text{Aut}(G) \rangle \), respectively (see [6]). Clearly, the absolute centre is a characteristic subgroup contained in the centre of \( G \) and the autocommutator subgroup is a characteristic subgroup containing the derived subgroup of \( G \). Hegarty [6] uses the notation \( G^* \) for \( K(G) \) and proves that if \( G/L(G) \) is finite, then so is \( K(G) \). Autocommutator subgroup and absolute centre are already studied in [3, 11].

Let \( n \) be a positive integer. The autocommutator \( [x, \alpha]_n \) is defined inductively by \( [x, \alpha]_1 = [x, \alpha] \) and \( [x, \alpha]_n = [[x, \alpha]_{n-1}, \alpha] \) for \( n \geq 2 \). The group \( G \) is said to be \( n \)-auto-Engel if \( [x, \alpha]_n = [\alpha, x]_n = 1 \) for all \( x \in G \) and every \( \alpha \in \text{Aut}(G) \), where \( [\alpha, x] = [x, \alpha]^{-1} \). Auto-Engel groups are already studied by Moghaddam et al. (see [9]).

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For any integer \( n \neq 0, 1 \), a group \( G \) is called \( n \)-auto-Bell if \( [x^n, \alpha] = [x, \alpha^n] \) for every \( x \in G \) and \( \alpha \in \text{Aut}(G) \). In particular, a group \( G \) satisfying the previous identity for all inner automorphisms \( \alpha \in \text{Inn}(G) \) is an \( n \)-Bell group. The study of \( n \)-Bell groups was the subject of several articles, see for instance Brandl and Kappe [1], Kappe and Morse [8], Delizia et al. [4] and Tortora [13].

A group \( G \) is called \( n \)-Kappe if the factor group \( G/R_2(G) \) has finite exponent dividing \( n \), where \( R_2(G) = \{ g \in G : [g, x, x] = 1 \text{ for all } x \in G \} \) is the set of all right 2-Engel elements of \( G \). It is well known that every \( n \)-Bell group is \( n(n-1) \)-Kappe (see Brandl and Kappe [1]).

In [9], it is proved that the set of all right 2-auto-Engel elements of \( G \), \( AR_2(G) = \{ g \in G : [g, \alpha, \alpha] = 1 \text{ for all } \alpha \in \text{Aut}(G) \} \) is a characteristic subgroup of \( G \). Here, we call a group \( G \) an \( n \)-auto-Kappe group when the factor group \( G/AR_2(G) \) has finite exponent dividing \( n \). In this paper, we study some connections of such groups with \( n \)-auto-Bell groups.

Also, Delizia et al. [4] proved that for an \( n \)-Bell group \( G \), the exponent of \( G/Z_3(G) \) divides \( 2n(n-1) \).

In [10], Moghaddam et al. studied the concept of lower autocentral series and its properties. We define the upper autocentral series by a similar manner. The \( n \)-th absolute centre of \( G \) is defined in the following way: \( L_1(G) = L(G) \) and \( L_n(G) = \{ x \in G : [x, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_i \in \text{Aut}(G) \} \). One obtains an ascending chain of characteristic subgroups of \( G \) as follows:

\[
1 = L_0(G) \leq L_1(G) \leq \cdots \leq L_n(G) \leq \cdots ,
\]

which we may call the upper autocentral series of \( G \).

In Section 3, we show that if \( G \) is an \( n \)-auto-Bell group, then the factor group \( G/L_3(G) \) has finite exponent dividing \( 2n(n-1) \).

2. Auto-Bell and auto-Kappe groups

First, we state a result about 2-auto-Engel groups, which is proved in [9].

**Lemma 2.1** ([9]). Let \( G \) be a 2-auto-Engel group. Then for every \( x, y \in G \), \( \alpha \in \text{Aut}(G) \) and \( n \in \mathbb{Z} \) the following properties hold:

- (a) \([x, x^\alpha] = 1\);
- (b) \([x, \alpha^n] = [x, \alpha]^n = [x^n, \alpha]\);
- (c) \([x^\alpha, y] = [x, y^\alpha] \);
- (d) \([\alpha, x, y] = [\alpha, y, x]^{-1}\).

By the above lemma, every 2-auto-Engel group is an \( n \)-auto-Bell group for any integer \( n \neq 0, 1 \). Now, suppose that \( G \) is a 2-auto-Bell group. Then the identity \([x^2, \alpha] = [x, \alpha^2]\) implies that \([x, \alpha][x, \alpha]^{\alpha^{-1}} = ([x, \alpha][x, \alpha]^\alpha]^\alpha^{-1} \). Hence \(([x, \alpha]^\alpha)^\alpha = [x, \alpha]\) and so \([x, \alpha, \alpha^{-1}, \varphi_2] = 1\), where \( \varphi_2 \) is the inner automorphism defined by \( x \). If we replace the automorphism \( \alpha \) by \( \varphi_2 \), then we have \([x, \alpha^{-1}, \varphi_2]^{\alpha^{-1}} = 1\). Hence \([x, \alpha, \alpha] = 1\) and since a right 2-auto-Engel element is also a left one (see [9]), \( G \) is a 2-auto-Engel group. Thus for any 2-auto-Bell group \( G \), we have \([G, \alpha] \subseteq C_G(\alpha)\) for every \( \alpha \in \text{Aut}(G) \).
and hence \(\text{Aut}(G), x, x = 1\) for every \(x\) in \(G\) (i.e., \(x\) is also a left 2-auto-Engel element). Therefore
\[
\text{Aut}(G) = A(G) = \{\alpha \in \text{Aut}(G) : xx^n = x^n x \text{ for all } x \in G\},
\]
the set of commuting automorphisms of the group \(G\) (see [2]). It is easy to see that every 2-auto-Bell group satisfies the identity \(\alpha(x)\alpha^{-1}(x) = x^2\). In Section 4, we discuss a family of infinitely many non-abelian finite 2-groups which are 2-auto-Bell.

In what follows, we determine the structure of the abelian 2-auto-Bell groups. Let \(G = \langle x, y : x^4 = y^4 = 1, xy = yx \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2\). Consider the automorphism \(\alpha\) of \(G\) given by \(\alpha(x) = xy\) and \(\alpha(y) = yx^2\). Clearly, \([x, \alpha, \alpha] = x^2\) and hence \(G\) is not a 2-auto-Bell group. Now, assume that \(G\) is a 2-auto-Bell abelian group, then for the automorphism \(\alpha : x \mapsto x^{-1}\), we have \(x^4 = [x, \alpha, \alpha] = 1\) for every \(x \in G\). Therefore \(G\) is a direct sum of cyclic groups of order 2 or 4. On the other hand, \([x, \alpha]^4 = [x, \alpha]^4 = 1\) and so \(\exp(\text{Aut}(G))\) divides 4. Using the above example and the fact that \(\mathbb{Z}_2 \times \mathbb{Z}_2\) and \(\mathbb{Z}_4 \times \mathbb{Z}_4\) have an automorphism of order 3, it follows that \(G \cong 1, \mathbb{Z}_2\) or \(\mathbb{Z}_4\). Recall that the structure of non-abelian 2-auto-Bell (2-auto-Engel) 2-groups is studied in [9].

Now, we discuss the relations between auto-Bell and auto-Kappe groups after some preliminary results.

**Lemma 2.2.** Let \(G\) be an \(n\)-auto-Bell group, \(x \in G\) and \(\alpha \in \text{Aut}(G)\). Then
\(\text{(i)}\) \([x^n, \alpha, x^{-n}] = 1\);
\(\text{(ii)}\) \(x^{n(1-n)} \in Z(x^{\text{Aut}(G)}), \text{ where } x^{\text{Aut}(G)} = \langle x^n : \alpha \in \text{Aut}(G) \rangle\).

**Proof.** (i) Since \(G\) is \(n\)-auto-Bell,
\[
[x^n, \alpha]^{-x^{-n}} = [x^{-n}, \alpha] = [x^{-1}, \alpha^n] = [x, \alpha^{-n}]^{-x^{-1}}.
\]
Conjugating with \(x\) and taking the inverse yields \([x^n, \alpha][x^n, \alpha, x^{-1-n}] = [x, \alpha^n]\). Hence \([x^n, \alpha, x^{-1-n}] = 1\).

(ii) Using the Jacobi identity, one obtains \([x, \alpha, y^n][y, x, \alpha^y]\alpha, y, x^n\] = 1 for every \(x\) and \(y\) in \(G\) and \(\alpha \in \text{Aut}(G)\), where \(\varphi_y\) is the inner automorphism of \(G\) defined by \(y\). From this identity and (i) it follows that
\[
1 = [\alpha, x^{1-n}, x^n\alpha] = [x^{(n-1)}\alpha, x^{-1-n}, x^n\alpha] = [x^{n-n^2}, x^n\alpha].
\]
Hence \(x^{n(1-n)} \in Z(x^{\text{Aut}(G)})\). \(\Box\)

**Proposition 2.3.** Every \(n\)-auto-Bell group is also \((1-n)\)-auto-Bell and hence \(n(1-n)\)-auto-Bell.

**Proof.** Since \(G\) is an \(n\)-auto-Bell group, \([x^n, \alpha^{-1}]^\alpha = [x, \alpha^{-n}]^\alpha\). Therefore \(x^{-na}x^n = x^{-a}x^{a^{-1}}\) and hence
\[
x^{-1}x^{(1-n)a}x^n = x^{n-1}x^{a^{-1}}.
\]
So \([x^{1-n}, \alpha] = [x, \alpha^{1-n}]\). Finally, \([x^{1-n}, \alpha][x^{1-n}, \alpha, x^n] = [x, \alpha^{1-n}]\) and by Lemma 2.2(i), \(G\) is a \((1-n)\)-auto-Bell group. Clearly, it follows that \(G\) is also \(n(1-n)\)-auto-Bell. \(\Box\)
Observe that an \( n \)-Bell group need not be a \((-n)\)-Bell group, in general. Clearly, by Proposition 2.3 an \( n \)-auto-Bell group need not be \((-n)\)-auto-Bell. In the following theorem, we show that every \( n \)-auto-Bell group is also \( n(n-1) \)-auto-Bell.

**Theorem 2.4.** Every \( n \)-auto-Bell group is also \( n(n-1) \)-auto-Kappe and hence \( n(n-1) \)-auto-Bell.

**Proof.** Let \( G \) be an \( n \)-auto-Bell group, \( x \in G \), \( \alpha \in \text{Aut}(G) \). Using Lemma 2.2(ii) and Proposition 2.3, we get \( x^{n(1-n)} \in Z(x^{\text{Aut}(G)}) \) and hence

\[
1 = [x^{n(1-n)}, x^\alpha] = [x, x^{n(1-n)\alpha}] = [x, [x^{n(1-n)}, \alpha]].
\]

Therefore

\[(1) \quad [\alpha^{n(1-n)}, x, x] = 1,
\]

and hence, \( \alpha^{n(n-1)} \in A(G) \). So, in every \( n \)-auto-Bell group, we have the following identity,

\[(2) \quad [x^{n(n-1)\alpha}, x] = 1 = [x^{\alpha}, x^{n(n-1)}].
\]

Now, put \( m = n(n-1) \). For the \( n \)-auto-Bell group \( G \), it is easy to see that

\[(3) \quad x^{(n-1)\alpha}x^{n(1-n)} = x^n.
\]

Replacing \( x \) by \( x^n \) yields

\[(4) \quad x^{n\alpha^{1-n}} = x^{-m\alpha}x^{n^2}.
\]

In the equation (4), if we replace \( \alpha \) by \( \alpha^{-1} \) and conjugate with \( \alpha \), we get \( x^{n\alpha^n} = x^{-m}x^{n^2\alpha} \). Now, conjugating the equation (3) with \( \alpha^n \) and using the latter equality yields

\[(5) \quad x^{(n-1)\alpha^{-1+n}} = x^{n\alpha^n}x^{-\alpha} = x^{-m}x^{n(n^2-1)\alpha}.
\]

By the equation (2), clearly \([x^m, \alpha, x] = 1 \) and so \([x^m, \alpha^{-1}, x]^{\alpha} = 1 \). It follows that

\[(6) \quad [x^m, \alpha, x^\alpha] = 1.
\]

Therefore by Proposition 2.3, equations (4), (5) and (6)

\[
[x^m, \alpha, x] = [x^{-m}, \alpha]^{x^{n\alpha^n}} = [x^n, \alpha^{1-n}]^{x^{(n-1)\alpha}, \alpha^n} = x^{-n}x^{n\alpha^{1-n}}x^{(1-n)\alpha}x^{(n-1)\alpha^{1-n}+1} = x^n x^{-m}x^{n^2}x^{(1-n)\alpha}x^{-n\alpha^n}x^{-\alpha} = x^m x^{-m}x^{(1-n)\alpha}x^{-m}x^{n(n^2-1)\alpha} = x^{(-n(n-1)+(1-n)+n^2-1)\alpha} = 1.
\]
Thus $G$ is $n(n-1)$-auto-Kappe. It also follows that $[x^{-1}, \alpha^{-m}] = [x^{-1}, \alpha^{-m}]$.
Therefore $[x, \alpha^n]^{1-1} = [x, \alpha^n]$ and hence by Proposition 2.3 and (1) we get:

$$[x^m, \alpha] = [(x^{-m})^{-1}, \alpha] = [x, \alpha^{-m}]^{x^{-m}}$$
$$= [x, \alpha^{-m}]^{-1} = [x, \alpha^{-m}]^{x^{-m}}$$
$$= [x, \alpha^n].$$

So $G$ is an $n(n-1)$-auto-Bell group. \Halmos

Remark 2.5. Some connections are held between Kappe and Bell groups, which may not be true for auto-Kappe and auto-Bell groups. For example in [4, Theorem 2.1], it is pointed out that every $n$-Kappe group is an $n^2$-Bell group. If $G$ is the elementary abelian 2-group of order 4, then $G$ is a 2-auto-Kappe, but as $G$ has an automorphism of order 3, it cannot be a 4-auto-Bell group.

We end this section by pointing a result, which gives some relations about auto-Bell groups.

**Proposition 2.6.** Let $G$ be a group and $n \neq 0, 1$ be an integer.

(i) If $G$ is an $(n-1)$-auto-Kappe and $n$-auto-Bell group, then $G$ is also $(n-1)$-auto-Bell.

(ii) If $G$ is an $n$-auto-Kappe and $n$-auto-Bell group, then $G$ is also an $(n+1)$-auto-Bell group.

**Proof.** (i) Let $x \in G$ and $\alpha \in \text{Aut}(G)$. Since $G$ is an $n$-auto-Bell group (and hence $(1-n)$-auto-Bell) and also an $(n-1)$-auto-Kappe, we get

$$[x^{1-n}, \alpha] = [x, \alpha^{1-n}] = [x, \alpha^{n-1}]^{-1} - [x, \alpha^{n-1}]^{-1}.$$

On the other hand, since $x^{n-1}$ is a right 2-auto-Engel element, it is also a left one and so $[\alpha, x^{n-1}, x^{n-1}] = 1$. Therefore $[x^{1-n}, \alpha] = [x^{n-1}, \alpha]^{-1}$. This implies that $[x^{n-1}, \alpha] = [x, \alpha^{n-1}]$ and hence $G$ is an $(n-1)$-auto-Bell group.

(ii) Since $G$ is an $n$-auto-Kappe, one may show that $[x^n, \varphi, \alpha^{-1}, \alpha] = 1$, where $\varphi_x$ is the inner automorphism defined by the element $x$. Replacing $\alpha$ by $\varphi^{-1} \varphi_x$ yields $[x^n, \alpha, \varphi^{-1} \varphi_x] = 1$. Thus $[x^n, \alpha]^{x^{-1}} x = x[x^n, \alpha]$ and hence $[x^n, \alpha]^{x^{-1}} = x^n[x^n, \alpha]$. Therefore $x^{-1}[x^n, \alpha] x^{-1} x^n = x^{-1} x^n [x^n, \alpha] x^n$ and from the fact that $G$ is an $n$-auto-Bell group, it follows that $[x^n, \alpha]^x [x, \alpha] = [x, \alpha][x, \alpha]$. This shows that $[x^{n+1}, \alpha] = [x, \alpha^{n+1}]$. Thus $G$ is an $(n+1)$-auto-Bell. \Halmos

3. Upper autocommutative series in auto-Bell groups

Given a group $G$, the $n$-th autocommutator subgroup of $G$ is

$$K_n(G) = \langle [x, \alpha_1, \alpha_2, \ldots, \alpha_n] : x \in G, \alpha_1, \ldots, \alpha_n \in \text{Aut}(G) \rangle.$$ 

It can be easily seen that for every $n \in \mathbb{N}$, the $n$-th autocommutator subgroup is a characteristic subgroup of $G$ containing $\gamma_{n+1}(G)$. Now, we obtain the following series of subgroups

$$G = K_0(G) \supseteq K_1(G) \supseteq K_2(G) \supseteq \cdots \supseteq K_n(G) \supseteq \cdots.$$
which is called the lower autocentral series of \( G \). In [10], it is proved that for any finite abelian group \( G \) and every natural number \( n \), there exists a finite abelian group \( H \) such that \( G \cong K_n(H) \).

Now, the \( n \)-th absolute centre of \( G \) is defined inductively by \( L_1(G) = L(G) \) and \( L_n(G) = \{ x \in G : [x, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_i \in \text{Aut}(G) \} \). Clearly, the \( n \)-th absolute centre of \( G \) is contained in the \( n \)-th centre of \( G \), \( Z_n(G) \). One obtains an ascending chain of characteristic subgroups of \( G \) as follows:

\[
1 = L_0(G) \leq L_1(G) \leq \cdots \leq L_n(G) \leq \cdots,
\]
which we may call the upper autocentral series of \( G \). In the following theorem, we prove that if \( G \) is an \( n \)-auto-Bell group, then \( [G^{2^n(n-1)}, \alpha, \beta, \gamma] = 1 \) for every \( \alpha, \beta, \gamma \in \text{Aut}(G) \).

**Theorem 3.1.** Let \( G \) be an \( n \)-auto-Bell group. Then the factor group \( G/L_3(G) \) has finite exponent dividing \( 2n(n-1) \).

**Proof.** First, we show that for any right 2-auto-Engel element \( x \) of \( G \), the subgroup \( x^{\text{Aut}(G)} = \langle x^\alpha : \alpha \in \text{Aut}(G) \rangle \) is abelian. Let \( \alpha \) and \( \beta \) be automorphisms of \( G \). Then

\[
[x^\alpha, x^\beta] = [x^{\alpha \beta^{-1}}, x]^\beta = [x[x, \alpha \beta^{-1}], x]^\beta = [[x, \alpha \beta^{-1}], x]^\beta.
\]

On the other hand, every right 2-auto-Engel element is also a left 2-auto-Engel element. Hence \( x^{\alpha \beta^{-1}, x, x} = 1 \) and this implies that \( [x^\alpha, x^\beta] = 1 \) and hence \( x^{\text{Aut}(G)} \) is abelian.

Now, by Theorem 2.4, \( g := x^{n(n-1)} \) is a right 2-auto-Engel element. So, for each \( \alpha \in \text{Aut}(G) \), we have \( [g, \alpha^{-1}] = [g, \alpha]^{-1} \). On the other hand, since \( g^{\text{Aut}(G)} \) is abelian, we get \( [g, \alpha \beta] = [g, \alpha][g, \beta][g, \alpha, \beta] \) (observe that \( [g, \alpha] \in g^{\text{Aut}(G)} \) for every \( \alpha, \beta \in \text{Aut}(G) \)). Hence \( [g, \alpha \beta^{-1}] = [g, \beta^{-1} \alpha^{-1}] \) and the above equality shows that

\[
[g, \alpha, \beta] = [g, \beta, \alpha]^{-1}.
\]

Now, suppose that \( \alpha, \beta \) and \( \gamma \) are arbitrary automorphisms of \( G \). One may check that the equality \( [g, \alpha, \beta \gamma] = [g, \beta \gamma, \alpha]^{-1} \) implies that \( [g, \alpha, \beta, \gamma]^2 = 1 \) and since \( g \) is a 2-auto-Engel element, we obtain \( [g^2, \alpha, \beta, \gamma] = 1 \). Therefore \( [x^{2^n(n-1)}, \alpha, \beta, \gamma] = 1 \) and this completes the proof. \( \Box \)

4. *Abelian* \( n \)-*auto-Bell groups

Clearly, every abelian group is an \( n \)-Bell group, but this statement is not true for \( n \)-auto-Bell groups. In what follows, we give some examples of auto-Bell groups and also discuss some results about \( n \)-auto-Bell abelian groups. Observe that by Proposition 2.3, in this section we may suppose that \( n \geq 2 \).

**Example 4.1.** (i) Let \( G \) be a non-periodic abelian group, and consider the inverting automorphism \( \alpha \in \text{Aut}(G) \) and a torsion-free element \( x \in G \). Then one can easily see that \( x^\alpha = x^{-2n} \) and \( [x, \alpha^n] = x^{(-1)^n-1} \). If \( G \) is an \( n \)-auto-Bell group, then we must have \( -2n = (-1)^n - 1 \), and this implies that \( n \in \{0, 1\} \). So, \( G \) cannot be an \( n \)-auto-Bell group for every integer \( n \neq 0, 1 \).
(ii) In [7], Jamali constructed the following family of groups. For \( m \geq 3 \), let \( G_m \) be a 2-group with the following presentation

\[
G_m = \langle a_1, \ldots, a_m, b : a_1^2 = a_2^2 = \cdots = a_m^2 = 1, a_{m-1}^2 = b^2, [a_1, b] = 1, \rangle.
\]

The group \( G_m \) is of order \( 2^{2m} \) with exponent 4 whose automorphism group is isomorphic to \( \mathbb{Z}_2^{m} \) and also \( Z(G_m) \cong \mathbb{Z}_2^m \). Clearly, for every \( m \geq 3 \), \( G_m \) is a non-abelian 2-auto-Bell (and hence an \( n \)-auto-Bell, for every \( n \geq 3 \)) group. By using GAP [5], one can check that \( G_3 \cong (\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2 \).

(iii) Let \( G = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \cong \langle x \rangle \times \langle y \rangle \times \langle z \rangle \). Consider the automorphism \( \alpha \) defined by \( \alpha(x) = xy, \alpha(y) = x^2yz \) and \( \alpha(z) = x^4y^2z \). One can easily check that \( 1 = [x^4, \alpha] \neq [x, \alpha^4] \) and so \( G \) is not a 4-auto-Bell group.

Recall that there are only two non-trivial abelian 2-auto-Bell groups, namely \( \mathbb{Z}_2 \) and \( \mathbb{Z}_4 \).

Observe that if \( G \cong H \times K \) is an \( n \)-auto-Bell group, then so are \( H \) and \( K \). Now, let \( G \) be an abelian \( n \)-auto-Bell group \( (n \geq 3) \) and \( \alpha \) be the inverting automorphism. Clearly, the identity \( [x^n, \alpha] = [x, \alpha^n] \) implies that \( \exp(G) \) divides \( 2n \) or \( 2(n - 1) \) when \( n \) is an even or an odd integer, respectively. By Proposition 2.3, \( G \) is also a \((1 - n)^2\)-auto-Bell group. Hence, the exponent of \( \text{Aut}(G) \) divides \( n(n - 2) \) or \((n - 1)^2 \) when \( n \) is an even or an odd integer, respectively.

By the above statement, it is easy to see that the 3-auto-Bell abelian groups are actually 2-auto-Bell. Assume that \( n = 4 \). Therefore \( G \) is a direct sum of cyclic groups of order 2, 4 or 8. Hence, Example 4.1(iii) and the fact that \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) and \( \mathbb{Z}_8 \times \mathbb{Z}_8 \) have an automorphism of order 3, show that \( G \) is isomorphic to one of the groups \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( \mathbb{Z}_4 \times \mathbb{Z}_4 \), \( \mathbb{Z}_8 \times \mathbb{Z}_8 \) or \( \mathbb{Z}_4 \times \mathbb{Z}_8 \).

Finally, let \( G \) be a 5-auto-Bell abelian group. It is easy to see that \( \exp(G) \) and \( \exp(\text{Aut}(G)) \) divide 8 and 16, respectively. One may check that the abelian 5-auto-Bell groups are actually 4-auto-Bell.

Remark 4.2. Let \( p \) be an odd prime, \( n \geq 6 \) and \( G \) be an abelian \( n \)-auto-Bell \( p \)-group (if any). By the above statement, it is easy to see that \( \exp(\text{Aut}(G)) \) divides \( n \) or \((n - 1)\) when \( n \) is an even or an odd integer, respectively.

The following theorem may be considered as a criterion for recognition of abelian \( p \)-groups which are not \( n \)-auto-Bell.

**Theorem 4.3.** Let \( G \) be a finite abelian \( n \)-auto-Bell group with \( |G| = \prod_{i=1}^{m} p_i^{r_i} \). Then for every \( 1 \leq j \leq m \), the numbers \( p_j(p_j - 1) \) and \( \prod_{i=1}^{j-1} p_i \) divide \( n \) or \((n - 1)\) when \( n \) is an even or an odd integer, respectively.

**Proof.** Suppose that an arbitrary prime \( p \) divides the order of \( G \). Clearly, the Sylow \( p \)-subgroup \( P \) of \( G \) is also an \( n \)-auto-Bell group. If \( p = 2 \) or \( 3 \) the result is true. Suppose that \( n \) is an even integer and \( p \geq 5 \). By considering the inverting automorphism, we get \( p|n \). Let \( \alpha : x \mapsto x^{-1} \) be an automorphism of \( P \), where \( 1 < \lambda < p \) and \( (\lambda, p - 1) = 1 \). Then the identity \( [x^n, \alpha] = [x, \alpha^n] \) implies that \( p|(\lambda^n - n\lambda + n - 1) \). Therefore \( \lambda^n \equiv 1 \pmod{p} \).
On the other hand, Euler’s theorem implies that \(\lambda^{p-1} \equiv 1 \pmod{p}\) and since \((\lambda, p-1) = 1\), we get \((p-1)|n\). Therefore \(p(p-1)|n\). Similarly, it may be shown that \(p(p-1)|(n-1)\), if \(n\) is an odd integer. Therefore the proof is complete.

The above theorem immediately yields the following corollary.

**Corollary 4.4.** There is no abelian \(n\)-auto-Bell \(p\)-group for \(n < p(p-1)\).

**Proposition 4.5.** If \(G\) is an abelian \(p(p-1)n\)-auto-Bell \(p\)-group (\(p\) odd and \(1 \leq n \leq p-1\)), then \(G \cong \mathbb{Z}_p\).

**Proof.** Clearly, \(\mathbb{Z}_p\) is a \(p(p-1)m\)-auto-Bell group for every \(m \in \mathbb{N}\). It is enough to show that \(\mathbb{Z}_p \times \mathbb{Z}_p\) and \(\mathbb{Z}_p^k\) \((k \geq 2)\) are not \(p(p-1)n\)-auto-Bell. Since \(\text{Aut}(\mathbb{Z}_p \times \mathbb{Z}_p) \cong GL(2, p)\) and \(\exp(GL(2, p)) = p(p^2 - 1)\), we obtain \(1 = [x^{p(p-1)n}, \alpha] \neq [x, \alpha^{p(p-1)n}]\) for some \(x \in \mathbb{Z}_p \times \mathbb{Z}_p\) and \(\alpha \in GL(2, p)\).

Also, since \(p^k\) does not divide \(p(p-1)n\), we get \([x^{p(p-1)n}, \alpha] \neq [x, \alpha^{p(p-1)n}] = 1\) and hence the cyclic group of order \(p^k\) \((k \geq 2)\) cannot be a \(p(p-1)n\)-auto-Bell \(p\)-group.

**Remark 4.6.** In the previous proposition, it is not difficult to show that if \(n = p\), then \(G \cong \mathbb{Z}_p\) or \(\mathbb{Z}_{p^2}\). If \(n = p+1\), then \(G \cong \mathbb{Z}_p\) or \(\mathbb{Z}_p \times \mathbb{Z}_p\) and finally if \(n = p+2\), then \(G \cong \mathbb{Z}_p\). Observe that if \(n > p+2\), then the structure of \(G\) may depend on the odd prime \(p\).

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