COSET OF A HYPERCOMPLEX NUMBER SYSTEM IN CLIFFORD ANALYSIS

JI EUN Kim AND KWANG HO Shon

Abstract. We give certain properties of elements in a coset group with hypercomplex numbers and research a monogenic function and a Clifford regular function with values in a coset group by defining differential operators. We give properties of those functions and a power of elements in a coset group with hypercomplex numbers.

1. Introduction

Many kinds of quaternion, specially, split quaternions and dual quaternions, etc., have applications in physics and computer systems. There are conventional and mathematical constructions of quaternions by multiplication rules of each elements. Leo [10] formulated special relativity by a quaternionic algebra on reals and showed that a complexified quaternionic version of special relativity is not a necessity. Hasebe [3] constructed quantum Hall effect on split quaternions and analyzed that a wave function and membrane-like excitations are derived explicitly. Brody and Graefe [1] introduced quaternionic and coquaternionic (split signature analogue of quaternions) extensions of Hamiltonian mechanics and offered complexified classical and quantum mechanics. Hucks [4] introduced basic properties and definitions for the hyperbolic complex numbers, and applied the Dirac equation in 4 dimensions to special relativistic physics. Sobczyk [12] explored an underlying geometric framework in which matrix multiplication arises from the product of numbers in a geometric (Clifford) algebra. Jaglom [5] generated the mathematical operations and representations between complex numbers and geometry. We [6, 7, 8, 9] have researched corresponding Cauchy-Riemann systems and properties of a regularity of functions with values in special quaternions on Clifford analysis and gave a regular function with values in dual split quaternions and relations

Received November 24, 2014; Revised January 2, 2015.

2010 Mathematics Subject Classification. 32W50, 32A99, 30G35, 11E88.

Key words and phrases. coset, differential operator, monogenic function, regular function, Clifford analysis.

Kwang Ho Shon was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2013R1A1A2008978).
between a corresponding Cauchy-Riemann system and a regularity of functions with values in dual split quaternions.

In the conventional mathematical construction of complex and multicomplex numbers, multiplication rules are suggested instead of being derived from a general principle. Petrache [11] proposed a systematic approach based on the concept of a coset product from the group theory. He showed that extensions of real numbers in two or more dimensions follow from the closure property of finite coset groups with the utility of multidimensional number systems expressed by elements of small group symmetries.

In this paper, we give the form of elements in a coset group with special unit matrix bases and the multiplication of those elements. Also, we consider certain properties of elements in a coset group with hypercomplex numbers and then investigate a monogenic function and a Clifford regular function with values in a coset group by defining differential operators. We give properties of those functions and a power of elements in a coset group with hypercomplex numbers.

2. Preliminaries

Throughout this paper, let \( \mathbb{R}, \mathbb{C}, \) and \( \mathbb{N} \) be the sets of real and complex numbers, and positive integers, respectively, and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Referring Petrache [11], we consider the coset group \( G = \{ \mathbb{R}, g_1 \mathbb{R}, g_2 \mathbb{R}, g_3 \mathbb{R} \} \), where \( g_m \) is an element of the set outside of \( \mathbb{R} \) but compatible with operations in \( \mathbb{R} \) and \( g_m (m = 1, 2, 3) \). Then we obtain the following numbers by generating a set within cosets:

\[
A = \{ \zeta = p + gq \mid p, q \in \mathbb{C} \},
\]

where \( g \) is an element of the set outside of \( \mathbb{C} \) for which addition and multiplication rules follow from the properties of \( g \): For any \( \zeta, \eta \in A \),

\[
\zeta + \eta = (p_1 + p_2) + g(q_1 + q_2)
\]

and

\[
\zeta \eta = (p_1 p_2 + \alpha q_1 q_2) + g(p_1 q_2 + p_2 q_1),
\]

where \( \alpha = g^2 \) is a complex number. From the above multiplication rule, the product \( \zeta \eta \) can be written in a matrix form:

\[
\begin{pmatrix}
p_1 p_2 + \alpha q_1 q_2 \\
p_1 q_2 + p_2 q_1
\end{pmatrix}
= \begin{pmatrix}
p_1 \\
q_1
\end{pmatrix}
\begin{pmatrix}
p_2 \\
q_2
\end{pmatrix},
\]

which gives the following matrix form:

\[
\zeta = \begin{pmatrix}
p \\
\alpha q
\end{pmatrix}
(\begin{pmatrix}
p \\
q
\end{pmatrix}, \begin{pmatrix}
p \alpha q \\
p
\end{pmatrix}).
\]
Since complex numbers \( p \) and \( q \) are also obtained by the above process, we obtain
\[
ζ = \left( \begin{array}{ccc}
x_0 & x_1 & x_2 \\
x_0α & x_0α & x_0α & x_3 \\
x_0α & x_0α & x_0α & x_3 \\
α^2x_3 & αx_2 & αx_1 & x_0 \\
\end{array} \right) = x_01 + x_1I + x_2J + x_3K,
\]
where \( 1 \) is the unit matrix,
\[
I = \left( \begin{array}{ccc}
0 & 1 & 0 \\
α & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & α \\
\end{array} \right), \quad J = \left( \begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
α & 0 & 0 \\
0 & α & 0 \\
\end{array} \right), \quad K = \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & α & 0 \\
0 & 0 & 0 \\
α^2 & 0 & 0 \\
\end{array} \right).
\]

We consider properties of \( 1, I, J \) and \( K \). By the multiplication of matrices, we obtain
\[
I^2 = J^2 = α1, \quad K^2 = α^2 1,
\]
\[
IJ = JI = K, \quad JK = KJ = αI, \quad IK = IK = αJ.
\]

If \( α = −1 + i0 \), where \( i = \sqrt{-1} \) is the imaginary unit in \( C \), then
\[
I^2 = J^2 = −1, \quad K^2 = 1,
\]
\[
IJ = JI = K, \quad JK = KJ = −I, \quad IK = IK = −J.
\]

Let \( C \) be a set of \( ζ \) with \( 1, I, J \) and \( K \) as follows:
\[
C = \{ z = z_1 + z_2J \mid z_1 = x_0 + x_1I, \ z_2 = x_2 + x_3I, \ x_r ∈ R \ (r = 0, 1, 2, 3) \}
\]
and the elements of \( C \) be said to be pseudo split quaternions.

We give the commutative multiplication of elements of \( C \): For any \( z, w ∈ C \),
\[
zw = (z_1w_1 − z_2w_2) + (z_1w_2 + z_2w_1)J
\]
\[
= (x_0y_0 − x_1y_1 − x_2y_2 + x_3y_3) + (x_0y_1 + x_1y_0 − x_2y_3 − x_3y_2)I
\]
\[
+ (x_0y_2 − x_1y_3 + x_2y_0 − x_3y_1)J + (x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0)K.
\]

When \( z_1 \) is a scalar multiplication of \( z_2 \), we have the conjugate number \( \overline{z} = \overline{z_1} − \overline{z_2}J \), the norm of \( z \) is \( |z|^2 = \overline{z}z = \sum_{r=0}^{3} x_r^2 \) and the inverse number of \( z \) is \( z^{-1} = \frac{\overline{z}}{|z|^2} \).

Also, when \( z_1 \) satisfies the equation \( z_1 = \Re z_2I \), where \( \Re \) is a scalar number, we have the conjugate number \( z^* = \overline{z_1} + \overline{z_2}J \), the modulus of \( z \) is \( N(z) = zz^* = x_0^2 + x_1^2 − x_2^2 − x_3^2 \) and the inverse number of \( z \) is \( z^{-1} = \frac{z^*}{N(z)} \).

Let \( Ω \) be an open set in \( C^2 \). We give a function \( f : C^2 → C \) such that
\[
f(z) = f(z_1, z_2) = f_1(z_1, z_2) + f_2(z_1, z_2)J,
\]
where \( f_1 = u_0 + u_1I \) and \( f_2 = u_2 + u_3I \) with \( u_r \ (r = 0, 1, 2, 3) \) are real valued functions. We give differential operators as follows:
\[
D := \frac{1}{2} \left( \frac{∂}{∂x_0}I \frac{∂}{∂x_1} − J \frac{∂}{∂x_2} + K \frac{∂}{∂x_3} \right) = \frac{∂}{∂z_1} − J \frac{∂}{∂z_2}.
\]
\[
\mathbf{D} = \frac{1}{2} \left( \frac{\partial}{\partial x_0} + I \frac{\partial}{\partial x_1} + J \frac{\partial}{\partial x_2} + K \frac{\partial}{\partial x_3} \right) = \frac{\partial}{\partial z_1} + J \frac{\partial}{\partial z_2}.
\]

When \( \frac{\partial f}{\partial z_1} \) is a scalar multiplication of \( \frac{\partial f}{\partial z_2} \), there is a Laplacian operator such that
\[
D \mathbf{D} f = \mathbf{D} D f = \frac{\partial^2 f}{\partial z_1 \partial z_1} + \frac{\partial^2 f}{\partial z_2 \partial z_2} = \frac{1}{4} \sum_{r=0}^{3} \frac{\partial f}{\partial x_r}.
\]

Also, we have
\[
D^* = \frac{1}{2} \left( \frac{\partial}{\partial x_0} + I \frac{\partial}{\partial x_1} - J \frac{\partial}{\partial x_2} - K \frac{\partial}{\partial x_3} \right) = \frac{\partial}{\partial z_1} - J \frac{\partial}{\partial z_2}.
\]

When \( \frac{\partial f}{\partial z_1} \) satisfies the equation
\[
\frac{\partial f}{\partial z_1} = K \frac{\partial f}{\partial z_2},
\]
where \( K \) is a scalar number, there is a Coulomb operator [2] such that
\[
D D^* f = D^* D f = \frac{\partial^2 f}{\partial z_1 \partial z_1} - \frac{\partial^2 f}{\partial z_2 \partial z_2} = \frac{1}{4} \left( \frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_3^2} \right).
\]

**Remark 2.1.** By the definition of differential operators, we have
\[
\mathbf{D} f = \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} \right) + \left( \frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial z_2} \right) J
\]
and
\[
D^* f = \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} \right) + \left( \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2} \right) J.
\]

**3. Properties of functions with values in \( C \)**

**Definition.** Let \( \Omega \) be an open set in \( \mathbb{C}^2 \). A function \( f(z) = f_1(z) + f_2(z)J \) is said to be L(R)-monogenic in \( \Omega \) if the following two conditions are satisfied:
(i) \( f_1(z) \) and \( f_2(z) \) are continuously differential functions on \( \Omega \), and
(ii) \( \mathbf{D} f(z) = 0 \) (resp. \( f(z) \mathbf{D} = 0 \)) on \( \Omega \).

**Definition.** Let \( \Omega \) be an open set in \( \mathbb{C}^2 \). A function \( f(z) = f_1(z) + f_2(z)J \) is said to be L(R)-Clifford regular in \( \Omega \) if the following two conditions are satisfied:
(i) \( f_1(z) \) and \( f_2(z) \) are continuously differential functions on \( \Omega \), and
(ii) \( D^* f(z) = 0 \) (resp. \( f(z) D^* = 0 \)) on \( \Omega \).

Since the equation \( \mathbf{D} f = 0 \) (resp. \( D^* f = 0 \)) is equivalent to the equation \( f \mathbf{D} = 0 \) (resp. \( f D^* = 0 \)), we don’t need to distinguish between left and right monogenic (resp. Clifford regular).

**Remark 3.1.** The equation \( \mathbf{D} f(z) = 0 \) is equivalent to the following equations:
\[
(3.1) \quad \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \quad \text{and} \quad \frac{\partial f_2}{\partial z_1} = - \frac{\partial f_1}{\partial z_2}.
\]

Also, the equation \( D^* f(z) = 0 \) is equivalent to the following equations:
\[
(3.2) \quad \frac{\partial f_1}{\partial z_1} = - \frac{\partial f_2}{\partial z_2} \quad \text{and} \quad \frac{\partial f_2}{\partial z_1} = \frac{\partial f_1}{\partial z_2}.
\]
The Equations (3.1) and (3.2) are the analogue of the Cauchy-Riemann systems in \( \mathbb{C} \).

**Remark 3.2.** Let \( \Omega \) be an open set in \( \mathbb{C}^2 \). If a function \( f(z) = f_1(z) + f_2(z)J \) is monogenic in \( \Omega \), then it satisfies

\[
D^* f = 2 \frac{\partial f}{\partial z_1} + 2J \frac{\partial f}{\partial z_2} = 0.
\]

Also, if a function \( f(z) = f_1(z) + f_2(z)J \) is Clifford regular in \( \Omega \), then it satisfies

\[
\mathcal{D} f = 2 \frac{\partial f}{\partial z_1} + 2J \frac{\partial f}{\partial z_2} = 0.
\]

**Proposition 3.3.** Let \( \Omega \) be an open set in \( \mathbb{C}^2 \). For \( n \in \mathbb{N}_0 \), a function

\[
f(z) = z^n = (z_1 + z_2 J)^n = f_1 + f_2 J,
\]

where

\[
f_1 = \sum_{k=0}^{n} \frac{(-1)^{\frac{k}{2}}}{k!} \binom{n}{k} z_1^{n-k} z_2^k \quad \text{and} \quad f_2 = \sum_{k=1}^{n} \frac{(-1)^{\frac{k-1}{2}}}{k!} \binom{n}{k} z_1^{n-k} z_2^k,
\]

is monogenic and Clifford regular in \( \Omega \).

**Proof.** By the definition of differential operators, we have

\[
\frac{\partial}{\partial z_i} z_p = 0,
\]

where \( m \in \mathbb{N}_0 \) and \( t, p = 1, 2 \). Hence, \( \mathcal{D} z^n = 0 \) and \( D^* z^n = 0 \). Therefore, we obtain \( z^n \) is monogenic and Clifford regular in \( \Omega \). \( \square \)

**Proposition 3.4.** Let \( \Omega \) be an open set in \( \mathbb{C}^2 \). A function \( f(z) = z^{-1} \) \( (z \neq 0) \) is monogenic and Clifford regular in \( \Omega \).

**Proof.** Since a function \( f(z) = z^{-1} \) is defined by one of two cases as follows:

(i) \( z_1 = \Re z_2 \), where \( \Re \) is scalar.
(ii) \( z_1 = \Re z_2 I \), where \( \Re \) is scalar.

If \( z \) satisfies the case (i), then

\[
\mathcal{D} z^{-1} = \mathcal{D} \left( \frac{\overline{z}}{z_2} \right) = \left( \frac{\partial}{\partial z_1} + J \frac{\partial}{\partial z_2} \right) \left( \frac{\overline{z_1} - \overline{z_2} J}{z_1 z_2 + z_2 z_2} \right) = 0.
\]

Also, if \( z \) satisfies the case (ii), then

\[
D^* z^{-1} = D^* \left( \frac{z}{z_2} \right) = \left( \frac{\partial}{\partial z_1} - J \frac{\partial}{\partial z_2} \right) \left( \frac{\overline{z} + \overline{z} J}{z_1 z_1 - z_2 z_2} \right) = 0.
\]

Therefore, we obtain the result. Furthermore, we have

\[
\mathcal{D} \left( \frac{z}{z_2} \right) = 0 \quad \text{and} \quad D^* \left( \frac{z}{z_2} \right) = 0.
\]

Therefore, a function of the inverse form is monogenic (resp. Clifford regular) in \( \Omega \), regardless of calculating operators. \( \square \)
Theorem 3.5. Let \( \Omega \) be an open set in \( \mathbb{C}^2 \) and a function \( f \) be monogenic in \( \Omega \). Then we have

\[
Df = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_3} K = -\frac{\partial f}{\partial x_1} I - \frac{\partial f}{\partial x_2} J.
\]

Proof. From Remark 1 and the system (3.1), we have

\[
Df = \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} \right) + \left( \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2} \right) J
\]

\[
= \frac{\partial}{\partial x_1}(u_1 - u_0 I + u_3 J - u_2 K) + \frac{\partial}{\partial x_3}(u_2 + u_3 I - u_0 J - u_1 K)
\]

\[
= -\frac{\partial f}{\partial x_0} I - \frac{\partial f}{\partial x_2} J;
\]

or

\[
Df = -\left( \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2} \right) + \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} \right) J
\]

\[
= \frac{\partial}{\partial x_0}(u_0 + u_1 I + u_2 J + u_3 K) + \frac{\partial}{\partial x_3}(u_3 - u_2 I - u_1 J + u_0 K)
\]

\[
= \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_3} K.
\]

Therefore, we obtain the result. \( \square \)

Corollary 3.6. Let \( \Omega \) be an open set in \( \mathbb{C}^2 \) and a function \( f \) be Clifford regular in \( \Omega \). Then we have

\[
Df = \frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial x_2} J = -\frac{\partial f}{\partial x_1} I + \frac{\partial f}{\partial x_3} K.
\]

Proof. Following the process of proof of Theorem 3.5, we also obtain the result. \( \square \)

Proposition 3.7. Let \( \Omega \) be an open set in \( \mathbb{C}^2 \) and let \( f \) and \( g \) be monogenic (resp. Clifford regular) in \( \Omega \). Then the following properties are satisfied:

(i) \( f \alpha \) and \( \alpha f \) are monogenic (resp. Clifford regular) in \( \Omega \), where \( \alpha \) is a constant in \( \mathbb{C} \).

(ii) \( fg \) is monogenic (resp. Clifford regular) in \( \Omega \).

Proof. From the property of multiplication in \( \mathbb{C} \) and the definition of monogenic (resp. Clifford regular) in \( \Omega \), the results are obtained. \( \square \)

We let

\[
\omega := dz_1 \wedge dz_2 \wedge dz_2 + J dz_1 \wedge dz_1 \wedge dz_2.
\]
Theorem 3.8. Let $\Omega$ be a domain in $\mathbb{C}^2$ and $U$ be any domain in $\Omega$ with a smooth boundary $bU$ such that $\overline{U} \subset \Omega$. If a function $f$ is monogenic in $\Omega$, then

$$\int_{bU} \omega f = 0,$$

where $\omega f$ is the product on $\mathcal{C}$ of the form $\omega$ on the function $f(z)$.

Proof. Since the function $f = f_1 + f_2J$ has the equation $\omega f = f_1dz_1 \wedge dz_2 + f_2dz_1 \wedge d\bar{z}_2$, we have

$$d(\omega f) = \frac{\partial f_1}{\partial z_1}dz_1 \wedge dz_2 + J\frac{\partial f_2}{\partial z_1}dz_1 \wedge d\bar{z}_2 - \frac{\partial f_2}{\partial z_2}dz_2 \wedge dz_1 + J\frac{\partial f_1}{\partial z_2}dz_2 \wedge d\bar{z}_1$$

$$= \left(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\right)dz_1 \wedge dz_2 + J\left(\frac{\partial f_1}{\partial z_2} - \frac{\partial f_2}{\partial z_1}\right)d\bar{z}_1 \wedge d\bar{z}_2,$$

where $d\bar{z}_1 = dz_1 \wedge dz_2 \wedge d\bar{z}_2$. Since $f$ is monogenic in $\Omega$, $f$ satisfies the equation (3.1). Hence, we have $d(\omega f) = 0$. Therefore, by Stokes’ theorem, we obtain the result. 

Corollary 3.9. Let $\Omega$ be a domain in $\mathbb{C}^2$ and $U$ be any domain in $\Omega$ with a smooth boundary $bU$ such that $\overline{U} \subset \Omega$. Suppose

$$\omega = dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 - Jdz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_1 \wedge dz_2$$

and a function $f$ is Clifford regular in $\Omega$. Then

$$\int_{bU} \omega f = 0,$$

where $\omega f$ is the product on $\mathcal{C}$ of the form $\omega$ on the function $f(z)$.

Proof. Using the process of proof of Theorem 3.8, we have

$$d(\omega f) = \left(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\right)dV + J\left(\frac{\partial f_1}{\partial z_2} - \frac{\partial f_2}{\partial z_1}\right)dV,$$

where $dV = dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2$. Since $f$ is Clifford regular in $\Omega$, $f$ satisfies the equation (3.2). Therefore, we have $d(\omega f) = 0$ and by Stokes’ theorem, we obtain the result. 

Example 3.10. Let $\Omega$ be a domain in $\mathbb{C}^2$ and $U$ be any domain in $\Omega$ with a smooth boundary $bU$ such that $\overline{U} \subset \Omega$ and let $\omega = dz_1 \wedge dz_2 \wedge d\bar{z}_2 + Jdz_1 \wedge d\bar{z}_1 \wedge dz_2$. Suppose $f(z) = z^n$ $(n \in \mathbb{N}_0)$ be monogenic in $\Omega$. Then

$$\int_{bU} \omega f = \int_U d(\omega f)$$

$$= \int_U \left(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\right)dV + \int_U J\left(\frac{\partial f_1}{\partial z_2} - \frac{\partial f_2}{\partial z_1}\right)dV = 0,$$
where $\omega f$ is the product on $\mathcal{C}$ of the form $\omega$ on the function $f(z)$ and $dV = dz_1 \wedge dz_2 \wedge d\xi_1 \wedge d\xi_2$.

**Example 3.11.** Let $\Omega$ be a domain in $\mathbb{C}^2$ and $U$ be any domain in $\Omega$ with a smooth boundary $bU$ such that $U \subset \Omega$ and let $\omega = dz_1 \wedge dz_2 + Jdz_1 \wedge d\xi_2 \wedge d\xi_2$. If $f(z) = z^n$ ($n \in \mathbb{N}_0$) is Clifford regular in $\Omega$, then

$$\int_{bU} \omega f = \int_U d(\omega f) = \int_U \left( \frac{\partial f_1}{\partial \xi_1} + \frac{\partial f_2}{\partial \xi_2} \right) dV + \int_U f \left( \frac{\partial f_2}{\partial \xi_1} - \frac{\partial f_1}{\partial \xi_2} \right) dV = 0,$$

where $\omega f$ is the product on $\mathcal{C}$ of the form $\omega$ on the function $f(z)$ and $dV = dz_1 \wedge dz_2 \wedge d\xi_1 \wedge d\xi_2$.

**References**


Ji Eun Kim
Department of Mathematics
Pusan National University
Busan 609-735, Korea
E-mail address: jeunkim@pusan.ac.kr

Kwang Ho Shon
Department of Mathematics
Pusan National University
Busan 609-735, Korea
E-mail address: khshon@pusan.ac.kr