FINITE GROUPS WITH SOME SEMI-$p$-COVER-AVOIDING OR $ss$-QUASINORMAL SUBGROUPS

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Abstract. Suppose that $G$ is a finite group and $H$ is a subgroup of $G$. $H$ is said to be an $ss$-quasinormal subgroup of $G$ if there is a subgroup $B$ of $G$ such that $G = HB$ and $H$ permutes with every Sylow subgroup of $B$. $H$ is said to be semi-$p$-cover-avoiding in $G$ if there is a chief series $1 = G_0 < G_1 < \cdots < G_t = G$ of $G$ such that, for every $i = 1, 2, \ldots, t$, if $G_i/G_{i-1}$ is a $p$-chief factor, then $H$ either covers or avoids $G_i/G_{i-1}$. We give the structure of a finite group $G$ in which some subgroups of $G$ with prime-power order are either semi-$p$-cover-avoiding or $ss$-quasinormal in $G$. Some known results are generalized.

1. Introduction

All groups considered in this paper are finite. $G$ always means a group, $|G|$ denotes the order of $G$ and $\pi(G)$ denotes the set of all primes dividing $|G|$.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation, provided that (1) if $G \in \mathcal{F}$ and $H \leq G$, then $G/H \in \mathcal{F}$, and (2) if $G/M$ and $G/N$ are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for any normal subgroups $M$, $N$ of $G$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation.

A famous topic in group theory is to study the influence of some subgroups with prime-power order on the structure of $G$. In [6], Li, Shen and Liu generalized $s$-quasinormal subgroups to $ss$-quasinormal subgroups. A subgroup $H$ of $G$ is said to be an $ss$-quasinormal subgroup of $G$ if there is a subgroup $B$ of $G$ such that $G = HB$ and $H$ permutes with every Sylow subgroup of $B$. Recently, Fan, Guo and Shum [1] introduced the semi-$p$-cover-avoiding property. A subgroup $H$ of $G$ is said to be semi-$p$-cover-avoiding in $G$ if there is a chief

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series $1 = G_0 < G_1 < \cdots < G_t = G$ of $G$ such that $H$ either covers or avoids $G_i/G_{i-1}$ whenever $G_i/G_{i-1}$ is a $p$-chief factor.

Some interesting results have been obtained about the structure of a group $G$ under assumption that some subgroups of $G$ are $ss$-quasinormal or semi-$p$-cover-avoiding in $G$ (see: [1, 3, 5, 6]).

There are examples to show $ss$-quasinormal and semi-$p$-cover-avoiding are two different properties of subgroups.

**Example 1.1.** Let $G = A_5$, the alternative group of degree 5. Then $A_4$ is an $ss$-quasinormal subgroup of $G$ but not semi-$p$-cover-avoiding in $G$.

**Example 1.2 ([2, Example 2.4]).** Let $A_4$ be the alternative group of degree 4 and $C_2 = \langle c \rangle$ a cyclic group of order 2, generated by an element $c$. Let $G = C_2 \times A_4$. Then $A_4 = K_4 \cdot \langle t \rangle$, where $K_4 = \langle a, b \rangle$ is the Klein four-group with generators $a$ and $b$ of order 2 and $\langle t \rangle$ is a cyclic group of order 3. Take $H = \langle ac \rangle$ to be the subgroup of $G$ generated by $ac$. It is clear that the following series

$$1 < K_4 < A_4 < C_2 \times A_4 = G$$

is a chief series of $G$ such that $H$ covers $G = A_4$ and avoids the rest. This is to say that $H$ has the semi-cover-avoiding property in $G$. Of course, $H$ is semi-$p$-cover-avoiding in $G$. However, $H$ is not $ss$-quasinormal in $G$.

The aim of this article is to unify and improve some earlier results using $ss$-quasinormal and semi-$p$-cover-avoiding subgroups. Our main theorems are as follows:

**Theorem 3.1.** Let $G$ be a group and $p$ a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Let $P$ be a Sylow $p$-subgroup of $G$. If all maximal subgroups of $P$ are either semi-$p$-cover-avoiding or $ss$-quasinormal subgroups in $G$, then $G$ is $p$-nilpotent.

**Theorem 3.6.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a solvable normal subgroup $H$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are either semi-$p$-cover-avoiding or $ss$-quasinormal in $G$, then $G \in \mathcal{F}$.

**Theorem 3.7.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F^*(H)$ are either semi-$p$-cover-avoiding or $ss$-quasinormal in $G$, then $G \in \mathcal{F}$.

2. Basic definitions and preliminary results

In this section, we give some results that are needed in this paper.

**Lemma 2.1 ([6]).** Let $H$ be an $ss$-quasinormal in a group $G$, $K \leq G$ and $N$ a normal subgroup of $G$.

(i) If $H \leq K$, then $H$ is $ss$-quasinormal in $K$;

(ii) $HN/N$ is $ss$-quasinormal in $G/N$;
Theorem 3.1. Let $N \leq K$ and $K/N$ is ss-quasinormal in $G/N$, then $K$ is ss-quasinormal in $G$.

Lemma 2.2 ([1, 3]). Let $H$ be a semi-p-cover-avoiding subgroup of a group $G$ and $N$ a normal subgroup of $G$. Then
(i) $H$ is semi-p-cover-avoiding in $K$ for every subgroup $K$ of $G$ with $H \leq K$;
(ii) $HN/N$ is semi-p-cover-avoiding in $G$ if one of the following holds:
(1) $N \leq H$;
(2) $\gcd(|H|, |N|) = 1$, where $\gcd(\cdot, \cdot)$ denotes the greatest common divisor.

Lemma 2.3 ([7]). Let $G$ be a group and $p$ a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. Let $P$ be a Sylow $p$-subgroup of $G$. If all maximal subgroups of $P$ are semi-p-cover-avoiding or $s$-quasinormally embedded subgroups in $G$, then $G$ is $p$-nilpotent.

Lemma 2.4 ([8]). Let $G$ be a group and $p$ a prime dividing $|G|$ with $(|G|, p - 1) = 1$.
(i) If $N$ is normal in $G$ of order $p$, then $N \leq Z(G)$;
(ii) If $G$ has a cyclic Sylow $p$-subgroup, then $G$ is $p$-nilpotent;
(iii) If $M \leq G$ and $|G:M| = p$, then $M \leq G$.

Lemma 2.5 ([6]). Let $H$ be a nilpotent subgroup of $G$. Then the following statements are equivalent.
(i) $H$ is $s$-quasinormal in $G$;
(ii) $H \leq F(G)$ and $H$ is ss-quasinormal in $G$;
(iii) $H \leq F(G)$ and $H$ is $s$-quasinormally embedded in $G$.

Lemma 2.6 ([7]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if one of the following holds:
(i) all maximal subgroups of all non-cyclic Sylow subgroups of $H$ are either semi-p-cover-avoiding or $s$-quasinormally embedded in $G$;
(ii) all maximal subgroups of all non-cyclic Sylow subgroups of $F^*(H)$ are either semi-p-cover-avoiding or $s$-quasinormally embedded in $G$.

Lemma 2.7 ([4, X.13]). Let $G$ be a group and $M$ a subgroup of $G$.
(i) If $M$ is normal in $G$, then $F^*(M) \leq F^*(G)$;
(ii) $F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \text{Soc}(F(G)C_G(F(G))/F(G))$;
(iii) $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.

3. Main results

Theorem 3.1. Let $G$ be a group and $p$ a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. Let $P$ be a Sylow $p$-subgroup of $G$. If all maximal subgroups of $P$ are either semi-p-cover-avoiding or ss-quasinormal subgroups in $G$, then $G$ is $p$-nilpotent.
Proof. If all maximal subgroups of $P$ are semi-$p$-cover-avoiding in $G$, then $G$ is $p$-nilpotent by Lemma 2.3. Hence there exists a maximal subgroup $P_1$ of $P$ such that $P_1$ is ss-quasinormal in $G$. Firstly, fix an $H$ which is a maximal subgroup of $P$ such that $H$ is ss-quasinormal in $G$.

Now we prove that there exists a Hall $p'$-subgroup $K$ of $G$ such that $HK$ is a subgroup of index $p$ in $G$.

By conditions, there is a subgroup $B \leq G$ such that $G = HB$ and $HX = XH$ for all $X \in \text{Syl}(B)$, and $H \cap B$ is of index $p$ in $B_p$, a Sylow $p$-subgroup of $B$ containing $H \cap B$. Thus $S \notin H$ and $S \cap H = B \cap H$ for all $S \in \text{Syl}_p(B)$. So $B \cap H = \bigcap_{b \in B} (S^b \cap H) \leq \bigcap_{b \in B} S^b = O_p(B)$.

We claim that $B$ has a Hall $p'$-subgroup. Because $|O_p(B) : B \cap H| = p$ or 1, it follows that $|B/O_p(B)| = p$ or 1. As $(|G|, p - 1) = 1$, then $B/O_p(B)$ is $p$-nilpotent by Lemma 2.4, and hence $B$ is $p$-solvable. So $B$ has a Hall $p'$-subgroup. Thus the claim holds. Now, let $K$ be a Hall $p'$-subgroup of $B$. $\pi(K) = \{p_2, \ldots, p_s\}$ and $P_i \in \text{Syl}_{p_i}(K)$. By the conditions, $H$ is ss-quasinormal in $G$, so $H$ permute with subgroup $(P_2, \ldots, P_s) = K$ and $HK \leq G$. Moreover, $[G : HK] = p$ as desired.

Now, for every $H_i$ which is a maximal subgroup of $P$ ($H_i$ is ss-quasinormal in $G$), there exists a Hall $p'$-subgroup $K_i$ of $G$ such that $M_i = H_iK_i$, which is a subgroup of index $p$ in $G$. As $(|G|, p - 1) = 1$, by Lemma 2.4, $M_i \leq G$. Obviously, $H_i$ is s-quasinormally embedded in $G$. Thus every maximal subgroup of $G$ is either semi-$p$-cover-avoiding or s-quasinormally embedded in $G$. By Lemma 2.3, $G$ is $p$-nilpotent.

Corollary 3.2. Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ a Sylow $p$-subgroup of $G$. If all maximal subgroups of $P$ are semi-$p$-cover-avoiding or ss-quasinormal subgroups in $G$, then $G$ is $p$-nilpotent.

Corollary 3.3. Suppose that $G$ is a group. If all maximal subgroups of all Sylow subgroups of $G$ are either semi-$p$-cover-avoiding or ss-quasinormal in $G$, then $G$ has Sylow tower of supersolvable type.

Proof. It is clear that $(|G|, p - 1) = 1$, if $p$ is the smallest prime dividing $|G|$. By the hypothesis, all maximal subgroups of all Sylow subgroups of $G$ are either semi-$p$-cover-avoiding or ss-quasinormal in $G$, so $G$ satisfies the condition of Theorem 3.1, and hence $G$ is $p$-nilpotent. Let $U$ be the normal $p$-complement of $G$, then $U$ satisfies the condition by induction, hence $G$ possesses Sylow tower property of supersolvable type.

Theorem 3.4. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If for every prime $p$ dividing $|H|$ and $P \in \text{Syl}_p(H)$, If all maximal subgroups of $P$ are either semi-$p$-cover-avoiding or ss-quasinormal in $G$, then $G \in \mathcal{F}$.

Proof. Assume that the theorem is not true and let $G$ be a minimal counter-example.
(1) $H$ has minimal normal subgroup $H_1$, $H_1 \leq Q \leq H$, $Q \in \text{Syl}_q(H)$ and $q$ is the largest prime in $\pi(H)$.

Obviously, $H$ satisfies the condition of Corollary 3.3, so $H$ possesses Sylow tower property of supersolvable type. Let $q$ is the largest prime dividing $|H|$ and $Q$ is a Sylow $q$-subgroup of $H$, then $Q \subseteq H$, so $H$ has minimal normal subgroup $H_1$, $H_1 \leq Q$ and $H_1$ is an elementary abelian $q$-group, as desired.

(2) $G/H_1 \in \mathcal{F}$, $H_1 \notin \Phi(G)$, $H_1 = Q \in \text{Syl}_q(H)$.

Obviously, $G/H_1 \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, so $H_1$ is the unique minimal normal subgroup of $G$ containing in $H$, $H_1 \notin \Phi(G)$. Moreover, $H_1 = F(H)$. Since $H$ is solvable, so $C_H(H_1) \leq F(H)$ and $C_H(H_1) = H_1 = F(H)$. Since $Q \leq H$, $Q \leq F(H)$, thus $H_1 = Q \in \text{Syl}_q(H)$.

(3) The final contradiction.

For any maximal subgroup $Q_1$ of $Q$, $Q_1$ is either semi-$p$-cover-avoiding or ss-quasinormal in $G$ by (2) and the hypothesis. Thus $Q_1$ is either semi-$p$-cover-avoiding or $s$-quasinormally embedded in $G$ by Lemma 2.5. Hence $G \in \mathcal{F}$ by Lemma 2.6(i). We get the final contradiction. □

**Corollary 3.5.** Let $G$ be a group, $H$ a normal subgroup of $G$ such that $G/H$ is supersolvable. If all maximal subgroups of all Sylow subgroups of $H$ are either semi-$p$-cover-avoiding or ss-quasinormal in $G$, then $G$ is supersolvable.

**Theorem 3.6.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a solvable normal subgroup $H$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are either semi-$p$-cover-avoiding or ss-quasinormal in $G$, then $G \in \mathcal{F}$.

**Proof.** As $H$ is solvable, by Lemma 2.7, $F(H) = F^*(H)$. For any Sylow subgroup $P$ of $F(H)$ and for any maximal subgroup $P_1$ of $P$, if $P_1$ is ss-quasinormal in $G$, then $P_1$ is $s$-quasinormally embedded in $G$ by Lemma 2.5. Applying Lemma 2.6(ii), we can get $G \in \mathcal{F}$. □

**Theorem 3.7.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F^*(H)$ are either semi-$p$-cover-avoiding or ss-quasinormal in $G$, then $G \in \mathcal{F}$.

**Proof.** Suppose that the theorem is false and let $G$ be a minimal counter-example.

Case 1. $\mathcal{F} = \mathcal{U}$.

Let $G$ be a minimal counter-example.

(1) Every proper normal subgroup $N$ of $G$ containing $F^*(H)$ is supersolvable.

Since $N/N \cap H \cong NH/H$ is supersolvable, we get $F^*(H) = F^*(F^*(H)) \leq F^*(N \cap H) \leq F^*(H)$ by Lemma 2.7. So $F^*(H) = F^*(N \cap H)$ and $N$, $N \cap H$ satisfy the hypothesis of the theorem. Hence $N$ is supersolvable by the minimal choice of $G$. 

(2) \( H = G \) and \( 1 \neq F^*(G) = F(G) < G \).

If \( H < G \), then \( H \) is supersolvable as \( H \) contains \( F^*(H) \) and \( F^*(H) = F(H) \), it follows that \( G \) is supersolvable by Theorem 3.6, a contradiction.

If \( F^*(G) = G \), then \( G \) is supersolvable by applying Corollary 3.5, a contradiction. Thus \( F^*(G) < G \), it is supersolvable by (1), so \( F^*(G) = F(G) \neq 1 \) by Lemma 2.7.

(3) The final contradiction.

For any Sylow subgroup \( P \) of \( F^*(H) \) and for any maximal subgroup \( P_1 \) of \( P \), \( P_1 \) is either semi-p-cover-avoiding or ss-quasinormal in \( G \) by the hypothesis. As \( P_1 \leq F(G) \), so \( P_1 \) is either semi-p-cover-avoiding or s-quasinormally embedded in \( G \) by Lemma 2.5. Applying Lemma 2.6(ii), we can get \( G \) is supersolvable, the final contradiction.

Case 2. \( \mathcal{F} \neq \emptyset \).

By Case 1, \( H \) is supersolvable, so \( H \) is solvable and \( F^*(H) = F(H) \) by Lemma 2.7. Then \( G \in \mathcal{F} \) by Theorem 3.6.

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211–223.

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