ON THE STABILITY OF RADICAL FUNCTIONAL EQUATIONS IN QUASI-\(\beta\)-NORMED SPACES

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability results controlled by considering approximately mappings satisfying conditions much weaker than Hyers and Rassias conditions for radical quadratic and radical quartic functional equations in quasi-\(\beta\)-normed spaces.

1. Introduction

In 1960, the stability problem of functional equations originated from the question of Ulam [44] concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [22] in Banach spaces. Hyers’s theorem was generalized by Aoki [2] for additive mapping and by Rassias [31] for linear mapping by considering unbounded Cauchy differences. Rassias [32], [35] provided a generalization of Hyers’ theorem by proving the existence of unique linear mappings near approximate additive mappings. On the other hand, Rassias [36], [37] considered the Cauchy difference controlled by a product of different powers of norm. The above results has been generalized by Forti [13] and Gåvruta [15] who permitted the Cauchy difference to become arbitrary unbounded. Gajda and Ger [14] showed that one can get analogous stability results for subadditive multifunctions. Gruber [21] remarked that Ulam’s problem is of particular interest in probability theory and in the case of functional equations of different types. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings in various spaces ([1], [3]-[10], [16], [17], [24], [26], [34], [40], [41]).

The quadratic function \(f(x) = cx^2\) satisfies the functional equation

\[
(f) \quad f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]
and therefore the equation $(E)$ is called the \textit{quadratic functional equation}. The Hyers-Ulam stability theorem for the quadratic functional equation was proved by Skof [42] and Czerwik [12]. Since then, the stability problem of various quadratic functional equations have been extensively investigated by a number of authors (\cite{11, 18, 20, 23, 27, 29, 30, 33, 39}).

Before we present our results, we introduce some basic facts concerning quasi-$\beta$-normed space and some preliminary results. We fix a real number $\beta$ with $0 < \beta \leq 1$ and let $K$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $X$ be a linear space over $K$. A \textit{quasi-$\beta$-norm} $\| \cdot \|$ is a real-valued function on $X$ satisfying the following:

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
2. $\|\lambda x\| = |\lambda|^{\beta} \cdot \|x\|$ for all $\lambda \in K$ and $x \in X$;
3. there exists a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \| \cdot \|)$ is called a \textit{quasi-$\beta$-normed space} if $\| \cdot \|$ is a quasi-$\beta$-norm on $X$. The smallest possible $K$ is called the \textit{module of concavity} of $\| \cdot \|$. A \textit{quasi-$\beta$-Banach space} is a complete quasi-$\beta$-normed space.

A quasi-$\beta$-norm $\| \cdot \|$ is called a ($\beta, p$)-\textit{norm} ($0 < p \leq 1$) if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasi-$\beta$-Banach space is called a ($\beta, p$)-\textit{Banach space}. For further details on quasi-$\beta$-normed spaces and ($\beta, p$)-Banach spaces, refer to the papers \cite{19, 25, 28, 38} and \cite{43}.

Recall that a function $\varphi : A \to B$ with a domain $A$ and a codomain $(B, \leq)$ which is closed under the addition is a \textit{subadditive (superadditive) function} if $\varphi(x + y) \leq (\geq) \varphi(x) + \varphi(y)$ and a \textit{subquadratic (superquadratic) function} with $\varphi(0) = 0$ if $\varphi(x + y) + \varphi(x - y) \leq (\geq) 2\varphi(x) + 2\varphi(y)$ for all $x, y \in A$, respectively.

Let $\ell \in \{-1, 1\}$ be fixed. If there exists a constant $L$ with $0 < L < 1$ such that a function $\varphi : A \to B$ satisfies

$$\ell \varphi(x + y) \leq \ell L^\ell (\varphi(x) + \varphi(y))$$

for all $x, y \in A$, then we say that $\varphi$ is \textit{contractively subadditive} if $\ell = 1$ and $\varphi$ is \textit{expansively superadditive} if $\ell = -1$. Similarly, if there exists a constant $L$ with $0 < L < 1$ such that a function $\varphi : A \to B$ with $\varphi(0) = 0$ satisfies

$$\ell \varphi(x + y) + \ell \varphi(x - y) \leq 2\ell L^\ell (\varphi(x) + \varphi(y))$$

for all $x, y \in A$, then we say that $\varphi$ is \textit{contractively subquadratic} if $\ell = 1$ and $\varphi$ is \textit{expansively superquadratic} if $\ell = -1$.

In this paper, we point out the generalized Hyers-Ulam stability results controlled by approximately mappings for the radical quadratic and radical quartic functional equations which is introduced in \cite{27},

\begin{equation}
(1.1) \quad f(\sqrt{ax^2 + by^2}) = af(x) + bf(y)
\end{equation}

and

\begin{equation}
(1.2) \quad f\left(\sqrt{ax^2 + by^2}\right) + f\left(\sqrt{ax^2 - ay^2}\right) = 2a^2 f(x) + 2b^2 f(y)
\end{equation}
in quasi-$\beta$-Banach spaces, and new theorems about the generalized Hyers-Ulam stability by using subadditive and subquadratic functions for those functional equations in $(\beta,p)$-Banach spaces.

2. Stability of the radical quadratic functional equation (1.1)

In this section, we are modified the generalized Hyers-Ulam stability of radical functional equations (1.1) in quasi-$\beta$-normed spaces and $(\beta,p)$-Banach spaces, respectively.

Let $X$ be a normed space and $\phi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ be a function. A function $f : \mathbb{R} \to X$ is called a $\phi$-approximately radical quadratic function if

$$
\left\|f(\sqrt{ax^2 + by^2}) - af(x) - bf(y)\right\|_X \leq \phi(x,y)
$$

for all $x, y \in \mathbb{R}$, where $a, b \in \mathbb{R}^+$ are such that $a + b \neq 1$.

First, using the idea of Gavruta, we prove the generalized Hyers-Ulam stability of radical functional equations (1.1) in the spirit of Ulam, Hyers and Rassias.

**Theorem 2.1.** Let $X$ be a quasi-$\beta$-Banach space and $f : \mathbb{R} \to X$ be a $\phi$-approximately radical quadratic function with $f(0) = 0$. If a function $\phi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ satisfies the following:

$$
\sum_{j=0}^{\infty} \frac{K}{2^j} \left(\phi\left(0, \sqrt{\frac{a}{b}} 2^j x\right) + \phi\left(2^j x, \sqrt{\frac{a}{b}} 2^j x\right) + \phi\left(2^j x, 0\right) + \phi\left(2^{j+1} x, 0\right)\right) < \infty,
$$

and

$$
\lim_{n \to \infty} \frac{1}{2^{3n+1}} \phi\left(2^n x, 2^n y\right) = 0
$$

for all $x, y \in \mathbb{R}$, then there exists a unique quadratic mapping $F : \mathbb{R} \to X$ satisfying the functional equation (1.1) and the following inequality:

$$
\left\|f(x) - F(x)\right\|_X \leq K^3 2^{2\beta} \sum_{j=0}^{\infty} \frac{K}{2^j} \left(\phi\left(0, \sqrt{\frac{a}{b}} 2^j x\right) + \phi\left(2^j x, \sqrt{\frac{a}{b}} 2^j x\right) + \phi\left(2^j x, 0\right) + \phi\left(2^{j+1} x, 0\right)\right)
$$

for all $x \in \mathbb{R}$.

**Proof.** Replacing $x$ and $y$ with $\frac{x}{\sqrt{a}}$ and $\frac{y}{\sqrt{b}}$ in (2.1), respectively, we get

$$
\left\|f(\sqrt{x^2 + y^2}) - af\left(\frac{x}{\sqrt{a}}\right) - bf\left(\frac{y}{\sqrt{b}}\right)\right\|_X \leq \phi\left(\frac{x}{\sqrt{a}}, \frac{y}{\sqrt{b}}\right)
$$

for all $x, y \in \mathbb{R}$. Setting $x = 0$ and $y = 0$ in (2.5), respectively, we get

$$
\left\|f(\sqrt{y^2}) - bf\left(\frac{y}{\sqrt{b}}\right)\right\|_X \leq \phi\left(0, \frac{y}{\sqrt{b}}\right), \quad \left\|f(\sqrt{x^2}) - af\left(\frac{x}{\sqrt{a}}\right)\right\|_X \leq \phi\left(\frac{x}{\sqrt{a}}, 0\right)
$$
for all $x, y \in \mathbb{R}$. Then we obtain

\[(2.6) \quad \left\| f(x) - \frac{b}{a} f\left(\sqrt[2]{\frac{a}{b}} x\right)\right\|_X \leq \frac{K}{a^3} \left(\phi(x, 0) + \phi(0, \sqrt[2]{\frac{a}{b}} x)\right)\]

for all $x \in \mathbb{R}$. Also, substituting $x$ and $y$ for $\frac{x+y}{\sqrt{2a}}$ and $\frac{x-y}{\sqrt{2b}}$ in (2.1), respectively, we get

\[(2.7) \quad \left\| f\left(\sqrt{2} x^2 + y^2\right) - a f\left(\frac{x+y}{\sqrt{2a}}\right) - b f\left(\frac{x-y}{\sqrt{2b}}\right)\right\|_X \leq \phi\left(\frac{x+y}{\sqrt{2a}}, \frac{x-y}{\sqrt{2b}}\right)\]

for all $x, y \in \mathbb{R}$. It follows from (2.5) and (2.7) that

\[(2.8) \quad \left\| f\left(\frac{x+y}{\sqrt{2a}}\right) + \frac{b}{a} f\left(\frac{x-y}{\sqrt{2b}}\right) - f\left(\frac{x}{\sqrt{a}}\right) - \frac{b}{a} f\left(\frac{y}{\sqrt{b}}\right)\right\|_X \leq \frac{K}{a^3} \left(\phi\left(\frac{x}{\sqrt{a}}, \frac{y}{\sqrt{b}}\right) + \phi\left(\frac{x+y}{\sqrt{2a}}, \frac{x-y}{\sqrt{2b}}\right)\right)\]

for all $x, y \in \mathbb{R}$. Letting $x = y = \sqrt{\alpha} x$ in (2.8), we get

\[(2.9) \quad \left\| f(\sqrt{2} x) - f(x) - \frac{b}{a} f\left(\sqrt[2]{\frac{a}{b}} x\right)\right\|_X \leq \frac{K}{a^3} \left(\phi(x, \sqrt[2]{\frac{a}{b}} x) + \phi(\sqrt{2} x, 0)\right)\]

for all $x \in \mathbb{R}$. It follows from (2.6) and (2.9) that

\[(2.10) \quad \left\| f(x) - \frac{1}{2} f(\sqrt{2} x)\right\|_X \leq \frac{K^2}{(2a)^2} \left(\phi(x, 0) + \phi(\sqrt{2} x, 0) + \phi(0, \sqrt[2]{\frac{a}{b}} x) + \phi(x, \sqrt[2]{\frac{a}{b}} x)\right).\]

Let $\Phi(x) = \frac{K^2}{(2a)^2} \left(\phi(x, 0) + \phi(\sqrt{2} x, 0) + \phi(0, \sqrt[2]{\frac{a}{b}} x) + \phi(x, \sqrt[2]{\frac{a}{b}} x)\right)$. Then, by the iterative method, we get

\[(2.11) \quad \left\| f(x) - \frac{1}{2^m} f(2^m x)\right\|_X \leq K \sum_{j=0}^{m-1} \left(\frac{K}{2}\right)^j \Phi(2^j x)\]

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. For all $k, m \in \mathbb{Z}^+$ with $m > k \geq 0$, we have

\[(2.12) \quad \left\| \frac{1}{2k} f(2^k x) - \frac{1}{2^m} f(2^m x)\right\|_X \leq K \sum_{j=k}^{m-1} \left(\frac{K}{2}\right)^j \Phi(2^j x)\]

for all $x \in \mathbb{R}$. By (2.2) and (2.12), the sequence $\{\frac{1}{2^m} f(2^m x)\}$ is a Cauchy sequence for all $x \in \mathbb{R}$. Since $X$ is the quasi-$\beta$-Banach space, it converges for all $x \in \mathbb{R}$. We can define a mapping $F : \mathbb{R} \rightarrow X$ by $F(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$ for all $x \in \mathbb{R}$. Then, by (2.2)

\[\left\| F(\sqrt{ax + by}) - a F(x) - b F(y)\right\|_X \leq \lim_{n \rightarrow \infty} \frac{1}{2^m} \phi(2^x x, f(2^y y) = 0\]

and $F(\sqrt{ax + by}) - a F(x) - b F(y) = 0$, that is, $F$ is a quadratic mapping [27]. Taking $m \rightarrow \infty$ in (2.12) with $k = 0$, it follows that $F$ satisfies (2.4) near the approximate function $f$ of (1.1).
Next, we assume that there exists another quadratic mapping \( G : \mathbb{R} \rightarrow X \) which satisfies the functional equations (1.1) and (2.4). Since \( G \) satisfies (1.1), we have \( G(2^\frac{a}{2}x) = 2^n G(x) \) for all \( x \in X \) and \( n \in \mathbb{Z}^+ \). Thus we get
\[
\left\| \frac{1}{2^n} f(2^\frac{a}{2}x) - G(x) \right\|_X \leq \frac{1}{2^{3n}} \Phi (2^\frac{a}{2}x)
\]
for all \( x \in \mathbb{R} \). Letting \( n \rightarrow \infty \), we establishes \( F(x) = G(x) \) for all \( x \in \mathbb{R} \). This completes the proof. \( \square \)

From Theorem 2.1, we obtain the following corollary concerning the stability for approximate mappings controlled by a sum of powers of norms and a product of powers of norms.

**Corollary 2.2.** Let \( X \) be a quasi-\( \beta \)-Banach space, let \( p, q \in \mathbb{R}^+ \cup \{0\}, \varepsilon \geq 0 \) and \( f : \mathbb{R} \rightarrow X \) be a function satisfying the following:
\[
\left\| f(\sqrt{ax^2 + by^2}) + af(x) - bf(y) \right\|_X \leq \left\{ \begin{array}{ll}
\varepsilon |x|^p |y|^q, & p + q < 2(\beta - \log_2 K) \\
\varepsilon (|x|^p + |y|^q), & p, q < 2(\beta - \log_2 K)
\end{array} \right.
\]
for all \( x, y \in \mathbb{R} \). If a function \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \cup \{0\} \) satisfies (2.2) and (2.3), then there exists a unique quadratic mapping \( F : \mathbb{R} \rightarrow X \) satisfying the functional equation (1.1) and the following inequality:
\[
\left\| f(x) - F(x) \right\|_X \leq \left\{ \begin{array}{ll}
x^p & p + q < 2(\beta - \log_2 K) \\
x^p \left( \frac{(2+2\beta)^2 |x|^p}{1-K^2 x^2} \right) & p, q < 2(\beta - \log_2 K)
\end{array} \right.
\]
for all \( x \in \mathbb{R} \).

**Theorem 2.3.** Let \( X \) and \( f \) be same as Theorem 2.1. If a function \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \cup \{0\} \) satisfies the following:
\[
\sum_{j=1}^{\infty} (2^j K)^j \left( \phi(0, \sqrt{\frac{a}{b}} 2^{-\frac{j}{2}} x) + \phi(2^{-\frac{j}{2}} x, \sqrt{\frac{a}{b}} 2^{-\frac{j}{2}} x) + \phi(2^{-\frac{j}{2}} x, 0) + \phi(2^{-\frac{j}{2}} x, 0) \right) < \infty
\]
and
\[
\lim_{n \rightarrow \infty} 2^{3n} \phi(2^{-\frac{n}{2}} x, 2^{-\frac{n}{2}} y) = 0
\]
for all \( x, y \in \mathbb{R} \), then there exists a unique quadratic mapping \( F : \mathbb{R} \rightarrow X \) satisfying the functional equation (1.1) and the following inequality:
\[
(2.13)
\]
\[
\left\| f(x) - F(x) \right\|_X \leq \frac{K^2}{2^{2n} a^2} \sum_{j=1}^{\infty} (2^j K)^j \left( \phi(0, \sqrt{\frac{a}{b}} 2^{-\frac{j}{2}} x) + \phi(2^{-\frac{j}{2}} x, \sqrt{\frac{a}{b}} 2^{-\frac{j}{2}} x) + \phi(2^{-\frac{j}{2}} x, 0) + \phi(2^{-\frac{j}{2}} x, 0) \right)
\]
for all \( x \in \mathbb{R} \).
Proof. If \( x \) is replaced with \( \frac{x}{2} \) in (2.10), then the proof follows from the proof of Theorem 2.1. \( \square \)

**Corollary 2.4.** Let \( X, p, q \) and \( \varepsilon \geq 0 \) be as Corollary 2.2. If a function \( f : \mathbb{R} \rightarrow X \) satisfies the following inequality:

\[
\left\| f(\sqrt{ax^2 + by^2}) + af(x) - bf(y) \right\|_X \leq \begin{cases} \varepsilon |x|^p |y|^q, & 2(\beta + \log_2 K) < p + q; \\ \varepsilon (|x|^p + |y|^q), & 2(\beta + \log_2 K) < p, q \\
\end{cases}
\]

for all \( x, y \in \mathbb{R} \), then there exists a unique quadratic mapping \( F : \mathbb{R} \rightarrow X \) satisfying the functional equation (1.1) and the following inequality:

\[
\left\| f(x) - F(x) \right\|_X \leq \begin{cases} \frac{K^3\Phi(x)}{2(1 + \beta)} \cdot \left( \frac{\left(2 + \varepsilon\beta\right)x^p}{2\alpha^\beta - K} \right), & 2(\beta + \log_2 K) < p, q; \\
\frac{K^3\Phi(x)}{2(1 + \beta)} \cdot \left( \frac{\left(2 + \varepsilon\beta\right)x^p}{2\alpha^\beta - K} \right), & 2(\beta + \log_2 K) < p, q \\
\end{cases}
\]

for all \( x \in \mathbb{R} \).

Now, we investigate the generalized Hyers-Ulam stability of radical functional equations (1.1) in \((\beta, p)\)-Banach spaces using contractively subadditive and expansively superadditive.

**Theorem 2.5.** Let \( X \) be a \((\beta, p)\)-Banach space and \( f : \mathbb{R} \rightarrow X \) be a \( \phi \)-approximatively radical quadratic function with \( f(0) = 0 \). Assume that the function \( \phi \) is contractively subadditive with a constant \( L \) satisfying \( 2^{1-\beta} L < 1 \). Then there exists a unique quadratic mapping \( F : \mathbb{R} \rightarrow X \) satisfying the functional equation (1.1) and the following inequality:

\[
\left( \mathcal{E} \right) \left\| f(x) - F(x) \right\|_X \leq \frac{\hat{\Phi}(x)}{\sqrt[4]{(4a)^{3p} - (2a^\beta L)^p}} 
\]

for all \( x \in \mathbb{R} \), where

\[
\hat{\Phi}(x) = \phi(x, 0) + \phi(\sqrt{2}x, 0) + \phi(0, \sqrt{b^\beta x}) + \phi\left( x, \frac{\sqrt{a}}{\sqrt{b}} x \right)
\]

and

\[
\hat{\Phi}(x) = K^3(2\beta \hat{\Phi}(x) + \hat{\Phi}(\sqrt{2}x)).
\]

Proof. It follows from (2.10) in the proof of Theorem 2.1 that

\[
\left\| 2f(x) - f(\sqrt{2}x) \right\|_X \leq \frac{K^2}{\alpha^p} \left( \phi(x, 0) + \phi(\sqrt{2}x, 0) + \phi(0, \sqrt{b^\beta x}) + \phi\left( x, \frac{\sqrt{a}}{\sqrt{b}} x \right) \right).
\]

Let \( \hat{\Phi}(x) = \phi(x, 0) + \phi(\sqrt{2}x, 0) + \phi(0, \sqrt{b^\beta x}) + \phi(x, \sqrt{b^\beta x}) \). Then we obtain

\[
\left\| f(x) - \frac{1}{4} f(2x) \right\|_X \leq \frac{1}{(4a)^\beta} \hat{\Phi}(x),
\]
where \( \hat{\Phi}(x) = K^{3}(2^{2} \hat{\Phi}(x) + \hat{\Phi}(\sqrt{2}x)) \). It follows from (2.16) with \( 2^{j}x \) in the place of \( x \) and the iterative method that

\[
\left\| \frac{1}{4^k} f(2^k x) - \frac{1}{4^m} f(2^m x) \right\|_X^p \leq \sum_{j=k}^{m-1} \frac{1}{4^{j\beta p}} \left\| f(2^j x) - \frac{1}{4} f(2^{j+1} x) \right\|_X^p
\]

\[
< \frac{1}{(4a)^{2p}} \sum_{j=k}^{m-1} \frac{1}{4^{j\beta p}} \hat{\Phi}(2^j x)^p
\]

\[
< \left( \frac{\hat{\Phi}(x)}{4^{\beta p}} \right)^p \sum_{j=k}^{m-1} (2^{1-2\beta L})^{jp}
\]

for all \( x \in \mathbb{R} \) and \( m, k \in \mathbb{Z}^+ \) with \( m > k \geq 0 \). Then the sequence \( \left\{ \frac{1}{4^m} f(2^m x) \right\} \) is a Cauchy sequence in a \((\beta, p)\)-Banach space \( X \) and so we can define a mapping \( F: \mathbb{R} \to X \) by

\[
F(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x)
\]

for all \( x \in \mathbb{R} \). Then we get

\[
\left\| F(\sqrt{ax^2 + by^2}) - aF(x) - bF(y) \right\|_X^p \leq \phi(x, y)^p \lim_{n \to \infty} (2^{1-2\beta L})^{np} = 0
\]

for all \( x, y \in \mathbb{R} \). Then \( F(\sqrt{ax^2 + by^2}) - aF(x) - bF(y) = 0 \), that is, \( F \) is a quadratic mapping. Taking \( m \to \infty \) in (2.17) with \( k = 0 \), we can show that \( F \) satisfies (2.14) near the approximate function \( f \) of the functional equation (1.1).

Next, we assume that there exists another quadratic mapping \( G: \mathbb{R} \to X \) which satisfies the functional equation (1.1) and (2.14). Then we have

\[
\left\| G(x) - \frac{1}{4^n} f(2^n x) \right\|_X^p \leq \frac{\hat{\Phi}(x)^p}{(4a)^{2p} - (2\alpha^2 L)^p} (2^{1-2\beta L})^{np}
\]

for all \( x \in \mathbb{R} \) and \( n \in \mathbb{Z}^+ \). Letting \( n \to \infty \), the uniqueness of \( F \) follows. This completes the proof. \( \square \)

**Theorem 2.6.** Let \( X, f, \hat{\Phi}(x) \) be same as in Theorem 2.5. Assume that the function \( \phi \) is expansively superadditive with a constant \( L \) satisfying \( 2^{2\beta - 1} L < 1 \). Then there exists a unique quadratic mapping \( F: \mathbb{R} \to X \) satisfying the functional equation (1.1) and the following inequality:

\[
\left\| f(x) - F(x) \right\|_X \leq \frac{\hat{\Phi}_2(x)}{\sqrt{(2\alpha^2 L^{-1})^p - (2a)^{2p}}} \left( \hat{\Phi}(2^{-1} x) + 2^3 \hat{\Phi}(2^{-1} x) \right)
\]

for all \( x \in \mathbb{R} \), where \( \hat{\Phi}_2(x) = K^{3} \left( \hat{\Phi}(2^{-1} x) + 2^3 \hat{\Phi}(2^{-1} x) \right) \).
Proof. It follows from (2.15) of the proof of Theorem 2.5 that

\[ ||f(x) - 4f(2^{-1}x)||_X \leq \frac{K^3}{a^3} \left( \frac{\Phi(2^{-\frac{1}{2}}x) + 2^2\Phi(2^{-1}x)}{a} \right) \]

for all \( x \in \mathbb{R} \). Then, in (2.19), replacing \( x \) by \( 2^{-j}x \) and using the iterative method, we have

\[ ||4^k f(2^{-k}x) - 4^m f(2^{-m}x)||_X \leq \left( \frac{\Phi_2(x)}{a^3} \right)^p \sum_{j=k}^{m-1} (2^{2\beta-1}L)^j \]

for all \( x \in \mathbb{R} \) and \( k, m \in \mathbb{Z}^+ \) with \( m > k \geq 0 \). The remains follow the proof of Theorem 2.5. This completes the proof. \( \square \)

3. Stability of the radical quartic functional equation (1.2)

In this section, we are modified the generalized Hyers-Ulam stability of radical functional equations (1.2) in quasi-\( \beta \)-normed spaces and \( (\beta, p) \)-Banach spaces, respectively.

Let \( X \) be a normed space and \( \psi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{ 0 \} \) be a function. A function \( f : \mathbb{R} \to X \) is called a \( \psi \)-approximatively radical quartic function if

\[ ||f(\sqrt{ax^2 + by^2}) + f(\sqrt{ax^2 - by^2}) - 2a^2 f(x) - 2b^2 f(y)||_X \leq \psi(x, y) \]

for all \( x, y \in \mathbb{R} \), where \( a, b \in \mathbb{R}^+ \) are fixed with \( a^2 + b^2 \neq 1 \).

First, we prove the generalized Hyers-Ulam stability of the radical functional equations (1.2) in quasi-\( \beta \)-normed spaces using the idea of Gavruta.

**Theorem 3.1.** Let \( X \) be a quasi-\( \beta \)-Banach space and \( f : \mathbb{R} \to X \) be a \( \psi \)-approximatively radical quartic function with \( f(0) = 0 \). If a mapping \( \psi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{ 0 \} \) satisfy the following:

\[ \sum_{j=0}^{\infty} \left( \frac{K}{4a^p} \right)^j \psi(0, \sqrt{\frac{a}{b} 2^j x}) + \psi\left( 2^j x, \sqrt{\frac{a}{b} 2^j x} \right) + \psi\left( 2^j x, 0 \right) + \frac{1}{2^j} \psi\left( 2^{j+1} x, 0 \right) < \infty, \]

and

\[ \lim_{n \to \infty} \frac{1}{2^{jn}} \psi\left( 2^{jn} x, 2^{jn} y \right) = 0 \]

for all \( x, y \in \mathbb{R} \), then there exists a unique quartic mapping \( H : \mathbb{R} \to X \) satisfying the functional equation (1.2) and the following inequality:

\[ ||f(x) - H(x)||_X \leq \frac{K^3}{(4a^2)^p} \sum_{j=0}^{\infty} \left( \frac{K}{4a^p} \right)^j \psi(0, \sqrt{\frac{a}{b} 2^j x}) + \psi\left( 2^j x, \sqrt{\frac{a}{b} 2^j x} \right) + \psi\left( 2^j x, 0 \right) + \frac{1}{2^j} \psi\left( 2^{j+1} x, 0 \right) \]

for all \( x \in \mathbb{R} \).
Proof. Replacing $x$ and $y$ with $\frac{x}{\sqrt{a}}$ and $\frac{y}{\sqrt{b}}$ in (3.1), respectively, we get
\begin{equation}
\left\| f(\sqrt{x^2 + y^2}) + f(\sqrt{|x^2 - y^2|}) - 2a^2 f\left(\frac{x}{\sqrt{a}}\right) - 2b^2 f\left(\frac{y}{\sqrt{b}}\right) \right\|_X \leq \psi\left(\frac{x}{\sqrt{a}}, \frac{y}{\sqrt{b}}\right)
\end{equation}
for all $x, y \in \mathbb{R}$. Setting $x = y = \sqrt{ax}$ in (3.5), we get
\begin{equation}
\left\| f(\sqrt{2ax^2}) - 2a^2 f(x) - 2b^2 \left(\frac{\sqrt{a}}{b} x\right) \right\|_X \leq \psi\left(x, \frac{\sqrt{a}}{b} x\right)
\end{equation}
for all $x \in \mathbb{R}$. Replacing $x$ and $y$ with $\sqrt{2ax}$ and 0 in (3.5), respectively, we obtain
\begin{equation}
\left\| f(\sqrt{2ax^2}) - a^2 f(\sqrt{2}x) \right\|_X \leq \frac{1}{2a} \psi\left(\sqrt{2}x, 0\right)
\end{equation}
for all $x \in \mathbb{R}$. It follows from (3.6) and (3.7) that
\begin{equation}
a^2 f(\sqrt{2}x) - 2a^2 f(x) - 2b^2 f\left(\frac{\sqrt{a}}{b} x\right) \leq K \left(\psi\left(x, \frac{\sqrt{a}}{b} x\right) + \frac{1}{2b} \psi\left(\sqrt{2}x, 0\right)\right)
\end{equation}
for all $x \in \mathbb{R}$. Substituting $x = \sqrt{ax}$ and $y = 0$ in (3.5), we get
\begin{equation}
\left\| 2f(\sqrt{ax^2}) - 2a^2 f(x) \right\|_X \leq \psi\left(x, 0\right)
\end{equation}
for all $x \in \mathbb{R}$. Also, substituting $x = 0$ and $y = \sqrt{ax}$ in (3.5), we get
\begin{equation}
\left\| 2f(\sqrt{ax^2}) - 2b^2 f\left(\frac{\sqrt{a}}{b} x\right) \right\|_X \leq \psi\left(0, \frac{\sqrt{a}}{b} x\right)
\end{equation}
for all $x \in \mathbb{R}$. It follows from (3.9) and (3.10) that
\begin{equation}
\left\| 2b^2 f\left(\frac{\sqrt{a}}{b} x\right) - 2a^2 f(x) \right\|_X \leq K \left(\psi\left(x, 0\right) + \psi\left(0, \frac{\sqrt{a}}{b} x\right)\right)
\end{equation}
for all $x \in \mathbb{R}$. It follows from (3.8) and (3.11) that
\begin{equation}
\left\| f(x) - \frac{1}{4} f(2\frac{x}{\sqrt{a^2}}) \right\|_X \leq \frac{K^2}{(4a^2)^\beta} \left(\psi\left(0, \frac{\sqrt{a}}{b} x\right) + \psi\left(x, \frac{\sqrt{a}}{b} x\right) + \psi\left(x, 0\right) + \frac{1}{2b} \psi\left(2\frac{x}{\sqrt{a^2}}, 0\right)\right)
\end{equation}
for all $x \in \mathbb{R}$. Let $\Psi(x) = \psi(0, \sqrt{\beta} x) + \psi(x, \sqrt{\beta} x) + \psi(x, 0) + \frac{1}{2b} \psi(2\sqrt{\beta}, 0)$. Then, for all $m, k \in \mathbb{Z}^+$ with $m > k \geq 0$, we get
\begin{equation}
\left\| \frac{1}{4^m} f\left(2^\frac{m}{\sqrt{a^2}}x\right) - \frac{1}{4^k} f\left(2^\frac{k}{\sqrt{a^2}}x\right) \right\|_X \leq \frac{K^3}{(4a^2)^\beta} \sum_{j=k}^{m-1} \left(\frac{K}{2^j}\right)^j \Psi\left(2^\frac{j}{\sqrt{a^2}}x\right)
\end{equation}
for all $x \in \mathbb{R}$. From (3.2) and (3.13), the sequence $\{\frac{1}{4^j} f\left(2^\frac{j}{\sqrt{a^2}}x\right)\}$ is a Cauchy sequence for all $x \in \mathbb{R}$. Since $X$ is the $(\beta, p)$-Banach space $X$, it converges and
so we can define a mapping $\mathcal{H} : \mathbb{R} \to X$ by
\[ \mathcal{H}(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^{\frac{n}{4}}x) \]
for all $x \in \mathbb{R}$. The remains are similar to that of Theorem 2.1. This completes the proof. □

**Theorem 3.2.** Let $f : \mathbb{R} \to X$ be a $\psi$-approximatively radical quadratic function. If a mapping $\psi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ satisfies the following:
\[ \sum_{j=1}^{\infty} (4^j K)^j \left( \psi \left( 0, \sqrt[4]{2} x \right) \right) \leq \psi \left( \frac{1}{2^n} \psi \left( 2^{-\frac{n}{4}} x, 0 \right) \right) + 1 \]
for all $x, y \in \mathbb{R}$, then there exists a unique quadratic mapping $\mathcal{H} : \mathbb{R} \to X$ satisfying the functional equation (1.2) and the following inequality:
\[ \left\| f(x) - \mathcal{H}(x) \right\|_X \leq \frac{K^2}{(4\beta)^p} \sum_{j=1}^{\infty} (4^j K)^j \left( \psi \left( 0, \sqrt[4]{2} x \right) \right) \]
and
\[ \lim_{n \to \infty} 2^n \psi \left( 2^{-\frac{n}{4}} x, 2^{-\frac{n}{4}} y \right) = 0 \]
for all $x, y \in \mathbb{R}$. The remains are similar to that of Theorem 2.1. This completes the proof.

**Proof.** If $x$ is replaced with $\frac{x}{4\beta}$ in the inequality (3.12), then the proof follows from that of Theorem 3.1. □

**Corollary 3.3.** For any $p, q \in \mathbb{R}^+ \cup \{0\}$ and $\beta \geq 0$, if a function $f : \mathbb{R} \to X$ satisfies the following inequality:
\[ \left\| f(\sqrt{ax^2 + by^2}) + f(\sqrt{|ax^2 - by^2|}) - 2a^2 f(x) - 2b^2 f(y) \right\|_X \leq \begin{cases} \varepsilon |x|^p |y|^q, & p + q < 4\beta - 2 \log_2 K; \\ \varepsilon (|x|^p + |y|^q), & p, q < 4\beta - 2 \log_2 K \end{cases} \]
for all $x, y \in \mathbb{R}$, then there exists a unique quartic mapping $\mathcal{H} : \mathbb{R} \to X$ satisfying the functional equation (1.2) and the following inequality:
\[ \left\| f(x) - \mathcal{H}(x) \right\|_X \leq \begin{cases} \frac{\varepsilon K^4 \sqrt{\beta \psi(x)^{p+q}}}{a^{2\beta} (4^\beta - K \sqrt{2^{p+q}})^2}, & p + q < 4\beta - 2 \log_2 K; \\ \frac{\varepsilon K^4}{a^{2\beta}} \left( \frac{2(\sqrt{\beta} \psi(x)^p + 2 \psi(x)^q)}{4^\beta - K \sqrt{2^{p+q}}} \right), & p, q < 4\beta - 2 \log_2 K \end{cases} \]
for all $x \in \mathbb{R}$.

Now, we prove the generalized Hyers-Ulam stability of the radical functional equations (1.2) in $(\beta, p)$-Banach spaces using contractively subquadratic and expansively superquadratic.
Theorem 3.4. Let $X$ be a $(\beta,p)$-Banach space and $f : \mathbb{R} \rightarrow X$ be a $\psi$-approximatively radical quadratic function with $f(0) = 0$. Assume that the function $\psi$ is contractively subquadratic with a constant $L$ satisfying $2^{2-4\beta}L < 1$. Then there exists a unique quartic mapping $H : \mathbb{R} \rightarrow X$ satisfying the functional equation (1.2) and the following inequality:

$$
\|f(x) - H(x)\|_X \leq \frac{\hat{\Psi}(x)}{\sqrt{(16a^2)^{2p} - (4a^{2\beta})^p}}.
$$

where

$$
\hat{\Psi}(x) = \psi(x,0) + \psi\left(0, \sqrt{\frac{a}{b}}x\right) + \psi\left(x, \sqrt{\frac{a}{b}}x\right) + \frac{1}{2^{\beta}}\psi(\sqrt{2}x,0)
$$

and

$$
\hat{\Psi}(x) = K^3 \left(4^{2\beta} \hat{\Psi}(x) + \hat{\Psi}(\sqrt{2}x)\right)
$$

for all $x \in \mathbb{R}$.

Proof. Using (3.12) in the proof of Theorem 3.1, we have

$$
\left\|f(x) - \frac{1}{16}f(2x)\right\|_X \leq \frac{K^3(4^{2\beta} \hat{\Psi}(x) + \hat{\Psi}(\sqrt{2}x))}{(4a^{2\beta})} = \frac{\hat{\Psi}(x)}{(4a^{2\beta})}
$$

for all $x \in \mathbb{R}$. Then, in (3.16), replacing $x$ by $2^{-j}x$ and using the iterative method, we have

$$
\left\|\frac{1}{16^k}f(2^kx) - \frac{1}{16^m}f(2^mx)\right\|_X \leq \sum_{j=k}^{m-1} \left\|\frac{1}{16^{j+1}}f(2^{j+1}x)\right\|_X \leq \left(\frac{1}{4a^{2\beta}}\right)^{2\beta p} \sum_{j=k}^{m-1} \left(\frac{\hat{\Psi}(2^jx)}{4a}\right)^{2^{\beta p}j^p} \leq \left(\frac{\hat{\Psi}(x)}{4a}\right)^{2^{\beta p}m^p} \sum_{j=k}^{m-1} \left(2^{2-4\beta}L\right)^{jp}
$$

for all $x \in \mathbb{R}$ and $m,k \in \mathbb{Z}^+$ with $m > k \geq 0$. The sequence \(\left\{\frac{1}{16^n}f(2^n)\right\}\) is a Cauchy sequence for all $x \in \mathbb{R}$. Since $X$ is a $(\beta,p)$-Banach space, it converges for all $x \in \mathbb{R}$. Then we can define a mapping $H : \mathbb{R} \rightarrow X$ by

$$
H(x) := \lim_{n \rightarrow \infty} \frac{1}{16^n}f(2^n x)
$$

for all $x \in \mathbb{R}$. The remains are similar to the proof of Theorem 2.5. This completes the proof. \(\square\)

Theorem 3.5. Let $X, f, \hat{\Psi}$ be same as in Theorem 3.4. Assume that the function $\psi$ is expansively superquadratic with a constant $L$ satisfying $2^{3\beta-2}L < 1$. 
Then there exists a unique quartic mapping \( H : \mathbb{R} \to X \) satisfying the functional equation \( (1.2) \) and the following inequality:

\[
\|f(x) - H(x)\|_X \leq \frac{\hat{\Psi}_2(x)}{\sqrt{(4a^2bL^{-1})^p - (16a^2)^{\beta p}}}
\]

for all \( x \in \mathbb{R} \), where \( \hat{\Psi}_2(x) = K^3 \left( \Phi(2^{-\frac{1}{2}}x) + 4^\beta \Phi(2^{-1}x) \right) \).

**Proof.** It follows from (3.12) in the proof of Theorem 3.1 that

\[
\|f(x) - 16f(2^{-1}x)\|_X \leq \frac{1}{a^{2\beta}} \hat{\Psi}_2(2^{-1}x)
\]

for all \( x \in \mathbb{R} \) and so

\[
\|16^k f(2^{-k}x) - 16^m f(2^{-m}x)\|_X \leq \left( \frac{\hat{\Psi}_2(x)}{a^{2\beta}} \right)^p \sum_{j=k}^{m-1} (2^{4\beta - 2L})^{jp}
\]

for all \( x \in \mathbb{R} \) and \( k, m \in \mathbb{Z}^+ \) with \( m > k \geq 0 \). The remains follow the proof of Theorem 3.1. This completes the proof. \( \Box \)

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