ON CHARACTERIZATIONS OF SET-VALUED DYNAMICS

HAHNG-YUN CHU AND SEUNG KI YOO

Abstract. In this paper, we generalize the stability for an $n$-dimensional cubic functional equation in Banach space to set-valued dynamics. Let $n \geq 2$ be an integer. We define the $n$-dimensional cubic set-valued functional equation given by

$$f(2 \sum_{i=1}^{n-1} x_i + x_n) \oplus f(2 \sum_{i=1}^{n-1} x_i - x_n) \oplus 4 \sum_{i=1}^{n-1} f(x_i) = 16 f(\sum_{i=1}^{n-1} x_i) \oplus 2 \sum_{i=1}^{n-1} (f(x_i + x_n) \oplus f(x_i - x_n)).$$

We first prove that the solution of the $n$-dimensional cubic set-valued functional equation is actually the cubic set-valued mapping in [6]. We prove the Hyers-Ulam stability for the set-valued functional equation.

1. Introduction

A well-known problem in functional analysis is to obtain the stabilities of functional equations. The stability problem of a functional equation was originated from a question of S. M. Ulam [22] concerning the stability of a group homomorphism. D. H. Hyers [8] affirmatively answered the stability of the linear functional equation $f(x + y) = f(x) + f(y)$. The Hyers’ theorem was generalized by T. Aoki [1]. Th. M. Rassias [19] considered a generalized version of the Hyers’ theorem permitted the Cauchy difference to become unbounded. Thereafter, P. Gavruta [7] proved the Rassias’ theorem by using the control function $\phi : G \times G \to [0, \infty)$ such that $\sum_{k=0}^{\infty} 2^{-k} \phi(2^k x, 2^k y) < \infty$ for all $x, y \in G$. Jun and Kim [9] established the Hyers-Ulam stability of the following functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).$$

It is easily proved that the function $f(x) = cx^3$ is a solution of the above functional equation. From the reason, the equation (1.1) is called a cubic

Received October 7, 2014; Revised January 12, 2016.

2010 Mathematics Subject Classification. Primary 39B82, 47H04, 47H10, 54C60.

Key words and phrases. Hyers-Ulam stability, $n$-dimensional cubic set-valued functional equation.

This research has been performed as a subproject of project Research for Applications of Mathematical Principles(No C21501) and supported by the National Institute of Mathematics Sciences(NIMS).

©2016 Korean Mathematical Society
In [10], Jung and Chang also investigated the Hyers-Ulam-Rassias stability of the following cubic type functional equation
\begin{equation}
(1.2) \quad f(x + y + 2z) + f(x + y - 2z) + f(2x) + f(2y) \\
= 2(f(x + y) + 2f(x + z) + 2f(x - z) + 2f(y + z) + 2f(y - z)).
\end{equation}

In [12], Chu et al. extended the cubic functional equation to the following generalized form
\begin{align*}
&f(2^{n-1} \sum_{i=1}^{n-1} x_i + x_n) + f(2^{n-1} \sum_{i=1}^{n-1} x_i - x_n) + 4 \sum_{i=1}^{n-1} f(x_i) \\
= 16f(\sum_{i=1}^{n-1} x_i) + 2 \sum_{i=1}^{n-1} (f(x_i + x_n) + f(x_i - x_n)),
\end{align*}

where \( n \geq 2 \) is an integer, and they also investigated the Hyers-Ulam stability for the generalized cubic functional equation.

In [14], Lu and Park defined the additive set-valued functional equations
\begin{align*}
f(\alpha x + \beta y) &= rf(x) + sf(y), \\
f(x + y + z) &= 2f(x + y) + f(x + z) + f(x - z) + 2f(y + z) + 2f(y - z).
\end{align*}

\begin{align*}
f(\sum_{j=1}^{n-1} x_j + 2x_n) \oplus f(\sum_{j=1}^{n-1} x_j - 2x_n) \oplus \sum_{j=1}^{n-1} f(2x_j) \\
= 2f(\sum_{j=1}^{n-1} x_j) \oplus 4 \sum_{j=1}^{n-1} (f(x_j + x_n) \oplus f(x_j - x_n)).
\end{align*}

In this paper, we start with the construction of an \( n \)-dimensional cubic set-valued functional equation. Using set-valued operations, we prove the Hyers-Ulam stability of the set-valued functional equation.

Now we introduce notations in this article. Let \( CB(Y) \) be the set of all closed bounded subsets of \( Y \) and \( CC(Y) \) the set of all closed convex subsets of \( Y \). Let
CBC(Y) be the set of all closed bounded convex subsets of Y. For elements A, A′ of CC(Y) and positive real values α, β, we denote A ⊕ A′ := A + A′. It is easy to show that αA + αA′ = α(A + A′) and (α + β)A ⊆ αA + βA. If A is convex, then we obtain that (α + β)A = αA + βA for all α, β ∈ ℝ+. We define the cubic set-valued functional equation induced from the original functional equation (1.1) as follows.

**Definition 1.1.** Let f : X → CBC(Y) be a mapping. The cubic set-valued functional equation is defined by

\[
(1.3) \quad f(2x + y) ⊕ f(2x - y) = 2f(x + y) ⊕ 2f(x - y) ⊕ 12f(x)
\]

for all x, y ∈ X. Every solution of the cubic set-valued functional equation is said to be a cubic set-valued mapping.

We consider an n-dimensional cubic set-valued functional equation as applying the n-dimensional cubic functional equation

\[
(1.4) \quad f(2 \sum_{i=1}^{n-1} x_i + x_n) ⊕ f(2 \sum_{i=1}^{n-1} x_i - x_n) ⊕ 4 \sum_{i=1}^{n-1} f(x_i) = 16f(\sum_{i=1}^{n-1} x_i) ⊕ 2 \sum_{i=1}^{n-1} (f(x_i + x_n) ⊕ f(x_i - x_n))
\]

for all x₁, . . . , xₙ ∈ X, where n ≥ 2 is an integer. Every solution of the n-dimensional cubic set-valued functional equation is called a n-dimensional cubic set-valued mapping.

In the next section, we introduce basic notions and definitions for the proof of the main theorems. Next we obtain the equivalence for the n-dimensional cubic set-valued mapping. And then we consider the stability problem for the generalized n-dimensional cubic set-valued functional equation. As applications, we get some results for the stability problem related to the fixed point theory. Throughout this paper, let X be a real vector space and Y a Banach space.

### 2. Stability of the set-valued functional equation

Firstly, we will give precise definitions and notations to prove the main theorems. We interest in the generalization of the stability for original functional equations to set-valued dynamics. For a subset A ⊆ Y, the distance function d(·, A) is defined by d(x, A) := inf{∥x − y∥ : y ∈ A} for x ∈ Y. For A, A′ ∈ CB(Y), the Hausdorff distance h(A, A′) between A and A′ is defined by

\[
h(A, A′) := \inf\{α ≥ 0 | A ⊆ A′ + αB_Y, A′ ⊆ A + αB_Y\},
\]

where B_Y is the closed unit ball in Y. In [5], it was proved that (CBC(Y), ⊕, h) is a complete metric semigroup. Rådström [21] proved that (CBC(Y), ⊕, h) is isometrically embedded in a Banach space. The followings are directly proved by using the notion of the Hausdorff distance.
Remark 2.1. Let $A, A', B, B', C \in CBC(Y)$ and $\alpha > 0$. Then we have that
\begin{enumerate}
\item $h(A \oplus A', B \oplus B') \leq h(A, B) + h(A', B')$;
\item $h(\alpha A, \alpha B) = \alpha h(A, B)$;
\item $h(A, B) = h(A \oplus C, B \oplus C)$.
\end{enumerate}

Remark 2.2. The completeness of a phase space plays important role to prove the stability in our manuscript. In detail, we construct a unique $n$-dimensional cubic set-valued mapping to a complete metric semigroup under addition and scalar multiplication using the completeness of the codomain.

The following proposition states that the $n$-dimensional cubic set-valued functional equation (1.4) is actually the generalized form of the cubic set-valued functional equation (1.3). In set-valued dynamics, it is useful to find different expressions of set-valued functional equations.

**Proposition 2.3.** Let $f : X \to CBC(Y)$ be a mapping. A mapping $f$ satisfies the set-valued functional equation (1.4) if and only if $f$ satisfies the set-valued functional equation (1.3). That is, every $n$-dimensional cubic set-valued mapping is actually a cubic set-valued mapping. Moreover the set-valued mapping $f$ is single-valued in the proof of each implication.

**Proof.** Suppose that a mapping $f$ satisfies the set-valued functional equation (1.4). Putting $x_i = 0$ ($i = 1, \ldots, n$), we have $f(0) = \{0\}$. Setting $x_i = 0$ ($i = 1, \ldots, n - 1$) and $x_n = x$ in (1.4), we get
\[
 f(x) \oplus f(-x) = 2((n - 1)f(x) \oplus (n - 1)f(-x)).
\]
Hence $f(x) \oplus f(-x) = \{0\}$ for all $x \in X$. By the definition of addition for subsets in $X$, we directly obtain that the set-valued mapping $f$ is a single-valued mapping. Let $x_1 = x$, $x_i = 0$ ($i = 2, \ldots, n - 1$) and $x_n = y$ in (1.2), we have
\[
 f(2x + y) \oplus f(2x - y) = 2f(x + y) \oplus 2f(x - y) \oplus 12f(x).
\]
Therefore, $f$ is a cubic set-valued mapping.

Conversely, suppose that a mapping $f$ satisfies the set-valued functional equation (1.3). We use the induction on $n \geq 2$ to prove (1.4). Clearly, (1.4) holds when $n = 2$. Consider the case $n = k$. By the induction hypothesis, we have
\[
 f(2\sum_{i=1}^{k-1} x_i + x_k) \oplus f(2\sum_{i=1}^{k-1} x_i - x_k) \oplus 4\sum_{i=1}^{k-1} f(x_i)
\]
\[
 = 16f(\sum_{i=1}^{k-1} x_i) \oplus 2\sum_{i=1}^{k-1} (f(x_i + x_k) \oplus f(x_i - x_k))
\]

(2.1)
for all $x_1, \ldots, x_k \in X$. Putting $x_1 = x_1 + y$ in (2.1), we get

\begin{equation}
\begin{aligned}
&f(\sum_{i=1}^{k-1} 2x_i + 2y + x_k) \oplus f(\sum_{i=1}^{k-1} 2x_i + 2y - x_k) \oplus 4f(x_1 + y) \oplus 4 \sum_{i=2}^{k-1} f(x_i) \\
&= 16f(x_1 + y) \oplus 2f(x_1 + y + x_k) \oplus 2f(x_1 + y - x_k) \\
&\oplus 2 \sum_{i=2}^{k-1} (f(x_i + x_k) \oplus f(x_i - x_k))
\end{aligned}
\end{equation}

(2.2)

for all $x_1, \ldots, x_k \in X$ and $y \in X$. Setting $x = 2x_1$, $y = 2y$ and $z = x_k$ in (1.2), we have

\begin{equation}
\begin{aligned}
&2f(x_1 + y + x_k) \oplus 2f(x_1 + y - x_k) \oplus 2f(2x_1) \oplus 2f(2y) \\
&= 4f(x_1 + y) \oplus f(2x_1 + x_k) \oplus f(2x_1 - x_k) \oplus f(2y + x_k) \oplus f(2y - x_k)
\end{aligned}
\end{equation}

(2.3)

for all $x_1, \ldots, x_k \in X$ and $y \in X$. By (2.2) and (2.3), we get

\begin{equation}
\begin{aligned}
&f(\sum_{i=1}^{k-1} 2x_i + 2y + x_k) \oplus f(\sum_{i=1}^{k-1} 2x_i + 2y - x_k) \oplus 4f(x_1 + y) \\
&\oplus 4 \sum_{i=2}^{k-1} f(x_i) \oplus 4f(x_1) \oplus 4f(y) \\
&= 16f(x_1 + y) \oplus 4f(x_1 + y) \oplus 2f(y + x_k) \oplus 2f(y - x_k) \\
&\oplus 2f(x_1 + x_k) \oplus 2f(x_1 - x_k) \oplus 2 \sum_{i=2}^{k-1} (f(x_i + x_k) \oplus f(x_i - x_k))
\end{aligned}
\end{equation}

(2.4)

for all $x_1, \ldots, x_k \in X$ and $y \in X$. Putting $y = 0$ in (2.4), we obtain the desired conclusion (1.4). In addition, the equation (1.3) guarantees that the set-valued mapping $f$ is single-valued in the part of the converse implication in this proof. □

Next, we prove the stability of the $n$-dimensional cubic set-valued functional equation.

**Theorem 2.4.** Let $n \geq 3$ be an integer and let $\phi : X^n \to [0, \infty)$ be a function such that

\begin{equation}
\tilde{\phi}(x_1, \ldots, x_n) := \sum_{i=0}^{\infty} \frac{1}{8^i} \phi(2^i x_1, \ldots, 2^i x_n) < \infty
\end{equation}

(2.5)
for all $x_1, \ldots, x_n \in X$. Suppose that $f : X \rightarrow (\text{CBC}(Y), h)$ is a mapping with $f(0) = \{0\}$ and

$$
(2.6) \quad h \left( f \left( \sum_{i=1}^{n-1} x_i + x_n \right) \oplus f \left( \sum_{i=1}^{n-1} x_i - x_n \right) \oplus 4 \sum_{i=1}^{n-1} f(x_i), 16f \left( \sum_{i=1}^{n-1} x_i \right) \oplus 2 \sum_{i=1}^{n-1} \left( f(x_i + x_n) \oplus f(x_i - x_n) \right) \right) \leq \phi(x_1, \ldots, x_n)
$$

for all $x_1, \ldots, x_n \in X$. Then for every $m \in \{1, 2, \ldots, n-1\}$ there exists a unique $n$-dimensional cubic set-valued mapping $T : X \rightarrow (\text{CBC}(Y), h)$ such that

$$
(2.7) \quad h(f(x), T(x)) \leq \frac{1}{16} \phi \left( \frac{x}{m}, \ldots, \frac{x}{m}, 0, \ldots, 0 \right)
$$

for all $x \in X$.

Proof. Set $x_i = x$ ($i = 1, 2, \ldots, m$) and $x_{m+1} = x_{m+2} = \cdots = x_n = 0$ in (2.6). Since the range of $f$ is convex, we have that

$$
(2.8) \quad h(f(2mx) \oplus f(2mx) \oplus 4mf(x), 16f(mx) \oplus 2mf(x) \oplus 2mf(x)) \leq \phi(x_1, \ldots, x, 0, \ldots, 0)
$$

and we get that

$$
(2.9) \quad h\left( f\left( \frac{2}{m}x \right), f(x) \right) \leq \frac{1}{16} \phi \left( \frac{x}{m}, \ldots, \frac{x}{m}, 0, \ldots, 0 \right)
$$

for all $x \in X$. Replacing $x$ by $\frac{x}{m}$ in (2.8), we have

$$
(2.10) \quad h\left( f\left( \frac{2}{m}x \right), f\left( \frac{2}{m}x \right) \right) \leq \frac{1}{16} \phi \left( \frac{2x}{m}, \ldots, \frac{2x}{m}, 0, \ldots, 0 \right)
$$

for all $x \in X$. Substituting $x$ by $2x$ and dividing by 8 in (2.9), we get

$$
(2.11) \quad h\left( f\left( \frac{2^2x}{8^2} \right), f(x) \right) \leq \frac{1}{16} \phi \left( \frac{x}{m}, \ldots, \frac{x}{m}, 0, \ldots, 0 \right) + \frac{1}{8^2} \phi \left( \frac{2x}{m}, \ldots, \frac{2x}{m}, 0, \ldots, 0 \right)
$$

for all $x \in X$. By (2.9) and (2.10), we have
ON CHARACTERIZATIONS OF SET-VALUED DYNAMICS

for all \( x \in X \). Using the induction on \( i \), we obtain that

\[
(2.12) \quad h\left(\frac{f(2^rx)}{8^r}, f(x)\right) \leq \frac{1}{16} \sum_{i=0}^{r-1} \frac{1}{8^i} \phi\left(\underbrace{\frac{2^ix}{m}, \ldots, \frac{2^ix}{m}, 0, \ldots, 0}_{m}\right)
\]

for all \( x \in X \) and \( r \in \mathbb{N} \). In order to prove the convergence of the sequence \( \{\frac{f(2^rx)}{8^r}\} \), we divide inequality (2.12) by \( 8^s \) and also replace \( x \) by \( 2^sx \). Hence it follows that

\[
(2.13) \quad h\left(\frac{f(2^{s+r}x)}{8^{s+r}}, f(2^sx)\right) \leq \frac{1}{16} \sum_{i=0}^{r-1} \frac{1}{8^i} \phi\left(\underbrace{\frac{2^{s+i}x}{m}, \ldots, \frac{2^{s+i}x}{m}, 0, \ldots, 0}_{m}\right)
\]

for all \( x \in X \) and \( r \in \mathbb{N} \). Since the right-hand side of the inequality (2.13) tends to zero as \( s \) tends to infinity, the sequence \( \{\frac{f(2^rx)}{8^r}\} \) is a Cauchy sequence in \( (CBC(Y), h) \). Therefore, from the completeness of \( CBC(Y), h) \), we can define \( T(x) := \lim_{r \to \infty} \frac{f(2^rx)}{8^r} \) for all \( x \in X \). It follows from (2.6) and the definition of \( T \) that

\[
(2.14) \quad h\left(\frac{f(2^nx)}{8^n}, f(x)\right) \leq \frac{1}{16} \sum_{i=0}^{n-1} \frac{1}{8^i} \phi\left(\underbrace{\frac{2^ix}{m}, \ldots, \frac{2^ix}{m}, 0, \ldots, 0}_{m}\right)
\]

which tends to zero as \( r \to \infty \) for all \( x \in X \). Therefore we get \( T(x) = T'(x) \) for all \( x \in X \) which completes this proof. \( \square \)

Remark 2.5. Let \( n \geq 3 \) be an integer, then using the method of the proof in the above theorem, we easily obtain the following result. Let \( \phi : X^n \to [0, \infty) \) be a function such that

\[
(2.15) \quad \tilde{\phi}(x_1, \ldots, x_n) := \sum_{i=0}^{\infty} 8^i \phi\left(\frac{x_1}{2^i}, \ldots, \frac{x_n}{2^i}\right) < \infty
\]
for all \( x_1, \ldots, x_n \in X \). Suppose that \( f : X \to (CBC(Y), h) \) is a mapping with
\[
\begin{align*}
(2.16) \quad h\left( f\left( 2 \sum_{i=1}^{n-1} x_i + x_n \right) \oplus f\left( 2 \sum_{i=1}^{n-1} x_i - x_n \right) \oplus 4 \sum_{i=1}^{n-1} f(x_i),
\right.
\left. 16f\left( \sum_{i=1}^{n-1} x_i \right) \oplus 2 \sum_{i=1}^{n-1} \left( f(x_i + x_n) \oplus f(x_i - x_n) \right) \right) \leq \phi(x_1, \ldots, x_n)
\end{align*}
\]
for all \( x_1, \ldots, x_n \in X \). Then for every \( m \in \{1, 2, \ldots, n - 1\} \), there exists a unique \( n \)-dimensional cubic set-valued mapping \( T : X \to (CBC(Y), h) \) such that
\[
(2.17) \quad h\left( f(x), T(x) \right) \leq \frac{1}{2^m} \phi\left( \frac{x_1}{m}, \ldots, \frac{x_n}{m}, 0, \ldots, 0 \right)
\]
for all \( x \in X \).

**Corollary 2.6.** Let \( 0 < p < 3 \) and \( \theta > 0 \) be real numbers, and let \( X \) be a real normed space. Suppose that \( f : X \to (CBC(Y), h) \) is a mapping satisfying
\[
\begin{align*}
&\quad \frac{1}{2} h\left( f\left( 2 \sum_{i=1}^{n-1} x_i + x_n \right) \oplus f\left( 2 \sum_{i=1}^{n-1} x_i - x_n \right) \oplus 4 \sum_{i=1}^{n-1} f(x_i),
\right.
\left. 16f\left( \sum_{i=1}^{n-1} x_i \right) \oplus 2 \sum_{i=1}^{n-1} \left( f(x_i + x_n) \oplus f(x_i - x_n) \right) \right) \leq \theta \sum_{i=1}^{n} \| x_i \|^{p}
\end{align*}
\]
for all \( x_1, x_2, \ldots, x_n \in X \). Then there exists a unique \( n \)-dimensional cubic set-valued mapping \( T : X \to (CBC(Y), h) \) that satisfies functional equation (2.1) and
\[
\begin{align*}
\quad h(f(x), T(x)) \leq \frac{\theta}{(2^n - 2^{p+1})^p} \| x \|^{p}
\end{align*}
\]
for all \( x \in X \).

**Proof.** The result follows from Theorem 2.4 by setting \( \phi(x_1, x_2, \ldots, x_n) = \theta \sum_{i=1}^{n} \| x_i \|^{p} \) for every \( x_1, x_2, \ldots, x_n \in X \). \( \square \)

**Remark 2.7.** In the case of \( p > 3 \), using Remark 2.5 by setting \( \phi(x_1, x_2, \ldots, x_n) = \theta \sum_{i=1}^{n} \| x_i \|^{p} \), we produce a unique \( n \)-dimensional cubic set-valued mapping \( T : X \to (CBC(Y), h) \) that satisfies functional equation (2.1) and
\[
\begin{align*}
\quad h(f(x), T(x)) \leq \frac{\theta}{(2^n - 2^{p+1})^p} \| x \|^{p}
\end{align*}
\]
for all \( x \in X \) under the same assumption in Corollary 2.6.

Now, we investigate the stability for the given \( n \)-dimensional cubic set-valued functional equation using the fixed point method. We first give the definition of a generalized metric on a set \( X \). A function \( d : X \times X \to [0, \infty) \) is called a *generalized metric* on \( X \) if \( d \) satisfies the following properties
Then for each element integers $n$ of the theory, which is variously applying to the theory of functional equations.

J: Let $T$ be a mapping $x$ for all $x, y \in X$.Using the alternative fixed point theorem, we investigate the stability of the $n$-dimensional cubic set-valued functional equation.

**Theorem 2.9.** Let $1 \leq m \leq n - 1$ be an integer. Suppose that a mapping $f : X \to (CBC(Y), h)$ with $f(0) = \{0\}$ satisfies the functional inequality

\begin{equation}
(2.18) \quad h\left( f\left( \sum_{i=1}^{n-1} x_i \right) + f\left( \sum_{i=1}^{n-1} x_i - x_n \right) \right) \leq 16f(x_1) \oplus 2 \sum_{i=1}^{n-1} f(x_i)
\end{equation}

for all $x_1, \ldots, x_n \in X$ and there exists a constant $L$ with $0 < L < 1$ for which the function $\phi : X^n \to \mathbb{R}^+$ satisfies

\begin{equation}
(2.19) \quad \phi\left( \frac{2x}{m}, \ldots, \frac{2x}{m}, 0, \ldots, 0 \right) \leq 8L\phi\left( \frac{x}{m}, \ldots, \frac{x}{m}, 0, \ldots, 0 \right)
\end{equation}

for all $x \in X$. Then there exists a unique $n$-dimensional cubic set-valued mapping $T : X \to (CBC(Y), h)$ given by $T(x) = \lim_{r \to \infty} \frac{r(x)}{m}$ such that

\begin{equation}
(2.20) \quad h(f(x), T(x)) \leq \frac{1}{16(1 - L)} \phi\left( \frac{2x}{m}, \ldots, \frac{2x}{m}, 0, \ldots, 0 \right)
\end{equation}

for all $x \in X$.

**Proof.** Let $S = \{ g \mid g : X \to CBC(Y), g(0) = \{0\}\}$. We define a generalized metric on $S$ defined by

\[ d(g_1, g_2) := \inf\{ \mu \in (0, \infty) \mid h(g_1(x), g_2(x)) \leq \mu\phi\left( \frac{x}{m}, \ldots, \frac{x}{m}, 0, \ldots, 0 \right), x \in X \}, \]
where, as usual, \( \inf \phi := \infty \). It is easy to show that \((S, d)\) is complete (see [11]).

Now, we define the mapping \( J : S \to S \) given by \( Jg(x) = \frac{1}{8}g(2x) \) for all \( x \in X \).

For \( g_1, g_2 \in S \), let \( d(g_1, g_2) = \mu \). Then
\[
\begin{align*}
b\left(\frac{1}{8}g_1(2x), \frac{1}{8}g_2(2x)\right) &\leq \frac{1}{8}\mu \phi\left(\frac{2x}{m}, \frac{2x}{m}, 0, \ldots, 0\right) \\
&\leq \frac{1}{8}\mu \phi\left(\frac{m}{x}, \frac{m}{x}, 0, \ldots, 0\right)
\end{align*}
\]

for all \( x \in X \). Then by (2.19), we have
\[
\begin{align*}h(Jg_1(x), Jg_2(x)) &\leq \mu L \phi\left(\frac{x}{m}, \frac{x}{m}, 0, \ldots, 0\right) \\
&\leq \mu L \phi\left(\frac{x}{m}, \frac{x}{m}, 0, \ldots, 0\right)
\end{align*}
\]

for all \( x \in X \). The above inequality shows that \( d(Jg_1, Jg_2) \leq Ld(g_1, g_2) \) for all \( g_1, g_2 \in S \). Hence \( J \) is a strictly contractive mapping with Lipschitz constant \( L \).

By (2.9), we obtain the inequality
\[
d(Jf, f) \leq \frac{1}{16}
\]

This means that the inequality (2.20) holds. It follows from (2.18) and (2.19) that
\[
\begin{align*}
h\left(T(2 \sum_{i=1}^{n-1} x_i + x_n) \oplus T(2 \sum_{i=1}^{n-1} x_i - x_n) \oplus 4 \sum_{i=1}^{n-1} T(x_i), \\
16T\left(\sum_{i=1}^{n-1} x_i \oplus 2 \sum_{i=1}^{n-1} (T(x_i + x_n) \oplus T(x_i - x_n))\right)\right) \\
&\leq \lim_{r \to \infty} L' \phi(2^r x_1, \ldots, 2^r x_n) = 0.
\end{align*}
\]

Therefore, \( T \) is a unique \( n \)-dimensional cubic set-valued mapping, as desired. \( \square \)

**Remark 2.10.** In Theorem 2.9, if we take a slight change for the inequality condition of the control mapping \( \phi \) as follows
\[
\phi\left(\frac{x}{m}, \ldots, \frac{x}{m}, 0, \ldots, 0\right) \leq \frac{L}{8} \phi\left(\frac{x}{2m}, \frac{x}{2m}, 0, \ldots, 0\right),
\]

then we obtain the same difference between the cubic-like set-valued function \( f \) and a unique \( n \)-dimensional cubic set-valued mapping \( T \) given by \( T(x) = \lim_{r \to \infty} 8^r f(\frac{x}{2^r}) \). Indeed, from (2.8), we get
\[
\begin{align*}h(f(x), 8f\left(\frac{x}{2}\right)) &\leq \frac{1}{2}\phi\left(\frac{x}{2m}, \frac{x}{2m}, 0, \ldots, 0\right) \\
&\leq \frac{L}{8} \phi\left(\frac{x}{2m}, \frac{x}{2m}, 0, \ldots, 0\right)
\end{align*}
\]
for all $x \in X$. Then we define the mapping $J : S \to S$ such that $Jg(x) = g(\frac{x}{2})$ for all $x \in X$. The rest of this proof is similar to the proof of Theorem 2.9.

**Corollary 2.11.** Let $0 < p < 3$ and $\theta \geq 0$ be real numbers, and let $X$ be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is a mapping satisfying

$$h\left( f\left(\sum_{i=1}^{n-1} x_i + x_n\right) \oplus f\left(\sum_{i=1}^{n-1} x_i - x_n\right) \oplus 4 \sum_{i=1}^{n-1} f(x_i),
16f\left(\sum_{i=1}^{n-1} x_i\right) \oplus 2 \sum_{i=1}^{n-1} \left( f(x_i + x_n) \oplus f(x_i - x_n) \right) \right) \leq \theta \sum_{i=1}^{n} \| x_i \|^p$$

for all $x_1, x_2, \ldots, x_n \in X$. Then there exists a unique $n$-dimensional cubic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies functional equation (1.4) and

$$h(f(x), T(x)) \leq \frac{\theta}{2^3 - 2^p} \| x \|^p$$

for all $x \in X$.

**Proof.** The proof follows from Theorem 2.9 by setting $\phi(x_1, x_2, \ldots, x_n) = \theta \sum_{i=1}^{n} \| x_i \|^p$ for every $x_1, x_2, \ldots, x_n \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. \qed

**Remark 2.12.** For the case of $p > 3$ in the above corollary, we also obtain a unique $n$-dimensional cubic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies the functional equation (1.4) and the difference between the cubic-like set-valued function $f$ and the $n$-dimensional cubic set-valued mapping $T$ as follows,

$$h(f(x), T(x)) \leq \frac{\theta}{2^p - 2^3} \| x \|^p$$

for all $x \in X$.

**Acknowledgement.** The authors are deeply grateful to the referees whose advices helped to improve our manuscript.

**References**


[18] , On selections of set-valued inclusions in a single variable with applications to several variables, Results Math. 64 (2013), no. 1-2, 1–12.


HAEUNG-YUN CHU
DEPARTMENT OF MATHEMATICS
CHUNGNA NATIONAL UNIVERSITY
DAEJEON 34134, KOREA
E-mail address: hychu@cnu.ac.kr

SEUNG KI YOO
DEPARTMENT OF MATHEMATICS
CHUNGNA NATIONAL UNIVERSITY
DAEJEON 34134, KOREA
E-mail address: skyoo@cnu.ac.kr