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Let \( H \) denote the class of functions analytic in the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Very recently, M. Nunokawa, S. Owa and J. Sokół published a paper [2] titled “A criterion for bounded functions” in Bull. Korean Math. Soc. 53 (2016), no. 1, pp. 215–225. In this paper, they prove the following theorem [2, Theorem 2.5].

**Theorem A.** Let \( h(z) = \{(1+z)/(1-z)\}^\alpha, \alpha \in (0,1], \) and \( p(z) \) be analytic in \( D \) with \( h(0) = p(0) = 1 \). Assume also that \( \phi(p(z)) \) is analytic in \( D \), moreover \( \Re \{ \phi(h(z)) \} \geq 0 \) in \( D \). If

\[
p(z) + zp'(z)\phi(p(z)) \prec h(z) \quad (z \in D),
\]

then \( p(z) \prec h(z) \) \( (z \in D) \).

In [2] Nunokawa et al. claimed that the above theorem is a new extension of a result by Hallenback and Ruscheweyh [1] because the function

\[
(1) \quad h(z) = \left( \frac{1+z}{1-z} \right)^\alpha \quad (h(0) = 1, \alpha \in (0,1])
\]

is not convex (see [2, p. 222, line 7]). However, such statement is incorrect.

In this note we shall prove that the function \( h(z) \) given by (1) is convex in \( D \). Further, we shall also show that the function

\[
(2) \quad g(z) = \left( \frac{1+z}{1-z} \right)^\alpha + \frac{2\alpha z}{1-z^2} \quad (g(0) = 1, \alpha \in (0,1])
\]

considered in [2] is close-to-convex in \( D \).

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Proposition 1. Let $0 < \alpha \leq 1$. Then the function $h(z)$ given by (1) is analytic and univalently convex in $D$ and

$$h(D) = \{ w : w \in \mathbb{C} \text{ and } -\frac{\alpha\pi}{2} < \arg w < \frac{\alpha\pi}{2} \}.$$ 

Proof. It is easy to see that the transformation

$$t = w^{\frac{1}{\alpha}}$$

maps the convex region

$$G = \{ w : w \in \mathbb{C} \text{ and } -\frac{\alpha\pi}{2} < \arg w < \frac{\alpha\pi}{2} \}$$

conformally onto the right-half $t$-plane $-\frac{\pi}{2} < \arg t < \frac{\pi}{2}$ so that $w = 1$ corresponding to $t = 1$. Since

$$z = \frac{t - 1}{t + 1}$$

maps the right-half $t$-plane $\text{Re}(t) > 0$ onto $D$, from (1), (3) and (4) we find that

$$w = t^\alpha = \left(\frac{1 + z}{1 - z}\right)^\alpha = h(z)$$

maps $D$ conformally onto $G = h(D)$ with $h(0) = 1$. This completes the proof.

Proposition 2. Let $0 < \alpha \leq 1$. Then the function $g(z)$ defined by (2) is close-to-convex in $D$.

Proof. Suppose that

$$g(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha + \frac{2\alpha z}{1 - z^2} := h(z) + Q(z).$$

Then the function $h(z)$ is convex in $D$ and satisfies

$$|\arg \{h(z)\}| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1).$$

The function $Q(z)$ is starlike in $D$. Thus

$$\text{Re}\left\{\frac{zg'(z)}{Q(z)}\right\} = \text{Re}\left\{\frac{h(z) + zQ'(z)}{Q(z)}\right\} > 0,$$

which shows that the function $g(z)$ is close-to-convex in $D$. Now the proof is complete.

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References


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