EAS Solid Element for Free Vibration Analysis of Laminated Composite and Sandwich Plate Structures

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Abstract: This study deals with an enhanced assumed strain (EAS) three-dimensional element for free vibration analysis of laminated composite and sandwich structures. The three-dimensional finite element (FE) formulation based on the EAS method for composite structures shows excellence from the standpoints of computational efficiency, especially for distorted element shapes. Using the EAS FE formulation developed for this study, the effects of side-to-thickness ratios, aspect ratios and ply orientations on the natural frequency are studied and compared with the available elasticity solutions and other plate theories. The numerical results obtained are in good agreement with those reported by other investigators. The new approach works well for the numerical experiments tested, especially for complex structures such as sandwich plates with laminated composite faces.

Key Words: EAS three-dimensional finite element, free vibration, laminates, composite structures, sandwich plates

1. Introduction

The structural members made of laminated composite and sandwich materials has been increased significantly due to their merits such as low density, high stiffnesses and high strengths. Analytical and numerical methods for free vibration of composite structures have been studied previously by a host of investigators using a variety of approaches. Srinivas et al. (1970) carried out a flexural vibration analysis of rectangular plates using an analytical solution. Noor and Burton (1990) developed an exact three-dimensional solution for stress and free vibration analysis of multilayered composite plates. Kant and Swaminathan (2001) presented analytical solution for free vibration of laminated composite and sandwich plates based on a higher-order shear deformation theory (HSDT). These works, based on analytical approaches, have limited capabilities in dealing with complex problems, primarily due to their limitations in handling different loading and
boundary conditions in the analysis. In order to overcome the limitations, many studies have been carried out using the finite element method for the free vibration analysis of rectangular composite plates based on different plate theories (Wu et al., 2010; Matsunaga, 2000; Lee and Wooh, 2004; Nayak et al., 2002). In general, finite elements with four or eight nodes per element can describe easily and accurately the kinematic behavior of a rectangular composite plate. However, the elements should be insensitive against mesh distortion that frequently occurs due to modern mesh generation tools or during finite deformations. Additionally such elements should not lock in thin structures and thus be applicable to shell problems. It is known that the best overall performance can be achieved by the assumed strain method developed by Simo and Hughes (1986). Andelfinger and Ramm (1993) presented the enhanced assumed strain (EAS) elements for two-dimensional, three-dimensional, plate and shell structures and their equivalence to HR-elements. Braes (1998) studied EAS elements and locking in membrane problems. Korelc et al. (2010) developed an improved EAS brick element for finite deformation in finite elasticity and plasticity. Computations with the EAS element are free from the shear locking and can yield accurate results for distorted element shapes (Han et al., 2006).

All these works are limited, in that they can analyze only the structural members made of isotropic materials. Thus, this study developed an EAS solid element for free vibration analysis of laminated composite and sandwich structures made of anisotropic materials. In order to validate the EAS solid element developed in this study, we compare our results with those published by various investigators.

2. Theoretical formulation

2.1 Enhanced assumed strain field

The variational basis of the finite element method with enhanced assumed strain (EAS) fields is based on the principle of Hu-Washizu in the following:

$$\Pi_{\text{EAS}}(u, \varepsilon, \sigma) = \int \left[ \frac{1}{2} \varepsilon : C : \varepsilon + \sigma : (\varepsilon^\prime - \varepsilon) \right] dV + \Pi_{\text{elas}}(u)$$  \hspace{1cm} (1)

where displacement field $u$, strains $\varepsilon$, and stresses $\sigma$ are the free variables, $C$ stands for the material stiffness matrix. The symbol $\rho$ is mass density, which is defined as the inertial force per unit acceleration per unit volume. Prescribed values are marked by an upper bar, namely body force $\mathbf{b}$, surface traction $\mathbf{T}$, and the boundary conditions $\overline{u}$ for prescribed displacements. $\mathbf{T}$ is the transformation matrix that the stresses and strains on the material axis can be transformed to those of the structural axis.

Following the idea of Simo and Rifai (1990), the assumed strains in the finite element calculations can be now split into a compatible part $\varepsilon^c$ that satisfies the geometric field equations in the strong sense and an enhanced part $\varepsilon^\%$:

$$\varepsilon = \varepsilon^c + \varepsilon^\% = \mathbf{B}u + \mathbf{M} \alpha$$  \hspace{1cm} (3)

where $\mathbf{B}$ is the compatible strain-displacement relation matrix, $\mathbf{M}$ is the interpolation matrix for the enhanced assumed strain fields, and $\alpha$ is the vector of the internal strain parameters corresponding to the enhanced strain.

By substituting Eq. (3) into Eq. (1) with three-field functional, we get

$$\Pi_{\text{EAS}}(u, \varepsilon, \sigma) = \int \left[ \frac{1}{2} \varepsilon^\% : C : \varepsilon^\% + \sigma : (\varepsilon^\prime - \varepsilon) \right] dV - \int \mathbf{b} : \varepsilon^\% dV - \int \mathbf{T} : \varepsilon^\% dS - \int \overline{u}^\prime \mathbf{T} dS + \int \mathbf{M} : \alpha dV$$  \hspace{1cm} (4)

The Euler equations for the stationarity of this functional Eq. (4), in which boundary condition, force term, and mass term is removed, are

$$\int \delta u^\prime \mathbf{B}^T [C : \varepsilon^\prime (\mathbf{B}u + \varepsilon^\%)] dV = 0$$  \hspace{1cm} (5)

$$\int \delta \varepsilon dV = 0$$  \hspace{1cm} (6)

$$\int \delta \sigma \varepsilon dV = 0$$  \hspace{1cm} (7)
The enhanced assumed strain, defined in the global coordinate, is interpolated according to Eq. (3):

\[ \hat{\varepsilon} = M \alpha \]

\[ M = \frac{\text{det} J}{\text{det} J} T^T M_i \]  

(8)

where \( \text{det} J \) denotes the determinant of the Jacobian matrix \( J \), \( \text{det} J_0 \) is the determinant of the Jacobian matrix \( J_0 = J_{\xi=\eta=\zeta=0} \) at center \((\xi = \eta = \zeta = 0)\) of the element in the natural coordinate, and \( M_i \) is the shape or interpolation function for the enhanced assumed strain, respectively. According to tensor calculus, \( T_i \) maps the polynomial shape functions of \( M_i \), defined in the natural coordinate, into the global coordinate (Simo and Raffai, 1990).

This transformation is restricted to the origin so that the components of \( T_i \) are constant and the chosen polynomial order is not increased. Then the matrix \( T_i \) contains the components \( J_{ij} \) of \( J_0 \) and can be written as \( J_i \) where are the components of Jacobian matrix \( J_0 \) at the center of the element in the natural coordinate. The Jacobian matrix at the center of the element \((\xi = \eta = \zeta = 0)\) does not originate the unexpected strain energy by the enhanced strain. The revised Jacobian at center of element then guarantees that the patch test is passed.

In Eq. (9), \( M_i \) must be assumed by the linear independent interpolation functions that satisfy the orthogonality of Eq. (6) (Simo and Raffai, 1990). Therefore, an optimal interpolation of \( M_i \) for the enhanced assumed strain can be found by inspecting the polynomial field of the compatible strain in the natural coordinate system. In order to decouple and enhance compatible strains, following complete trilinear 30-parameters interpolation function in the natural coordinate was chosen (Andelfinger and Ramm, 1993):

\[ T_i = \begin{bmatrix}
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33} \\
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33} \\
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33} \\
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33} \\
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33} \\
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33} \\
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33} \\
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33} \\
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33} \\
J_{11} & J_{12} & J_{13} & J_{21} & J_{22} & J_{23} & J_{31} & J_{32} & J_{33}
\end{bmatrix}
\]

(10)

\[ M = \begin{bmatrix}
\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \zeta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \xi & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \eta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta
\end{bmatrix}
\]

(9)

\[ \begin{align*}
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}
\end{align*}
\]

(11)

### 2.2 Finite element formulation

Eqs. (8) and (9) are introduced into the energy principle of Eq. (4), and variation with respect to the unknown parameters \( d_i \) and \( a_i \) results in the following system of equations:

\[ \begin{bmatrix}
K_{cc} & K_{cn} \\
K_{cn}^T & K_{nn}
\end{bmatrix}
\begin{bmatrix}
d_i \\
a_i
\end{bmatrix}
= \begin{bmatrix}
F \\
0
\end{bmatrix}
\]

(12)

where \( F \) is the vector of applied nodal forces used in the displacement method, \( d_i \) is the nodal displacements of node \( i \) in the global coordinate system, and the stiffness matrix \( K_{cc}, K_{cn}, K_{nn} \) are described as

\[ K_{cc} = \int_B B^T Q B dV \]

(13)

\[ K_{cn} = \int_B B^T Q M dV \]

(14)

\[ K_{nn} = \int_B M^T Q M dV \]

(15)

where \( n \) is the number of Gauss points, and \( Q \) is the material stiffness matrix as following:

\[ Q = T C T^T \]

(16)

where \( C \) is the material stiffness matrix as Eq. (17).

\[ C = \begin{bmatrix}
1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\
-\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\
-\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G_{xx} & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G_{yy} & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G_{zz}
\end{bmatrix}
\]

(17)
The stress-strain for the structural axis is obtained by

\[
\sigma = T \sigma_m = T C \varepsilon_m = T C T \varepsilon_s
\]  \hspace{1cm} (18)

Fig. 1 shows the relationship between the structural or problem axis (1−2−3) and the material axis (x−y−z) for a lamina. Finally the stress-strain relations for an orthotropic material in the structural axis can be also expressed as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{31}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\
Q_{21} & Q_{22} & Q_{23} & 0 & 0 \\
Q_{31} & Q_{32} & Q_{33} & 0 & 0 \\
0 & 0 & 0 & Q_{11} & Q_{12} \\
0 & 0 & 0 & Q_{21} & Q_{22} \\
0 & 0 & 0 & Q_{31} & Q_{32}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{23} \\
\varepsilon_{31}
\end{bmatrix}
\]  \hspace{1cm} (19)

Here, \( \alpha_i \) must be removed from Eq.(12), because it is an artificial parameter used for an enhanced strain. Then the static condensation for the strain parameter \( \alpha_i \) finally yields the element stiffness matrix as following:

\[
K' = [K_{11} - K_{16}K_{66}K_{66}]
\]  \hspace{1cm} (20)

In this study, the finite element obtained by these procedures is named by “EAS-SOLID8”.

Assuming uniform distribution of mass whose density is \( \rho \), measured per unit volume, the consistent element mass matrix \( M' \) in terms of nodal displacements, \( \mathbf{u} = [u \ v \ w]^T \), is

\[
M' = \int N^T \rho N \, dV = \sum_{i=1}^{8} \sum_{j=1}^{8} R_i R_j N^T \rho N_j \left| J \right|
\]  \hspace{1cm} (21)

In Eq. (21), \( N \) is the shape function matrix and the matrix \( \left| J \right| \) and \( N_{ji} \) are evaluated at each integration point \((\xi_j, \eta_j, \zeta_j)\) \( N \). The shape function matrix can be written as

\[
N_{[i;3,4]} = \begin{bmatrix}
N_1 & 0 & 0 & N_2 & \ldots & N_8 \\
N_1 & 0 & 0 & N_2 & \ldots & N_8 \\
N_1 & 0 & 0 & N_2 & \ldots & N_8
\end{bmatrix}
\]  \hspace{1cm} (22)

where,

\[
N_i = \frac{1}{8} \left[ (1+\xi_i) (1+\eta_i) (1+\zeta_i) \right]
\]  \hspace{1cm} (23)

The basic equation of vibration analysis of undamped system in the form of an eigen problem is

\[
(K' - \omega^2 M') \mathbf{d} = 0
\]  \hspace{1cm} (23)

where \( \omega_0 \) is the fundamental frequencies, \( \mathbf{d} \) is the displacement vector.

3. Numerical examples

The finite element formulation described earlier is now implemented to compare the results of our technique with those published by other investigators and also to study the influences on the analysis of two or three-dimensional composite structures. Fig. 2 shows the dimensions of a simply supported composite plate analyzed by the aforementioned formulations for the materials whose properties are listed in Table 1.
Note that the properties of Materials I, II and III are normalized by $E_2$. The units of $E_1$, $E_2$, $G_{12}$, $G_{23}$, $G_{13}$ are GPa and that of $\rho$ is kg/m$^3$, respectively.

### 3.1 Cross-ply

Table 2 shows natural frequencies of an antisymmetric cross-ply square laminated plates with $a/h=5.0$ and $a/b=1.0$. The sides of the plate made of Material I are simply supported and the natural frequencies are normalized as $\omega = \sqrt{\frac{a \times b^3}{h^3} \rho \frac{E_2}{E_1}}$. The mesh divisions of $4 \times 4$ or $8 \times 8$ or $10 \times 10$ for a quarter plate model are applied for different number of layers ($2$, $4$, $8$, and $10$) and $E_1/E_2$ ratios. From $8 \times 8$ mesh division, the numerical results obtained from this study using EAS-SOLID8 are in good agreement with those reported by other investigators.

In order to validate the FEM code developed for different side-to-thickness ratios, the natural frequency of a symmetric cross-ply composite laminate made of Material II is computed and compared with various results reported by other investigators. Table 3 shows non-dimensionalized fundamental frequencies of a simply supported symmetric cross-ply square laminated plates for different side-to-thickness ratios. It can be observed from the table that present results are in very good agreement with analytical solutions of Kant-Swaminathan regardless of side-to-thickness ratios. It is known in general that for thin plates a somewhat difference exists between the results obtained from solid and 2-D models (Noh et al., 2012). On the other hand, the frequencies obtained by the EAS-SOLID8 give better accurate because of the significant influences of the enhanced assumed strain.

### 3.2 Angle-ply

Non-dimensionalized natural frequencies of simply supported anti-symmetric angle-ply square laminated plates with different fiber angles and thickness-length ratios ($h/a$). The material properties of the individual layers of plates are given by Material III and the induced natural frequencies are normalized as $\omega = \sqrt{\frac{a \times b^3}{h^3} \rho \frac{E_2}{E_1}}$. Note that a two-dimensional exact solution are used by Noor-Burton and Xiaoping, while a $12 \times 12$ mesh of EAS three-dimensional elements is used in this study.

Table 4 represents Non-dimensionalized natural frequencies of a simply supported anti-symmetric angle-ply square laminated plates with $h/a$ made of Material III. The natural frequencies obtained by the EAS three-dimensional element are mostly higher than those by two-dimensional solution. The differences between two and three-dimensional formulations depend on many parameters such as ply angles, number of layers, length-to-thickness ratio, and boundary conditions.

Table 5 shows natural frequencies for a simply supported [$45/-45$]$_4$ square laminated plates with $a/h$. The present model is also compared with the classical plate theory (CPT), the uniform shear deformational theory with shear correction factor 5/6 (USDT) and the parabolic shear deformation theory (PSDT) in predicting the natural frequencies of composite plates made of Material II. It is observed from the table that the results of USDT, PSDT, and Xiaoping theory are in good agreement with those produced by present model regardless of thickness-length ratios.

### 3.3 Five-layer sandwich plate

Figs 3–4 show natural frequencies of a simply supported sandwich plate with anti-symmetric cross-ply face sheets for different side-to-thickness ratios($a/h$) and aspect ratios($a/b$). The material properties of the individual layers of plates are given by Material III and the induced natural frequencies are normalized as $\omega = \sqrt{\frac{a \times b^3}{h^3} \rho \frac{E_2}{E_1}}$. y

### Table 1. Mechanical and physical properties of the materials used in this study

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$G_{12}$</th>
<th>$G_{13}$</th>
<th>$G_{23}$</th>
<th>$\nu_{12}$</th>
<th>$\nu_{13}$</th>
<th>$\nu_{23}$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material I</td>
<td>open</td>
<td>-</td>
<td>$E_2$</td>
<td>0.6 $E_2$</td>
<td>0.6 $E_2$</td>
<td>0.5 $E_2$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>1.0</td>
</tr>
<tr>
<td>Material II</td>
<td>40 $E_2$</td>
<td>-</td>
<td>$E_2$</td>
<td>0.6 $E_2$</td>
<td>0.6 $E_2$</td>
<td>0.5 $E_2$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>1.0</td>
</tr>
<tr>
<td>Material III</td>
<td>15 $E_2$</td>
<td>-</td>
<td>$E_2$</td>
<td>0.6 $E_2$</td>
<td>0.5 $E_2$</td>
<td>0.5 $E_2$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>1.0</td>
</tr>
<tr>
<td>Material IV (Sandwich) Face</td>
<td>13.1</td>
<td>0.00689</td>
<td>10.34</td>
<td>0.00689</td>
<td>0.00345</td>
<td>0.00345</td>
<td>0.00345</td>
<td>0</td>
<td>0</td>
<td>1627</td>
</tr>
<tr>
<td>Core</td>
<td>0.00689</td>
<td>0.00689</td>
<td>0.00345</td>
<td>0.00345</td>
<td>0</td>
<td>0</td>
<td>97</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2. Nondimensionalized fundamental frequencies for a simply supported antisymmetric cross-ply square laminated plates with $a/h=5$, $a/b=1.0$.

<table>
<thead>
<tr>
<th>Lamination and number of layers</th>
<th>Source</th>
<th>Mesh</th>
<th>$E_1/E_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>[0/90]_1;</td>
<td>Present study (EAS-SOLID8)</td>
<td>4X4</td>
<td>6.3846</td>
</tr>
<tr>
<td></td>
<td>8X8</td>
<td>6.3159</td>
<td>7.0576</td>
</tr>
<tr>
<td></td>
<td>10X10</td>
<td>6.3078</td>
<td>7.0484</td>
</tr>
<tr>
<td>3D Elasticity (Noor, 1973)</td>
<td>Exact</td>
<td>6.2578</td>
<td>6.9845</td>
</tr>
<tr>
<td>[0/90]_2;</td>
<td>Present study (EAS-SOLID8)</td>
<td>4X4</td>
<td>6.6234</td>
</tr>
<tr>
<td></td>
<td>8X8</td>
<td>6.5525</td>
<td>8.1927</td>
</tr>
<tr>
<td></td>
<td>10X10</td>
<td>6.5440</td>
<td>8.1824</td>
</tr>
<tr>
<td>[0/90]_3;</td>
<td>Present study (EAS-SOLID8)</td>
<td>4X4</td>
<td>6.6706</td>
</tr>
<tr>
<td></td>
<td>8X8</td>
<td>6.5992</td>
<td>8.4205</td>
</tr>
<tr>
<td>Senthilnathan et al. (1987)</td>
<td>HSDT</td>
<td>6.5552</td>
<td>0.4041</td>
</tr>
<tr>
<td>[0/90]_4;</td>
<td>Present study (EAS-SOLID8)</td>
<td>4X4</td>
<td>6.6953</td>
</tr>
<tr>
<td></td>
<td>8X8</td>
<td>6.6235</td>
<td>8.5407</td>
</tr>
<tr>
<td></td>
<td>10X10</td>
<td>6.6150</td>
<td>8.5300</td>
</tr>
</tbody>
</table>

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Table 3. Nondimensionalized fundamental frequencies of a simply supported symmetric cross-ply square laminated plates for different side-to-thickness ratios.

<table>
<thead>
<tr>
<th>Source</th>
<th>MESH</th>
<th>Side-to-thickness ratio (a/h)</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>EAS-SOLID8</td>
<td>4X4</td>
<td>5.1544</td>
<td>7.9926</td>
<td>10.6005</td>
<td>11.2642</td>
<td>11.4765</td>
<td>11.5080</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8X8</td>
<td>5.1197</td>
<td>7.9200</td>
<td>10.4698</td>
<td>11.1411</td>
<td>11.3198</td>
<td>11.3503</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10X10</td>
<td>5.1155</td>
<td>7.9113</td>
<td>10.4543</td>
<td>11.0963</td>
<td>11.3013</td>
<td>11.3316</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Nondimensionalized natural frequencies of a simply supported anti-symmetric $[\theta/-\theta/\cdots]$ angle-ply square laminated plates with h/a (Material III).

<table>
<thead>
<tr>
<th>Lay-up</th>
<th>Source</th>
<th>MESH</th>
<th>$h/a$</th>
<th>0.01</th>
<th>0.1</th>
<th>0.2</th>
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Table 5. Nondimensionalized natural frequencies for a simply supported [45/-45]_4 square laminated plates with a/h (Material II).

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Fig. 3 Non-dimensionalized natural frequency versus side-to-thickness ratio (a/h) of a simply supported five-layer sandwich plate with antisymmetric cross-ply face sheets.

Fig. 4 Non-dimensionalized natural frequency versus aspect ratio (a/b) of a simply supported five-layer sandwich plate with antisymmetric cross-ply face sheets.

It is observed from the figures that the recent results of Rao et al. (2004) and Wu (2010) are in good agreement with those produced by present model regardless of side-to-thickness and aspect ratios. On the other hand, the results of Senthilnathan et al. (1987) and Kan-Swaminathan (2001) theories overestimate the natural frequencies for different side-to-thickness ratios and aspect ratios. In dealing with such complex sandwich types, it is critical to provide high computational efficiency and stable result. EAS-SOLID8 may make contributions to meet the demands.

4. Summary and conclusion

In this study, the solid finite element (EAS-SOLID8) with an enhanced assumed strain field is developed to further study free vibration of laminated composite and sandwich plates. The EAS-SOLID8, in comparison with another conventional element, is more attractive not only because it can analyze general thick plates but also it shows accurate results for thin structures. For laminated composite plates, the results by using the EAS-SOLID8 are observed to be in excellent agreement with the three-dimensional elasticity solutions and other finite element results. For sandwich plates, the recent results of Rao et al and Wu are in good agreement with those produced by the EAS-SOLID8 regardless of side-to-thickness and aspect ratios.

It is concluded from this study that the approach works well for the numerical experiments tested, especially for complex structures such as anisotropic plates. However, the examples might be too simplistic to extract conclusions for varies parameters. In order to prove the effectiveness of the technique, it will be necessary to prove the concept from more complicated...
parameters such as delamination problems.

감사의 글

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References


