Continuum Based Plasticity Models for Cubic Symmetry Lattice Materials Under Multi-Surface Loading

Seon, Woo-Hyun* · Hu, Jong-Wan**†

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ABSTRACT

The typical truss-lattice material successively packed by repeated cubic symmetric unit cells consists of sub-elements (SE) proposed in this study. The representative continuum model for this truss-lattice material such as the effective strain and stress relationship can be formulated by the homogenization procedure based on the notation of averaged mechanical properties. The volume fractions of micro-scale struts have a significant influence on the effective strength as well as the relative density in the lattice plate with replicable unit cell structures. Most of the strength contribution in the lattice material is induced by axial stiffness under uniform stretching or compression responses. Therefore, continuum based constitutive models composed of homogenized member stiffness include these mechanical characteristics with respect to strength, internal stress state, material density based on the volume fraction and even failure modes. It can be also recognized that the stress state of micro-scale struts is directly associated with the continuum constitutive model. The plastic flow at the micro-scale stress can extend the envelope of the analytical stress function on the surface of macro-scale stress derived from homogenized constitutive equations. The main focus of this study is to investigate the basic topology of unit cell structures with the cubic symmetric system and to formulate the plastic models to predict pressure dependent macro-scale stress surface functions.

Key Words: Homogenization, Lattice materials, Cubic symmetry, Finite Element (FE)

1. INTRODUCTION

Truss-lattice topologies with the low relative density can be manufactured from high strength alloys due to a simple high press forming performance. Truss-lattice structures are fabricated by packing bent nodes with an interwoven polynomial punched layer (Biagi and Smith 2007). The robust welding is applied to nodal connections. These lattice materials can be used as the cores of sandwich plates with solid flat faces. They are satisfied with the wide range of requirements. Truss-lattice materials have been characterized by providing very high stiffness with regard to light weight ratios. In addition, they can be used in multi functional applications ranged from the ultra light weight materials to automobile, aerospace parts, construction materials, furniture, and sporting tools with both lower producing cost and higher strength for compression. These lattice materials considered herein are generated by a uniform periodic replication of the characteristic truss unit cell.
The individual unit cell takes advantage of the successive packing of each layer by sharing all frame members with each unit cell. This packing system enables lattice materials to maximize the stiffness effect and light weight effect within the limited space.

Several analytical and experimental studies have been performed on the elastic property of the unit cell at the macroscopic level. Equivalent continuum elastic modeling of three dimensional (3D) octet truss (Fuller 1961) as the form of space periodic structure is generated by using the axial stiffness of its struts. This methodology for calculating this axial stiffness is presented by Nayfeh and Hefzy (1978); this approach is similar to 3D crystallographic techniques in order to define feasible geometric symmetric groups associated with regularly oriented cells that organize the homogenized truss structure. Octet-truss can be satisfied with a deterministic equation for stretching dominated pin-jointed frame condition (Gibson and Ashby 1997). Therefore, each parallel group of octahedral lattice struts mostly carries the axial load and axial stiffness due to stretching or compression response provides strength contribution to the elastic constitutive model (Doyoyo and Hu 2006).

In this study, the macroscopic continuum constitutive models are created to predict the mechanical behavior of the unit cell structure subjected to multi-axial loading. The relative density proportional to the strength of the unit cell structure can be controlled by the volume fractions of individual truss members. Thus, the relative density can be calculated by using this constitutive model. The elastic moduli under uni-axial states are investigated within the reasonable range of the relative density determined by the geometric section area of component truss members. On the basis of constitutive equations, analytical stress functions on the macro-scale stress surface are determined by examining whether the micro-scale lattice struts arrive at the certain limit of micro-scale stress state under multi axial loading. These analytical functions to predict accurate stress states are verified with the numerical experimental results obtained by finite element analysis (FEA).

The envelope of analytical stress functions on the macro-scale stress surface can be extended to use the hardening materials applied to the micro-scale strut members. The macro-scale flow rule normal to the pressure dependent yield surfaces is investigated in this study. The generalized plastic constitutive models are formulated by this material plasticity. The relative forces obtained by analytical calculations at each performance stress level are compared with those obtained by the FE tests. The main objective of this research is to develop the homogenized models, evaluate the stress function on the micro-scale stress surface, and investigate the envelope extension due to the plastic flow for lattice strut members.

### 2. BASIC TOPOLOGIES FOR UNIT CELL ELEMENTS

Sub-element (SE) defined as the least sub-structure which consists of the unit-cell structure represents the basic material property of a symmetry base lattice structure with only three truss members. The typical structural systems of SE are shown in Figure 1. All discrete truss members rotate the same angle between the longitudinal axis of each truss member and the global X, Y, or Z coordinate axis. In addition, three members share the same length (either $L$ or $\sqrt{2}L$) within the unit volume and have the connected nodal points at all edges. Therefore, strength and stiffness of its own truss member is uniform along the each global axis. For example, all truss members of cubic SE (Figure 1. (a)) are aligned with X, Y and Z axis, so this SE system takes advantages of the strong axial stiffness along the global coordinate axis. The longitudinal direction of octet truss members (Figure 1. (b)) is normal to that of diagonal truss members (Figure 1. (c)) at the same surface plane of unit volume. However, the stiffness distributions due to axial and bending response along the global axes are the same with both octet and diagonal SE systems. These SE can be stacked as the cubic symmetry base element which results in the unit cell structure (i.e. Octet unit cell). In words, the unit cell element is easily formulated by replicating these SE with rotational mirror symmetry. The structural characteristics of the unit cell elements reflect those of SE. The volume fraction of the truss member to the unit space. The volume fraction for the truss members located on the surface boundary varies because they occupy the different number of the unit space in comparison with the frame members.
located on the inside domain. Discrete SE as a component system is adequate to reflect on this boundary effect within the unit cell system. Thus, the exact volume fraction can be easily estimated by using SE system.

Fig. 1 The basic topology for the sub-element (SE) consisting to the cubic symmetric unit cell

Lattice elements consisting of the typical SE may result in the periodic truss structure by itself. Each lattice element is fabricated by either identical kinds of SE or two more different kinds of SE as shown in Figure 2. For this reason, lattice elements contain the same characteristics with SE in terms of structural topologies, deformation mechanism and effective material properties. These representative lattice elements are able to combine as the one eighth parts of the whole periodic unit cell having cubic symmetry such as octet unit cell and diagonally strengthened cubic unit cell. The sandwich plate can be easily constructed by the successive packing of the periodic lattice materials which tailor in the unit volume.

The whole unit cell element can be obtained by replicating these sub-lattice elements with the mirror symmetry or with 90 degree rotation at one reference axis. The detail packing of lattice elements to produce the cubic symmetric unit cell element is illustrated in Figure 3. For example, octet unit cell element is packed by the eight sub-structures composed of the tetrahedral lattice elements. Eight lattice elements as the substructure of the unit cell element fabricate the effective material property of basic cubic symmetry structure on the 3D global axis coordinate. In word, both relative density and continuum constitutive modeling in the unit cell structure reflect on the structural characteristics of lattice elements. Geometric symmetries reduce the number of independent elastic constants at the constitutive model. In addition, they can produce the distinct axial resistance along any axial direction at the unit cell subjected to multi-axial loading.

Fig. 2 The assemblage of the SE element to formulate a lattice element

3. ELASTIC CONSTITUTIVE EQUATIONS

The high mirror symmetry group such as the cubic symmetry makes the process of stiffness and strength tailoring simple. In case of the cubic symmetric materials, the constitutive equations given in the matrix form are generally characterized by three independent elastic constants. These macro-scale elastic constants appear as
the equivalent uni-axial stiffness \((C_i)\) including the volume fraction \((\nu_i)\) which each strut occupies

\[
C_i = E_i \nu_i \varepsilon_i
\]

\[
\nu_i = \alpha \frac{A_i L_i}{V_{\text{unit}}}
\]

where, \(\alpha\) represents the occupancy factor of the strut at one unit space \((V_{\text{unit}})\). \(A_i\) \((A_i = \pi r^2)\) and \(L_i\) denotes the cross sectional area and the length of the strut member, respectively. Hence, the axial stiffness of each strut member can be uniformly distributed into the unit volume space. Each strut aligned with the local member coordinate should be transformed into the global coordinate associated with its elastic constant using the linear transformation tensor \((\mathbf{N})\).

\[
\mathbf{C} = E_i \nu_i \mathbf{N}^T \mathbf{N}
\]

The transformation tensor can be converted into the dyadic product expressed as matrix form and tensorial form, respectively,

\[
N = n_x \otimes n_y \otimes n_z
\]

\[
N_{ij} = (n_x \otimes n_y)_{ij}
\]

where, \(n_x\) indicates the unit direction vector which is aligned with the current direction of the longitudinal axis of the strut under the condition of small-strain formulation. Therefore, the macro-scale constant stiffness tensor for SE elements can be calculated as follows;

\[
C_{ijkl} = \sum_{k=1}^{3} E_i \nu_i (k) \mathbf{N}(k) \mathbf{N}(k)_{ij} = 1 \cdots 3
\]

The characteristic for the cubic symmetry associated with each geometric symmetry group can be found on this constant stiffness tensor. Similarly, we can convert this tensorial equation into the specific matrix equation as follows;

\[
\sum_{11} = \begin{pmatrix}
C_{1111} & C_{1112} & C_{1122} & 0 & 0 & 0 \\
C_{1112} & C_{1122} & C_{1111} & 0 & 0 & 0 \\
C_{1122} & C_{1111} & C_{1122} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{1212} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{1212}
\end{pmatrix}
\]

\[
\sum_{12} = \frac{E_i \pi r^2}{2 \sqrt{2} L^2}
\]

\[
\sum_{21} = \begin{pmatrix}
C_{1111} & C_{1112} & C_{1122} & 0 & 0 & 0 \\
C_{1112} & C_{1122} & C_{1111} & 0 & 0 & 0 \\
C_{1122} & C_{1111} & C_{1122} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{1212} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{1212}
\end{pmatrix}
\]

For \(i \neq j\) case, \(E_{ij}\) (i.e. \(E_{12}\)) represents the total shear strain. The unit cell element can be generated by assembling these sub-elements (SE) as shown in Figure 2. Similarly, the elastic stiffness tensor for the unit cell element can be formulated by superposing constant elastic stiffness tensors for component SE. The complete elastic constitutive equations for all unit cell elements given in this study are summarized as follows;

\[
\sum_{11} = \begin{pmatrix}
C_{1111} & C_{1112} & C_{1122} & 0 & 0 & 0 \\
C_{1112} & C_{1122} & C_{1111} & 0 & 0 & 0 \\
C_{1122} & C_{1111} & C_{1122} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{1212} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{1212}
\end{pmatrix}
\]

The quantities of elastic constants, \(C_{1111}\), \(C_{1122}\) and \(C_{1212}\) are given as follows;

For the octet unit cell element:

\[
C_{1111} = 2
\]

\[
C_{1122} = 1
\]

\[
C_{1212} = 2
\]

For the diagonally strengthened cubic unit cell element

\[
C_{1111} = (2 + \sqrt{2})/2
\]

\[
C_{1122} = 1/2
\]

\[
C_{1212} = 1/2
\]

For the strengthened octet unit cell element

\[
C_{1111} = (2 + \sqrt{2})
\]

\[
C_{1122} = 1
\]

\[
C_{1212} = 1
\]
The same size of the cross sectional area is applied to all strut members. Only three independent elastic constants are required for the basic constitutive equation as shown in Equation (11) due to the symmetric arrangement of the strut members. The discrete strut members in the micro-scale level can be converted into the continuum media by using their volume fraction in the unit cell elements. Therefore, the micro-scale elastic constitutive model is considered as the continuum system because component elastic constants possess the volume fractions of all strut members. It implies that the constitutive equations possess the information of the relative density for the unit cell element.

The relative density ($\rho^\ast$) results in the significant mechanical property of the lattice materials to determine their light weight effect (박원태 2010). It is determined by the ratio of the macro-scale material density to the solid basis material density within the same unit volume space. Hence, it is calculated by the sum of all volume fractions as given in below

$$\rho^\ast = \sum_{k=1}^{n} (\nu_k)^{(k)} = \frac{1}{E} \sum_{k=1}^{n} (C)^{(k)} = \frac{3C_{1111} + 6C_{1122}}{E} \quad (15)$$

We can describe the specific values of relative densities for all unit cell cases

Octet unit cell : $\rho^\ast = 3\sqrt{2} \pi r^2 / L^2 \quad (16a)$

Diagonally strengthened cubic unit cell : $\rho^\ast = (6\sqrt{2} + 3) \pi r^2 / 4L^2 \quad (16b)$

Strengthened octet unit cell element : $\rho^\ast = (3\sqrt{2} + 3) \pi r^2 / L^2 \quad (16c)$

The relative density may be expressed as the square ratio of the radius of the solid strut member to the length of the unit cell ($r^2 / L^2$). When investigating the continuum elastic moduli (Typ. $E^\ast$ and $G^\ast$) for the lattice materials, it is necessary to introduce the relative density corresponding to their mechanical property in order to estimate their optimal strength as considering the weight effect. Continuum elastic moduli for each unit cell element according to the increment of the relative density are illustrated in Figure 4. As increasing the relative density proportional to the square ratio, elastic modulus ratio of the continuum elastic modulus of the unit cell element to that of the solid strut member increase very rapidly. Solid strut members are connected by pin-joint conditions, so the axial stiffness under stretching or compression only contributes to formulating the elastic constants in the constitutive equations. The bending at the cubic lattice member due to the global shear force is negligible in comparison with the stretching due to the global axial force. Shear modulus ratios between case A (Octet unit cell element) and case C (Strengthened octet unit cell element) are perfectly identical.

![Fig. 4 Comparisons of elastic moduli for cubic symmetric unit cell materials](attachment:image.png)

4. APPLICATION EXAMPLES

4.1 Basic Modeling and Approach

The geometry and packing of unit cell elements with the cubic symmetry are sketched in Figure 5. These lattice materials are generated by the uniformly periodic replication of the characteristic unit cell elements, so the mechanical behavior of the component unit cell can represent that of the whole lattice material. As a result, the numerical tests performed on one unit cell element can be replaced for those with lattice plates. Three typical unit cell elements, which are the octet unit cell (Case A), diagonally strengthened cubic unit cell (Case B), and strengthened octet unit cell (Case C) were selected to investigate the mechanical behavior of the cubic symmetry lattice material.
All strut members were designed with the same cross section area, so the ratio of the radius of the strut member to the length of the unit volume space \((r/L)\) is identical to all model cases. It is typically taken as the value of 0.1 \((r/L=0.1)\).

The material property models of the solid basis strut member are summarized in Figure 6. The previous study for the lattice materials (Deshpande et al. 2001) accepted the perfectly elasto-plastic model. The mechanism-based multi-surface plastic models under the use of this material model were developed to represent the complex behavior of cellular solids. Numerical tests were performed by using nonlinear finite element (FE) program ABAQUS (2006). The material property of LM 25 aluminum (Blue dotted line shown in Figure 6) measured by the material pull test is assigned with FE models for unit cells. The behavior of LM 25 is simulated by the Ramberg-Osgood model (Red dotted line shown in Figure 6) in order to apply the accurate material model to analytical predictions for the macro-scale stress surface. The stress and strain constitutive law for the strut member at the micro-scale level is employed to the analytical predictions as follows;

\[
\varepsilon = \frac{\sigma}{E_s} + \frac{\alpha \sigma}{E_s} \left( \frac{\sigma}{\sigma_y} \right)^n \quad (17)
\]

where, \(\sigma/E_s\) indicates the elastic strain and \(\alpha \sigma/E_s\) indicates the yield offset typically taken as the value of 0.002. The power coefficient \(n\) is assumed to be 8.9. The elastic modulus \(E_s\) and the ultimate stress \(\sigma_y\) were taken equal to 70 GPa and 170 MPa.

The plastic extension of the yield stress surface at the macro-scale level is investigated through the analytical predictions and FE test results at the stress performance level. The stress performance level shown in Figure 7 consists of the yield point, hardening point, and ultimate point for the micro-scale stress at the strut members.

In the FE models for cubic symmetric unit cells, each cylindrical strut member is modeled by the quadratic beam element (B32element in the ABAQUS program). The size and material property of strut members were assigned into this element. In order to avoid the rigid body motion and rotation, some edge points at the both end tip of structures were constrained. The uniform loads based on the displacement control were applied to highlighted nodal points in red at the same surface. The uni-axial loads as well as the multi-axial loads with the same loading strain rates \((i.e. \dot{\varepsilon}/E_s)\) were applied to FE models. The values of the applied forces were automatically calculated by using the history out-put command in the ABAQUS program. For the small strain formulation, the geometric linearity is applied during FE tests. The member stress of the cylindrical strut and the strain of the unit cell were obtained by the FE test. The
macro-scale stresses were computed by substituting these strains at the specific micro-scale stress level into the constitutive equations.

4.2 Plastic Stress Surface

The analytical prediction for the macro-scale stress surface function can be derived by the macro-scale and micro-scale stress relationship shown in Equation (18). We recall the equation as the tensorial form.

\[
\sum \sigma_{kk} = \sum_{k=1}^{n} \sigma(N_{kk} E_{kk})^{(k)} (n_{o} \otimes n_{o})^{(k)}
\]

(18)

\[
\sum \sigma_{kk} = \sum_{k=1}^{n} \sigma(N_{kk} E_{kk})^{(k)} (n_{o} \otimes n_{o})^{(k)}
\]

(19)

The strain rate at the micro-scale level can be predictable on the ground that all component strut members are transformed into the applied macro-scale strain with the same unit vector. For instance, all strut members placed on the planes (XX-YY and XX-ZZ) parallel to the loading direction have the same unit vector \( n = \sqrt{2}/2 \) in the tetrahedral lattice element. Similarly, the stress state of the strut member can be estimated by using the micro-scale stress strain relationship (See Figure 6). Therefore, the analytical predictions to estimate the macro-scale stress state include the internal variable states based on the micro-scale stress or strain. The extension of the stress surface due to the plastic hardening deformation of the strut members are investigated as compared analytical predictions with FE test results at the performance level of micro-scale stress.

For the octet unit cell (Case A), all strut members are aligned with the same unit directional vector. It is the cause that the strain rate, strain, and stress are constant at the strut members on the same stress plane surface. In addition, the regular models of the failure occurred at the surface of the unit cell under the multi-axial loads. Based on these characteristics of the octet unit cell, the analytical prediction at each performance stress level is derived to determine the stress function in the macro-scale stress surface.

Basic equation:

\[
|\Sigma_{ij}| + |\Sigma_{jj}| = 2 \varepsilon^{(ext)} \sigma^{(ext)} \quad \text{ (ii-jj stress surface)}
\]

(20a)

\[
|\Sigma_{ij}| + 2|\Sigma_{jj}| = 2 \varepsilon^{(ext)} \sigma^{(ext)} \quad \text{ (ii-ij stress surface)}
\]

(20b)

1D Yield State: \( \varepsilon^{(ext)} = \varepsilon_{y} \) and \( \sigma^{(ext)} = \sigma_{y} \)

(20c)

1D Yield State: \( \varepsilon^{(ext)} = \varepsilon_{u} \) and \( \sigma^{(ext)} = \sigma_{u} \)

(20d)

The general volume fraction for each truss member embedded in the infinitive unit cell volume are given to

\[
u_{v} = \frac{\sqrt{2} A_{L} L}{2 V_{\text{unit}}} = \frac{\sqrt{2} \pi (\frac{r}{L})^{2}}{2}
\]

(21)

The applied macro-scale strains can be transformed into the micro-scale strains with the unit direction vector. Comparison between the analytical predictions and FE test results is illustrated in Figure 8. All results show good agreements at the extended macro-scale stress surface. The uni-axial stresses are located on the interceptions of the stress surface and stress surface functions are enclosed by interpolating each two uni-axial stress points. The stress surface is extended as increasing the plastic deformation of the strut members. The shape modes determined by the truss members with the performance level of macro-scale stress are dependent on the loading combinations.

\[
\rho = \frac{3 (\Sigma_{11} + \Sigma_{22} + \Sigma_{33})}{\sigma^{(ext)}} = 3 \sqrt{2} \pi (\frac{r}{L})^{2}
\]

(22)

This relative density is identical to Equation (16a).
The diagonally strengthened cubic unit cell (Case B) consists of two different sub-elements (SE), diagonal SE and cubic SE. The truss members in the cubic SE were transformed with the unit direction vectors parallel to the global axes for the axial loads. The axial strength is significantly upgraded due to the cubic lattice members. However, the truss members parallel to the axial loads are susceptible to the plastic failure because of the fast strain rate. In words, the truss members in the cubic SE possess faster strain flow rate rather than those in the diagonal SE. The relationships between micro-scale and macro-scale strains are defined by

\[ \varepsilon_{\text{macro}} = N_{\text{sub}} \varepsilon_{\text{sub}} \]

\[ \sigma_{\text{macro}} = N_{\text{sub}} \sigma_{\text{sub}} \]

\[ \varepsilon_{\text{sub}} = \varepsilon_{\text{macro}} \]

\[ \sigma_{\text{sub}} = \sigma_{\text{macro}} \]

\[ \varepsilon_{\text{sub}} = \varepsilon_{\text{sub}}(\sigma_{\text{sub}}) \]

\[ \sigma_{\text{sub}} = \sigma_{\text{sub}}(\varepsilon_{\text{sub}}) \]

\[ \varepsilon_{\text{sub}} = \varepsilon_{\text{sub}}(\sigma_{\text{sub}}) \]

\[ \sigma_{\text{sub}} = \sigma_{\text{sub}}(\varepsilon_{\text{sub}}) \]

The analytical prediction at each performance stress level is derived to determine the stress function in the macro-scale stress surface

\[ |\Sigma_{ii}| + |\Sigma_{jj}| = \frac{\Sigma_{ii}}{\sigma_{\text{yield}}} + \frac{\Sigma_{jj}}{\sigma_{\text{yield}}} = 1 \]

Internal variables from both sub-element systems such as micro-scale stress and volume fraction are found in the analytical prediction. The bending effect happening at the cubic lattice member due to the shear force is negligible in comparison with the stretching effect. As a result, the macro-scale stress function in ii-ij surface is formulated as the elliptical form. Using the constitutive equation for 1D truss member (\( \sigma = \sigma(e) \)), micro-scale strain and stress relationships for both diagonal truss member (\( \varepsilon_{\text{sub}}(\sigma_{\text{sub}}) \) and \( \sigma_{\text{sub}}(\varepsilon_{\text{sub}}) \)) and cubic truss member (\( \varepsilon_{\text{sub}}(\sigma_{\text{sub}}) \) and \( \sigma_{\text{sub}}(\varepsilon_{\text{sub}}) \)) are converted into

1D Yield State: \( \varepsilon_{\text{yield}} = \varepsilon_{\text{sub}}(\sigma_{\text{yield}}) = \sigma_{\text{yield}}(\varepsilon_{\text{sub}}) \)

1D Strain Hardening: \( \varepsilon_{\text{yield}} = \varepsilon_{\text{sub}}(\sigma_{\text{yield}}) = \sigma_{\text{yield}}(\varepsilon_{\text{sub}}) \)

1D Ultimate State: \( \varepsilon_{\text{yield}} = \varepsilon_{\text{sub}}(\sigma_{\text{yield}}) = \sigma_{\text{yield}}(\varepsilon_{\text{sub}}) \)
The general volume fraction for each truss member embedded in the infinitive unit cell volume are given to

\[
\begin{align*}
\nu_{V_{\text{(dia)}}}^{(\text{dia})} & = \frac{\sqrt{2} A L}{2 V_{\text{unit}}} = \frac{\sqrt{2} \pi}{2} \left( \frac{r}{L} \right)^2 \\
\nu_{V_{\text{(sub)}}}^{(\text{sub})} & = \frac{A L}{4 V_{\text{unit}}} = \frac{\pi}{4} \left( \frac{r}{L} \right)^2
\end{align*}
\]

Comparison between the analytical predictions and FE test results is illustrated in Figure 10. All results show good agreements at the extended macro-scale stress surface. The shape modes were based on the stress state of the diagonal truss members. Similarly to the octet unit cell, the relative density can be computed at the state of the hydrostatic pressure

\[\rho^* = \frac{3\sqrt{2}}{2} \pi \left( \frac{r}{L} \right)^2 + \frac{3}{4} \pi \left( \frac{r}{L} \right)^2\]

Finally, the strengthened octet unit cell (Case C) consists of the tetrahedral lattice elements with three different sub-elements (SE) as shown in Figure 2 (b). The strut members in the octet SE and the diagonal SE were aligned with the same transformation angle to the axes. Two different transformation angles, incline or parallel to the axes, exist in this type of unit cell structure. Similar to the diagonally strengthened cubic unit cell, the strain rates of the cubic truss members flow more quickly in comparison with those of the octet truss members. The initial yield or ultimate state generally occurs at the cubic truss member because of different strain rates. As a result, the octet truss members are identical to the diagonal truss members in terms of stress distribution, the volume fraction, and unit direction vector. The shape modes were based on the stress state of the octet truss members. The relationships between micro-scale and macro-scale strains are defined by

For cubic SE:

\[e^{(\text{cub})} = N^{(\text{cub})} \bar{E} \]

For diagonal SE:

\[e^{(\text{diag})} = N^{(\text{diag})} \bar{E} \]

The strain in the cubic SE can be calculated by substituting Equation (27b) into Equation (27a) as follows;

\[e^{(\text{cub})} = \frac{N^{(\text{cub})}}{N^{(\text{diag})}} e^{(\text{diag})} \]

The analytical prediction at each performance stress level is derived to determine the stress function in the macro-scale stress surface

\[
\frac{1}{2} (\Sigma_{ij} + \Sigma_{ji})^2 + \frac{1}{2} (\sigma_{ij})^2 = 1
\]

\[2\nu_{\text{(sub)}} \sigma_{\text{(sub)}} \sigma_{\text{(sub)}} + 4\nu_{\text{(sub)}} \sigma_{\text{(sub)}} \sigma_{\text{(sub)}}
\]
Using the constitutive equation for 1D truss member \( \sigma = \sigma(\varepsilon) \) micro-scale strain and stress relationships for both octet truss member \( \varepsilon^{(oct)} \) and \( \sigma^{(oct)} \) and cubic truss member \( \varepsilon^{(cub)} \) and \( \sigma^{(cub)} \) are converted into

1D Yield State: 
\[
\varepsilon^{(oct)} = \varepsilon_y, \quad \sigma^{(oct)} = \sigma(\varepsilon_y), \quad \sigma^{(cub)} = \sigma(N^{(cub)}/N^{(oct)}\varepsilon_y) \quad (29b)
\]

1D Strain Hardening: 
\[
\varepsilon^{(oct)} = \varepsilon_y, \quad \sigma^{(oct)} = \sigma(\varepsilon_y), \quad \sigma^{(cub)} = \sigma(N^{(cub)}/N^{(oct)}\varepsilon_y) \quad (29c)
\]

1D Ultimate State
\[
\varepsilon^{(oct)} = \varepsilon_u, \quad \sigma^{(oct)} = \sigma(\varepsilon_u), \quad \sigma^{(cub)} = \sigma(N^{(cub)}/N^{(oct)}\varepsilon_u) \quad (29d)
\]

The general volume fraction for each truss member embedded in the infinitive unit cell volume are given to

\[
v^{(oct)} = \frac{\sqrt{2} A_L}{2V_{unit}} = \frac{\sqrt{2} \pi}{2} \left( \frac{r}{L} \right)^2 \quad (30a)
\]

\[
v^{(cub)} = \frac{A_L}{4V_{unit}} = \frac{\pi}{4} \left( \frac{r}{L} \right)^2 \quad (30b)
\]

The volume fraction of the octet truss member is identical to that of the diagonal truss member shown in Equation (25a). Similarly, the relative density can be calculated at the state of the hydrostatic pressure as follows

\[
\rho^* = 3\sqrt{2} \pi \left( \frac{r}{L} \right)^2 + 3\pi \left( \frac{r}{L} \right)^2 \quad (31)
\]

Comparison between the analytical predictions and FE test results is illustrated in Figure 11. The high relative density increases the envelope size. All results show good agreements at the extended macro-scale stress surface. Therefore, good agreements confirm the accuracy and adequacy of the analytical predictions for the macro-scale stress surface.

5. CONCLUSION

Analytical derivations and FE tests were performed to understand the mechanical property and behavior of the unit cell element with the cubic symmetry. The cubic symmetric unit cell element is formulated by the periodic replication of the characteristic least sub-elements. These component sub-elements (SE) contribute to producing the typical property of their packed unit cell element. The effective mechanical properties for the cubic symmetric lattice materials were investigated through elastic constitutive equations. The packing characteristic, relative density, stiffness contribution, and component sub-elements could be analyzed by tailoring the elastic constant stiffness tensor in the equivalent continuum system.

This equivalent continuum property from discrete truss members is simulated by the homogenization. The localized stresses at the micro-scale level were uniformly

(a) plastic surface in the axial-axial (ii-ii) space for strengthened octet unit cell material

(b) plastic surface in the axial-shear (ii-ij) space for strengthened octet unit cell material
distributed in the unit volume space by using the volume fraction of the truss member. The relationships between micro-scale and micro-scale variables were established through the homogenization theory. The analytical predictions to determine the macro-scale stress surface function were expressed as the localized stresses for the discrete truss members. The macro-scale stress surface function is extended as increasing the plastic deformation of internal truss members. The normality flow rule is used for the plastic formulation at the micro-scale level. The macro-scale stress surfaces were extended as the plastic deformation increase due to the hardening material effect. Analytical predictions to determine the stress surface function show good agreements with FE test results.

REFERENCE