An Adaptive Mesh Generation Scheme for the Finite Element Method

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Abstract

Ideal structures in hazard mitigation systems must have automated state identification capabilities in order to function acceptably in case of a severe hazard. This capability must be efficient in terms of real time computations. The finite element method has become the most widely used method of structural analysis and in an automated finite element structural analysis, an adaptive mesh generation scheme has become an essential component. Without an adaptive scheme, the same mesh is used throughout the analysis and this causes computational inefficiency due to fine mesh where they are not needed and inaccuracy due to overly distorted elements formed during the analysis that may proceed undetected as only the initial and the final element shapes are generally checked in practice. This is especially true for dynamic and nonlinear problems in structural analyses. Thus, the finite element mesh for these types of analyses must be dynamically adaptive and computationally efficient. In this paper, an efficient adaptive finite element mesh generation scheme for dynamic analyses of planar problems is described. Representative strain values are used for error estimates and refinements of meshes use combinations of the h-method (node movement) and the r-method (element division) based on a dispersion parameter. A coefficient that depends on the shape of element is used to correct overly distorted elements. A case study shows the validity and computational efficiency of the scheme. The study also demonstrates the potential applicability of the scheme's effective use in complex structural problems such as those under severe environmental hazards such as seismic loads, erratic wind loads, and general nonlinear problems.

Key words : Structural hazard mitigation, Finite element method, Dynamically adaptive finite element mesh

요 지

재난 상황에서 방재시스템의 구조물이 제대로 역할을 수행하려면 구조의 자동화된 상태 파악 기능이 필수적이다. 이 기능은 실시간 연산면에서 매우 효율적이어야 한다. 유한요소법은 구조해석법으로 가장 많이 사용되는 방법으로 자리 잡고 있으며, 자동화된 구조물의 유한요소 해석에서의 요소망은 적응적 형성 기법으로 생산하여야 한다. 적응적 요소망 기능이 없으면, 같은 요소망으로 유한요소 해석을 수행하는데, 이 경우에는 불필요한 부분의 많은 요소에 의한 연산 비효율성이 있을 수 있고, 일반적으로 초기와 최종 요소망 모양만 검토하기 때문에 해석 중 생길 수 있는 과도하게 변형된 요소에 의한 오차 증가도 초래할 수 있다. 다소 복잡한 동적 및 비선형 해석에서 이러한 문제가 자주 발생한다. 본 연구는 시간영역 동적해석에서 전 단계 해석 결과를 사용하여 계산된 대표 변형률 값을 오차 평가에 사용하고, 요소 세분화는 절점 이동인 r-법과 요소 분할인 h-법의 효율적인 조합을 분산 매개변수(dispersion parameter)를 활용하여 계산하고, 모양계수(shape factor) 개념으로 과도하게 변형된 요소 모양을 방지하는 동적인 적응적 유한 요소망 형성 전략을 제시한다. 적용한 예제를 통하여 정확성과 연산 효율성을 검증하고 나아가 방법의 간단함이 지진 하중, 풍하중 및 비선형 해석이 필요한 방재 구조시스템의 복잡한 해석에도 효율적으로 사용될 수 있는 가능성을 보여 준다.

핵심어 : 구조물 방재시스템, 유한요소법, 동적인 적응적 유한요소망

1. Introduction

Ideal structures in hazard mitigation systems must have automated state identification capabilities in order to function acceptably in case of a severe hazard. The finite element method has become the most widely used method of structural analysis and in an automated finite element structural analysis,

an adaptive mesh generation scheme has become an essential component. Without an adaptive scheme, the same mesh is used throughout the analysis and this causes computational inefficiency due to fine mesh where they are not needed and inaccuracy due to overly distorted elements formed during the analysis that may proceed undetected as only the initial and the final element shapes are generally checked in practice. This is

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especially true for dynamic and nonlinear problems in structural analyses. Thus, the finite element mesh for these types of analyses must be dynamically adaptive and computationally efficient. When the method is applied to dynamics analyzed in the time domain, the meshes may need to be modified at each time step as the finite element results of an analysis largely depend on the mesh and the element types used (Bathe and Wilson, 1976; Belytschko *et al.*, 1996; Heesom and Mahdjoubi, 2001; Jang and Lee, 2011; Jeong and Yoon, 2003; Zienkiewicz *et al.*, 2005).

In this paper, a dynamically adaptive finite element mesh generation scheme for dynamic analyses of structures is described. The adaptive mesh generation process involves estimation of error given a mesh and generation of a new improved mesh based on the error (Choi and Jung, 1998; Choi and Yu, 1998; Zienkiewicz and Zhu, 1987). The dynamically adaptive scheme uses computationally simple error estimation based on representative strains (Jeong and Yoon, 2003). The mesh generating scheme combines the r-method (moving an existing node) and the h-method (dividing an element into smaller elements) using a dispersion parameter (de Las Casa, 1988; Yoon, 2005). The scheme includes a check for limiting distortion for each element using shape factor that is easily computed for a particular shape (Yoon and Park, 2010). The dynamic analysis considered is in time domain, based on direct integration. The results are based on the basic procedure outlined in reference Yoon (2013) but the data presented are from diverse refinements on the software and tuning of parameters more clearly described in this paper. A case study using four node quadrilateral isoparametric elements for plane strain is considered. The results of the analyses of the case show that the proposed dynamically adaptive finite element mesh scheme is efficient computationally and have reasonable confidence level for accuracy for complex dynamic problems in structural analysis.

2. Time Domain Dynamic Analysis

The direct numerical integration iterative formulas in the time domain for a typical dynamic structural analysis based on the widely used Newmark- σ method may be summarized as follows (Bathe and Wilson, 1976; Newmark, 1959):

$$u_{i+1} = u_i + (\Delta t)\dot{u}_i + (\Delta t)^2 \left[\left(\frac{1}{2} - \beta \right) \ddot{u}_i + \beta \ddot{u}_{i+1} \right] \quad (1)$$

$$\dot{u}_{i+1} = \dot{u}_i + (\Delta t) [(1 - \gamma)\ddot{u}_i + \gamma\ddot{u}_{i+1}] \quad (2)$$

Here, u_i , $\dot{u}_i$, and $\ddot{u}_i$ are the displacement, velocity, and accel-

eration vectors in the i th step and u_{i+1} , $\dot{u}_{i+1}$, $\ddot{u}_{i+1}$ are the similar quantities in the $i+1$ th step; Δt is the time step duration; β and γ are parameters selected by the user and the recommended values of $\beta=1/4$ and $\gamma=1/2$ are commonly used for efficient convergence.

The matrix equilibrium equations for the $i+1$ th step may be represented as follows:

$$K'_{u_{i+1}} = F'_{i+1} \quad (3)$$

The matrices K' and F'_{i+1} represent the following:

$$K' = K + \frac{1}{\beta(\Delta t)^2} M \quad (4)$$

$$F'_{i+1} = F_{i+1} + \frac{M}{\beta(\Delta t)^2} \left[u_i + (\Delta t)^2 \dot{u}_i + \left(\frac{1}{2} - \beta \right) (\Delta t)^2 \ddot{u}_i \right] \quad (5)$$

Here, K is the stiffness matrix, M is the mass matrix, and is the force vector for the $i+1$ th time step.

3. Error Estimations and R-H Refinements

3.1 Error Estimations with Representative Strains

An adaptive mesh generation scheme automates mesh generation by repeated improvement of the previous mesh based on error estimations of the previous mesh and improvement strategies based on the errors (Zienkiewicz *et al.*, 1987; Zhu *et al.*, 1991). Since error estimations must be computed many times the computation time must be efficient. Norm of a matrix is generally used to represent error of a mesh where the matrix includes values of stress, strain and displacements. Norm of error $\|E\|$ in the domain Ω may be represented as follows:

$$\|E\| = \left[\int_{\Omega} (\varepsilon - \hat{\varepsilon})^T (\sigma - \hat{\sigma}) d\Omega \right] \quad (6)$$

Here, σ and ε are the exact solutions for stress and strain and $\hat{\sigma}$ and $\hat{\varepsilon}$ are the finite element solutions. Since the exact solutions are not generally known, Eq.(6) is only used to define the error and various attempts are generally devoted to estimating errors that resemble Eq.(6). The error defined below are based on the representative strains of element based on the standard deviations of the strains at the Gauss points in the element computed during the previous finite element analysis (Jeong and Yoon, 2003; Yoon and Park, 2010). The rationale for this is that as solution approaches the exact solution, the strains within an element must approach a constant value and thus the standard deviations must approach zero at the Gauss points. For planar problems using quadrilateral elements, the representative strain

value of element i may be represented by the following equations:

$$\|e\|_{ik} = \sqrt{\frac{\sum_{j=1}^{n_{gk}} (\varepsilon_{jk} - \varepsilon_k^*)^2}{n_{gk} - 1}}, k = x, y, xy \quad (7)$$

$$\|e\|_i = \{\|e\|_{ix} + \|e\|_{iy} + \|e\|_{ixy}\} \times \frac{A_i}{A_t} \quad (8)$$

Here, $\|e\|_{ik}$ is the k directional (x , y and xy for planar problems where xy is the shear component) standard deviation of the strain, n_{gk} is the number of Gauss points in the k direction, ε_{jk} is the k directional strain of Gauss point j , ε_k^* is the k directional strain, and $\|e\|_i$ is the representative strain of element i which is used as error for element i . In addition, A_i is the area of element i and A_t is the total area of the entire mesh.

In Eq.(8), the representative error of element i is weighted by the element's area with respect to the total area and thus the relative order of errors for all elements are computed to identify elements to be adaptively changed. Previous studies have shown that this is computationally very efficient for achieving this(Yoon, 2012; Yoon, 2013).

3.2 The R-method of Adaptive Mesh Generation with Shape Factors

The adaptive mesh generation based on the error estimation with representative strains is formulated combining the r-method and the h-method. The r-method moves existing node at coordinates (x, y) to the new adapted coordinates x_a, y_a :

$$x_a = \frac{\sum_{i=1}^{n_a} x_{ci}(\|e\|_i)}{\sum_{i=1}^{n_a} (\|e\|_i)} \quad (9)$$

$$y_a = \frac{\sum_{i=1}^{n_a} y_{ci}(\|e\|_i)}{\sum_{i=1}^{n_a} (\|e\|_i)} \quad (10)$$

Here, x_{ci}, y_{ci} are the coordinates of the centroid of element i , and n_a is the number of elements sharing the considered node. For nodes on the boundary, Eq.(9) and Eq.(10) are moved to the closest point on the boundary.

Possible distortion of an element shape beyond the tolerable limit is checked by limiting the shape factor. Shape factor S_i of a quadrilateral element i with total boundary length L_i may be

defined as follows(Yoon, 2005):

$$S_i = \frac{\sqrt{A_i}}{0.25L_i} \quad (11)$$

The shape factor is defined so that the maximum value is 1 which is for a square. Shape factors for quadrilaterals approaching a triangular shape are less than around 0.8285(shape factor for a right equilateral triangle). A robust shape of a quadrilateral element has approximately equal side lengths and angles. The shape factors of these are close to 1. An attempt to keep shape factors close to 1 throughout the analysis produces reliable results. Generally, shape factors between 0.9 and 1.0 are acceptable and thus the r-method must restrict the node movements with a limitation on the shape factor to be above 0.9 or 0.95.

3.3 The H-method of Adaptive Mesh Generation with Discretization Parameter

The h-method divides an element into smaller elements of the same shape. The elements to be subdivided are based on the discretization parameter d :

$$d = \frac{\alpha \cdot \text{mean}[\|e\|_{\text{initial}}]}{P_{\text{max}}} \quad (12)$$

Here, α is a constant, $\text{mean}[\|e\|_{\text{initial}}]$ is the mean representative strain value of the mesh and P_{max} is the maximum value of the applied load. A parametric study recommends a value between 12.0 and 15.0 for optimal value of α for general dynamics problems. Significant deviations of α from this range result in computational inefficiency for no apparent increase in accuracy(Yoon, 2012).

3.4 The R-H Method of Adaptive Mesh Generation with Dispersion Parameter

The effectiveness of the r-h method depends on strategies that combine the r-method and the h-method. Various means of combining have been studied (de Las Casas, 1988; Jang and Lee 2011; Yoon, 2012). To obtain an optimal combination of the r-method and the h-method, first the representative strain values are normalized at each step so that the minimum and maximum values are from 0 to 100.

A dispersion parameter D is defined as the absolute value of the difference between the mean of the normalized representative strains and the mode of the distribution of normalized representative strains:

$$D = |\text{mean}[\text{normalized}\|e\|_i] - \text{mode}[\text{normalized}\|e\|_i]| \quad (13)$$

A constant need to be set for D to shift between the r-method

and the h-method. A reasonable value for this constant is around 18-20 where if D is larger than this value, the r-method is used, and in other cases, the h-method is used. A value around 19 will entail the ratio of the r-method and the h-method to be about 2 to 3. A higher value for this constant will reduce this ratio. Note that in general, the h-method is much more efficient in reducing overall error as the element sizes are proportionately reduced (e.g. 1 element to 4 or 16 elements of the same shape reduced to 1/4 or 1/16 of the original size). However, the r-method is essential in creating new shapes and orientations that are not overly distorted by using the shape factors given by Eq.(11). The r-h method of adaptive refinement of the mesh is terminated when the change in the sum of the representative strain values is less than the preset tolerance (typically set at 0.01%-0.02%).

4. A Case Study

Fig. 1 shows a two dimensional 600 cm wide and 600 cm high portal steel frame that is considered for a sample case study. The beam and columns are all rectangular 100cm deep steel sections where Young's modulus is $210 \times 10^5 \text{ N/cm}^2$, Poisson's ratio is 0.3, and the unit mass is $7.85 \times 10^{-3} \text{ kg/cm}^3$. Unit thickness is assumed for modeling where 4 node isoparametric quadrilateral bilinear plane strain elements are used. The only applied dynamic load is a concentrated load P at the center of the frame (point B in Fig. 1) where between time $t=0$ and $t=1$ second, one period of a sinusoidal load given by $-500\sin(2\pi t)$ Newton is applied.

Free vibration response continues after the steady state response for 1 second, and the total response time considered is 5 seconds. The time step Δt selected for analysis is 0.005 seconds, yielding 1000 steps for 5 seconds. The adaptive meshes are generated with dispersion parameter $D=19$, discretization parameter $d=13.5$, and the limit on shape factor S_i is set to 0.95. The initial mesh to start the adaptive algorithm is a regular mesh composed of 144 identical square elements. The tolerance set to terminate the adaptive refinements is set at 0.015%

Fig. 1. Geometry of the Frame Example.

for change in the sum of the representative strain from the previous step. A new mesh is generated at every four time steps, i.e., at every 0.02 seconds. Table 1 shows the representative strain values in the initial steps of the dynamic adaptive mesh where the transitions of the r-method and the h-method are shown. Overall, the adaptive mesh scheme satisfies the preset tolerance after about five adaptive steps.

Fig. 2 shows the adaptive mesh at 0.270 seconds when the vertical deflection of point A is at the maximum which is during the steady state response; the number of elements generated here is the maximum at 840. Fig. 2 also shows the mesh when the number of elements is the minimum at 760, occurring at $t=3.760$. Fig. 3 shows some samples of generated meshes during the free vibration stages. Note that the meshes are symmetric, fine meshes are under the load during the steady state response only and around the inner corners, and coarse meshes are at the outer corners and away from the fixed ends where the deformations are minimal and fine elements are not needed. At the boundary between fine and coarse elements, the

Table 1. Representative Strain Values for the Adaptive Scheme

Adaptive Step	Method	Maximum Representative Strain	Sum of Representative Strain	Change in Sum of Representative Strain from previous Step (%)
0		0.0000292	0.0006492	-
1-initial	h	0.0000051	0.0004361	32.82
2	h	0.0000010	0.0002533	41.92
3	r	0.0000009	0.0002510	0.9080
4	r	0.0000008	0.0002501	0.3585
5-final	h	0.0000002	0.0001420	43.22

(a) Maximum number of elements-0.275sec.
[840 elements, 997 nodes], maximum deflection of A

(b) Minimum number of elements-3.760sec.
[760 elements, 903 nodes]

Fig. 2. Maximum and Minimum Number of Elements.

(a) 2.000sec. [820 elements, 957 nodes]

(b) 4.000sec. [832 elements, 969 nodes]

Fig. 3. Samples of Generated Meshes During Free Vibration.

node in the middle of the coarse mesh is slaved to the end nodes for displacement compatibility.

The solution obtained by the dynamic adaptive meshes is named the *strategic* solution. In general practice, a regularly discretized mesh is used throughout the analysis; a regular mesh of square shapes with 576 identical elements are used to simulate this and the solution from this is called the *general* solutions. To estimate a more accurate engineering solution without an adaptive scheme, a finer regular mesh with 2304 elements is generated by dividing each element used in the general solution into four identical square elements. The solution obtained by using this fine mesh is the *engineering* solution. The finite element program is run on Personal Computer with Pentium Dual Core CPU at 2.60 GHz, 2.0 GB RAM, and Windows XP. Fig. 4, Fig. 5 and Fig. 6 show respectively, the comparisons of the vertical displacement of point A, the horizontal (x directional) normal stress of point C, and the vertical (y directional) normal stress of point C of the *engineering*, the *general*, and the *strategy* solutions. The figures show close agreement among the three solutions. However numeric values show that if the engineering solutions are assume to be the most accurate, the errors from the general solutions are much

larger than the errors from the strategic solutions. Table 2 shows the comparative computation times and errors among the *engineering*, the *general*, and the *strategic* solutions. The total error is the square root of the sum of the errors at selected critical points (A, B, C in Fig. 1) where the engineering solutions are assumed to have no error. The data on the table show

Fig. 4. Vertical Displacement of Bottom Mid Point A of the Frame.

Fig. 5. Horizontal Normal Stress at Mid Point C of the Frame.

Fig. 6. Vertical Normal Stress at Mid Point C of the Frame.

that with the dynamic adaptive scheme, errors have been reduced significantly (3.59% to 0.48% for total displacement and 2.82% to 0.24% for stress) with a 350.00% increase in real run time whereas the fine mesh for the entire structure for the engineering solutions required enormous increase in real run time of 1032.93%. The run time includes the computation time to generate adaptive meshes for the strategic solution; this time is approximately 4 minutes 42 seconds or 31.97% of the total real run time. This percentage will be much smaller for realistically large three dimensional models for a typical dynamic and nonlinear analyses and thus the real run time will be compara-

tively much less than the corresponding engineering solution for such problems. In addition, if the adaptive time interval (4 in this case study) is set to be larger, the time for adaptation will be reduced at a cost for a slight increase in error. In any case, although the computing speed of computers is continuously increasing in general, the enormous amount of computing needed in the complex analysis of structures requires computing efficiency in every aspect of the algorithm in order for the method to be practical (Zienkiewicz *et al.*, 2005).

5. Conclusions

An efficient adaptive mesh generation scheme for dynamic analyses of structures in time domain is presented. The scheme uses representative strain values from each element computed from the previous time step for estimation of error and an optimal combination of the r-method and the h-method based on a dispersion parameter for mesh refinement. Shape factors used to correct overly distorted elements are proven to function effectively as the generated meshes only include approximately square elements. Analyses of the applications of the scheme for a case study show that the representative strain values which efficiently compute relative errors among elements successfully identify elements to be refined and that the algorithm is efficient computationally and have reasonable confidence level for accuracy for dynamic problems in structural analysis.

The proposed adaptive mesh algorithm produces robust meshes for finite element analyses; the dynamics analyses in time domain are considered for the case study but the scheme may also be used for general nonlinear analyses where similar to dynamic analyses, many linear analyses are repeated. In dynamic and nonlinear analyses, shape factors play a key role as overly distorted elements are prevented during iterations, that otherwise may have proceeded without being recognized. The scheme is especially appropriate for real time computations of large complex structures under erratic time dependent loads such as earthquakes and turbulent winds. Efficient automated analyses of these dynamic and nonlinear problems are essential part of automated hazard mitigation structural systems. Some aspects of the scheme still may be improved for

Table 2. Comparative Computation Times and Error

Solution Method	Total Displacement Error vs Engineering	Total Stress Error vs Engineering	Computation Time (Increase vs General)
Engineering (2304 elements)	0.00% (assumed exact)	0.00% (assumed exact)	43 min. 23 sec (1032.93%)
General (576 elements)	3.59 %	2.82 %	4 min. 12 sec. (0.00 %)
Strategy (760-840 elements)	0.48%	0.24%	14 min. 42 sec. (350.00 %)

more efficiency; a powerful expert system that generates a reasonable initial mesh that starts the proposed adaptive mesh generation scheme will make the algorithm computationally more efficient.

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