SYMMETRIC SOLUTIONS FOR A FOURTH-ORDER MULTI-POINT BOUNDARY VALUE PROBLEMS WITH ONE-DIMENSIONAL $p$-LAPLACIAN AT RESONANCE

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1. Introduction

In this paper, we are interested in the fourth-order symmetric multi-point BVP with the one-dimensional $p$-Laplacian

\[(\phi_p(x''(t)))'' = f(t, x(t), x'(t), x''(t)), \text{ a.e. } t \in [0, 1],\]

(1.1)

\[x''(0) = 0, \ (\phi_p(x''(t)))'|_{t=0} = 0, \ x(0) = \sum_{i=1}^{n} \mu_i x(\xi_i),\]

(1.2)

\[x(t) = x(1-t) \text{ a.e. } t \in [0, 1],\]

(1.3)

Received March 23, 2011. Revised June 20, 2011. Accepted July 15, 2011. *Corresponding author. †This work was supported by NNSF of China (11071014).

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where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < \frac{1}{2}$, $\mu_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, with
\[
\sum_{i=1}^{n} \mu_i = 1. \tag{1.4}
\]

Throughout we assume:
(A1) $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is symmetric on $[0, 1]$, i.e.
\[
f(t, u, v, w) = f(1 - t, u, -v, w) \quad \text{for} \quad t \in [0, 1]
\]
and satisfies the $L^1$-Carathéodory conditions, and $f(t, b, 0, 0) \not\equiv 0$, $\forall b \in \mathbb{R}$;
(A2) $\sum_{i=1}^{n} \mu_i \xi_i(2q - (2\xi_i)^{2q-1}) \neq 0$.

Due to the condition (1.4), the differential operator on the left side of (1.1) is not invertible. In the literature, BVPs of this type are referred to as problems at resonance.

Boundary value problems with a $p$-Laplacian have received a lot of attention in recent years. They often occur in the study of the $n$-dimensional $p$-Laplacian equation, non-Newtonian fluid theory and the turbulent flow of a gas in porous medium. Many works have been carried out to discuss the existence of solutions or positive solutions, multiple solutions for the local or nonlocal BVPs [3,5,14,16,19].

Multi-point BVPs of ordinary differential equations arise in a variety of different areas of Applied Mathematics and Physics. The study of multi-point BVPs for linear second-order ordinary differential equations was initiated by Il’in and Moiseev [6]. Since then many authors have studied more nonlinear multi-point BVPs [10-19]. The methods used therein mainly depend on the degree theory, fixed-point theorems, upper and lower techniques, and monotone iteration.

Recently, there is an increasing interest in considering some higher order BVPs, we refer the readers to [3-5,19] for details. However, as far as we know, the study of symmetric solutions for fourth-order $p$-Laplacian BVPs has rarely appeared.

Motivated by the papers mentioned above, we aim at studying the BVPs (1.1)-(1.3) at resonance. Due to the fact that the classical Mawhin’s continuation theorem can’t be directly used to discuss the BVP with nonlinear differential operator, in this paper, we investigate the multi-point BVP (1.1)-(1.3) by applying an extension of Mawhin’s continuation theorem due to Ge [2]. Furthermore, an example is given to illustrate the result.

2. Preliminaries

For the convenience of readers, we present here some background definitions and lemmas.

**Definition 2.1.** $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is called a $L^1$-Carathéodory function, if the following conditions hold:
(B1) for each $(u, v, w) \in \mathbb{R}^3$, the mapping $t \mapsto f(t, u, v, w)$ is Lebesgue measurable;
(B2) for a.e. $t \in [0, 1]$, the mapping $(u, v, w) \mapsto f(t, u, v, w)$ is continuous on $\mathbb{R}^3$;
(B3) for each $r > 0$, there exists $\alpha_r \in L^1[0, 1]$ such that for a.e. $t \in [0, 1]$ and every $(u, v, w)$ such that $\max\{|u|, |v|, |w|\} \leq r$, we have $|f(t, u, v, w)| \leq \alpha_r(t)$.

**Proposition 2.1** ([7]). $\phi_p$ satisfies the following properties

(C1) $\phi_p$ is continuous, monotonically increasing and invertible.

Moreover, $\phi_p^{-1} = \phi_q$ with $p > 1$ a real constant satisfying $\frac{1}{p} + \frac{1}{q} = 1$;

(C2) for $\forall u, v \geq 0$, $\phi_p(u + v) \leq \phi_p(u) + \phi_p(v)$, if $1 < p < 2$;

$\phi_p(u + v) \leq 2^{p-2}(\phi_p(u) + \phi_p(v))$, if $p \geq 2$.

**Definition 2.2** ([2]). Let $X$ and $Z$ be two Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Z$, respectively. A continuous operator $M : \text{dom} M \to Z$ is said to be quasi-linear if

(D1) $\text{Im} M$ is a closed subset of $Z$;

(D2) $\ker M = \{ x \in \text{dom} M : Mx = 0 \}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.

**Definition 2.3** ([7]). Let $X$ be a Banach spaces and $X_1 \subset X$ a subspace. A linear operator $P : X \to X_1$ is said to be a projector provided that $P^2 = P$.

The operator $Q : X \to X_1$ is said to be a semi-projector provided that $Q^2 = Q$ and $Q(\lambda x) = \lambda Qx$ for $x \in X$, $\lambda \in \mathbb{R}$.

Let $X_1 = \ker M$ and $X_2$ be the complementary space of $X_1$ in $X$, then $X = X_1 \oplus X_2$. On the other hand, suppose $Z_1$ is a subspace of $Z$ and $Z_2$ is the complementary of $Z_1$ in $Z$, so that $Z = Z_1 \oplus Z_2$. Let $P : X \to X_1$ be a projector and $Q : Z \to Z_1$ be a semi-projector, and $\Omega \subset X$ an open and bounded set with the origin $\theta \in \Omega$, where $\theta$ is the origin of a linear space.

Suppose $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0, 1]$ is a continuous operator. Denote $N_1$ by $N$. Let $\sum_\lambda = \{ x \in \overline{\Omega} : Mx = N_\lambda x \}$.

**Definition 2.4** ([2]). $N_\lambda$ is said to be $M$-compact in $\overline{\Omega}$ if

(D3) there is a vector subspace $Z_1$ of $Z$ with $\dim Z_1 = \dim X_1$ and an operator $R : \overline{\Omega} \times [0, 1] \to X_2$ continuous and compact such that for $\lambda \in [0, 1]$,

$$(I - Q)N_\lambda(\overline{\Omega}) \subset \text{Im} M \subset (I - Q)Z, \quad (2.1)$$

$$QN_\lambda x = 0, \quad \lambda \in (0, 1) \iff QN_\lambda x = 0, \quad (2.2)$$

$R(\cdot, 0)$ is the zero operator and

$$R(\cdot, \lambda)\sum_\lambda = (I - P)\sum_\lambda, \quad (2.3)$$

$$M[P + R(\cdot, \lambda)] = (I - Q)N_\lambda, \quad (2.4)$$

**Theorem 2.1** ([2]). Let $X$ and $Z$ be two Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Z$, respectively, and $\Omega \subset X$ an open and bounded set. Suppose $M : \text{dom} M \to Z$ is a quasi-linear operator and $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0, 1]$ is $M$-compact. In addition, if
It is clear that the operator and (1.3), then
Define $M$ denoted by $Z$ and
Thus, in view of (1.3) and 
and then
Let $\Omega$ be a bounded domain with the norm $|| \cdot ||$ and $Z = \{ z \in L^1[0,1] : z(t) = z(1-t), t \in [0,1] \}$ with the usual Lebesgue norm denoted by $|| \cdot ||_1$.
Define $M : \text{dom} M \to Z$ by $Mx(t) = (\phi_p(x''(t)))''$ with

For any open and bounded $\Omega \subset X$, we define $N_\lambda : \overline{\Omega} \to Z$ by $N_\lambda x(t) = \lambda f(t, x(t), x'(t), x''(t))$, $t \in [0,1]$. Then the BVP (1.1)-(1.3) can be written as $Mx = Nx$.

Lemma 3.1. The operator $M : \text{dom} M \to Z$ is quasi-linear.

Proof. It is clear that $X_1 = \ker M = \{ x \in \text{dom} M : x = a \in \mathbb{R} \}$.
Let $x \in \text{dom} M$ and consider the equation $(\phi_p(x''(t)))'' = z(t)$ subject to (1.2) and (1.3), then $z \in Z$. It follows from (1.2) and the symmetric conditions that

\[ x'(t) = - \int_t^2 \phi_q \left( \int_0^s (s - \tau)z(\tau)d\tau \right) ds, \quad \text{(3.1)} \]

and then

\[ x(t) = - \int_0^t \int_s^2 \phi_q \left( \int_0^\tau (\tau - k)z(k)dk \right) d\tau ds + x(0). \quad \text{(3.2)} \]

In view of (1.3) and $\sum_{i=1}^n \mu_i = 1$, we get

\[ \sum_{i=1}^n \mu_i \int_0^t \int_s^2 \phi_q \left( \int_0^s (s - \tau)z(\tau)d\tau \right) dsdt = 0. \quad \text{(3.3)} \]

Thus,

\[ \text{Im} M \subset \left\{ z \in Z : \sum_{i=1}^n \mu_i \int_0^t \int_s^2 \phi_q \left( \int_0^s (s - \tau)z(\tau)d\tau \right) dsdt = 0 \right\}. \quad \text{(3.4)} \]
Conversely, if (3.3) holds for \( z \in Z \), we take \( x \in \text{dom} \ M \) as given by (3.2) and establish that it is symmetric and \((\phi_p(x''(\cdot)))'\) is absolutely continuous along with derivative, then \((\phi_p(x''(t)))'' = z(t)\) for \( t \in [0, 1] \) and (1.2) and (1.3) are satisfied. Together with (3.4), we have

\[
\text{Im} M = \left\{ z \in Z : \sum_{i=1}^{n} \mu_i \int_{0}^{s} \int_{t}^{2} \phi_q \left( \int_{0}^{s} (s - \tau)z(\tau)d\tau \right) ds dt = 0 \right\}. \tag{3.5}
\]

So, \( \text{dim} \ker M = 1 < \infty \), \( \text{Im} M \subset Z \) is closed. Therefore, \( M \) is a quasi-linear operator. \( \Box \)

**Lemma 3.2.** The operator \( \bar{N}_\lambda : \bar{\Omega} \to Z \) is \( M \)-compact in \( \bar{\Omega} \).

**Proof.** We recall the condition (A2) and define the continuous operator \( Q : Z \to Z_1 \) by

\[
Qz(t) = 2\phi_p \left( \sum_{i=1}^{n} \mu_i \int_{0}^{s} \int_{t}^{2} \phi_q \left( \int_{0}^{s} (s - \tau)z(\tau)d\tau \right) ds dt \right). \tag{3.6}
\]

It is easy to check that \( Q^2 z = Qz \) and \( Q(\lambda z) = \lambda Qz \) for \( z \in Z \), \( \lambda \in \mathbb{R} \), that is, \( Q \) is a semi-projector and \( \text{dim} X_1 = \text{dim} Z_1 = 1 \). In addition, (3.5) and (3.6) imply that \( \text{Im} M = \ker Q \).

Let \( \Omega \subset X \) be an open and bounded subset with \( \theta \in \Omega \). For \( \forall x \in \Omega \), we have \( Q[(I - Q)\bar{N}_\lambda(x)] = 0 \). So \( (I - Q)\bar{N}_\lambda(x) \in \ker Q = \text{Im} M \). For \( \forall z \in \text{Im} M \), one gets \( Qz = 0 \). Thus, \( z = z - Qz = (I - Q)z \in (I - Q)Z \). Therefore, (2.1) holds. Obviously, (2.2) is satisfied.

Define \( R : \bar{\Omega} \times [0, 1] \to X_2 \) by

\[
R(x, \lambda)(t) = -\int_{0}^{t} \int_{s}^{1} \phi_q \left( \int_{0}^{s} (s - \tau)z(\tau)d\tau \right) ds dt.
\tag{3.7}
\]

where \( X_2 \) is the complementary space of \( X_1 = \ker M \) in \( X \). Clearly, \( R(\cdot, 0) = \theta \).

Now we prove that \( R : \bar{\Omega} \times [0, 1] \to X_2 \) is compact and continuous.

We first show that \( R \) is relatively compact for \( \forall \lambda \in [0, 1] \). Since \( \Omega \subset X \) is a bounded set, then there exists \( r > 0 \) such that \( \bar{\Omega} \subset \{ x \in X : ||x||_X \leq r \} \).

Because the function \( f \) satisfies the \( L^1 \)-Carathéodory conditions, there exists \( \alpha_r \in L^1[0, 1] \) such that for a.e. \( t \in [0, 1] \), \( |f(t, x(t), x'(t), x''(t))| \leq \alpha_r(t) \) for
\( x \in \Omega \). Then for any \( \Omega \), \( \lambda \in [0, 1] \), we obtain

\[
|R(x, \lambda)(t)| \leq \int_0^1 \left| \int_0^t \phi_\eta \left( \int_0^\tau (\tau - k)\lambda(f(k, x(k), x'(k), x''(k)) - (Qf)(k))dk \right) \, d\tau \right| \, ds
\]

\[
\leq \int_0^1 \phi_\eta \left( \int_0^1 |\alpha_r(s)| \, ds + \int_0^1 |(Qf)(s)| \, ds \right) \, dt
\]

\[
= \phi_\eta(||\alpha_r||_1 + ||Qf||_1) = L,
\]

\[
|R'(x, \lambda)(t)| = \left| \int_0^t \phi_\eta \left( \int_0^t (s - \tau)\lambda(f(\tau, x(\tau), x'(\tau), x''(\tau)) - (Qf)(\tau)) \, d\tau \right) ds \right|
\]

\[
\leq \int_0^1 \phi_\eta \left( \int_0^1 |\alpha_r(s)| \, ds + \int_0^1 |(Qf)(s)| \, ds \right) \, dt = L,
\]

\[
|R''(x, \lambda)(t)| = \left| \phi_\eta \left( \int_0^t (t - s)\lambda(f(s, x(s), x'(s), x''(s)) - (Qf)(s)) ds \right) \right|
\]

\[
\leq \phi_\eta \left( \int_0^1 |\alpha_r(s)| \, ds + \int_0^1 |(Qf)(s)| \, ds \right)
\]

\[
= \phi_\eta(||\alpha_r||_1 + ||Qf||_1) = L,
\]

that is, \( R'(\cdot, \lambda)\Omega \) is uniformly bounded. Meanwhile, for \( \forall t_1, t_2 \in [0, 1] \),

\[
|R(x, \lambda)(t_2) - R(x, \lambda)(t_1)| = \left| \int_{t_1}^{t_2} R'(x, \lambda)(s) \, ds \right| \leq L|t_2 - t_1| \to 0, \text{ as } |t_2 - t_1| \to 0.
\]

Similarly,

\[
|R(x, \lambda)'(t_2) - R(x, \lambda)'(t_1)| = \left| \int_{t_1}^{t_2} R''(x, \lambda)(s) \, ds \right| \leq L|t_2 - t_1| \to 0, \text{ as } |t_2 - t_1| \to 0.
\]

Also,

\[
|\phi_\eta(R''(x, \lambda)(t_2)) - \phi_\eta(R''(x, \lambda)(t_1))|
\]

\[
= \left| \int_0^{t_2} (t_2 - s)\lambda(f(s, x(s), x'(s), x''(s)) - (Qf)(s)) ds \right|
\]

\[
- \left| \int_0^{t_1} (t_1 - s)\lambda(f(s, x(s), x'(s), x''(s)) - (Qf)(s)) ds \right|
\]

\[
\leq \left| \int_0^{t_2} (t_2 - t_1)\lambda(f(s, x(s), x'(s), x''(s)) - (Qf)(s)) ds \right|
\]

\[
+ \left| \int_0^{t_2} (t_1 - s)\lambda(f(s, x(s), x'(s), x''(s)) - (Qf)(s)) ds \right|
\]

\[
\leq \int_0^1 (\alpha_r(s) + |(Qf)(s)|) \, ds \cdot |t_2 - t_1| + \int_{t_1}^{t_2} (t_1 - s) (\alpha_r(s) + |(Qf)(s)|) \, ds,
\]

\[
\leq (||\alpha_r||_1 + ||Qf||_1)|t_2 - t_1| + \int_{t_1}^{t_2} (\alpha_r(s) + |(Qf)(s)|) ds \to 0 \text{ as } |t_2 - t_1| \to 0.
\]
In view of the continuity of $\phi_p$, we have $|R''(x, \lambda)(t_2) - R''(x, \lambda)(t_1)| \to 0$, as $|t_2 - t_1| \to 0$. So, $R(\cdot, \lambda)\overline{\Omega}$ is equicontinuous on $[0,1]$. Thus, Arzela-Ascoli Theorem implies that $R(\cdot, \lambda)\overline{\Omega}$ is relatively compact.

Since $f$ is a $L^1$-Carathéodory function, the continuity of $R$ on $\overline{\Omega}$ follows from the Lebesgue dominated convergence theorem.

Define $P : X \to X_1$ by $(Px)(t) = x(0)$ for $t \in [0,1]$. $\forall x \in \sum_{\lambda}$, we have $\lambda f(t, x(t), x'(t), x''(t)) = \phi_p(x''(t)))'' \in \text{Im}M = \ker Q$. So

$$R(x, \lambda)(t) = \int_0^t \int_s^t \phi_q \left( \int_0^r (r-k) \lambda \left( f(k, x(k), x'(k), x''(k)) - (Qf)(k) \right) dk \right) dr ds$$

which implies (2.3). $\forall x \in \overline{\Omega}$, we have

$$M[Px + R(x, \lambda)(t)]$$

which yields (2.4). Therefore, $N_\lambda$ is $M$-compact in $\overline{\Omega}$. \hfill \Box

4. Main results

**Theorem 4.1.** Suppose that

(H1) there exists a constant $A > 0$ such that for $\forall \lambda \in \text{dom} M \setminus \ker M$ satisfying $|x(t)| > A$ for all $t \in [0,1]$, we have $QN_x \neq 0$;

(H2) there exist functions $\alpha, \beta, \gamma, \rho \in L^1[0,1]$ such that for $\forall (x, y, z) \in \mathbb{R}^3$ and a.e. $t \in [0,1]$, we have

$$|f(t, x, y, z)| \leq \alpha(t)|x|^{p-1} + \beta(t)|y|^{p-1} + \gamma(t)|z|^{p-1} + \rho(t), \quad (4.1)$$

we denote $\alpha_1 = ||\alpha||_{1}$, $\beta_1 = ||\beta||_{1}$, $\gamma_1 = ||\gamma||_{1}$, $\rho_1 = ||\rho||_{1}$;

(H3) there exist a constant $B > 0$ such that for $\forall b \in \mathbb{R}$ with $|b| > B$, we have either

$$b \cdot \sum_{i=1}^{n} \frac{n}{\mu(2q - (2\xi_i)^{2q-1})} \cdot \sum_{i=1}^{n} \mu_i \int_0^{\xi_i} \int_0^{1/2} \phi_q \left( \int_0^{s} (s-\tau) f(\tau, b, 0, 0) d\tau \right) ds \, dt < 0 \quad (4.2)$$

or

$$b \cdot \sum_{i=1}^{n} \frac{n}{\mu(2q - (2\xi_i)^{2q-1})} \cdot \sum_{i=1}^{n} \mu_i \int_0^{\xi_i} \int_0^{1/2} \phi_q \left( \int_0^{s} (s-\tau) f(\tau, b, 0, 0) d\tau \right) ds \, dt > 0; \quad (4.3)$$

(H4) $2^{q-3}(\alpha_1 + \beta_1 + 2^{p-1-1}) < 1$ for $p < 2$, \quad (4.4)
\[
\frac{1}{2}(2^{p-2}\alpha_1 + \beta_1 + 2^{p-1}\gamma_1)^{q-1} < 1 \text{ for } p \geq 2. \quad (4.5)
\]

Then the BVP (1.1)-(1.3) has at least one nonconstant symmetric solution.

**Lemma 4.1.** \(U_1 = \{x \in \text{dom } M : Mx = Nx \text{ for some } \lambda \in (0,1)\}\) is bounded.

**Proof.** For \(\forall x \in U_1\), we have \(Nx = Mx \in \text{Im } M = \text{ker } Q\) and then \(QNx = 0\). It follows from (H1) that there exists \(t_0 \in [0,1]\) such that \(|x(t_0)| \leq A\). Now, \(|x(t)| = |x(t_0) + \int_{t_0}^{t} x'(s)ds| \leq A + ||x'||_{\infty}\), that is, \(|x||_{\infty} \leq A + ||x'||_{\infty}\). Since \(x\) is symmetric on \([0,1]\), then

\[
|x'(t)| = |x'(1) + \int_{t}^{1} x''(s)ds| = |\int_{t}^{1} x''(s)ds| \leq \frac{1}{2}||x'||_{\infty}. 
\]

that is, \(|x'||_{\infty} \leq \frac{1}{2}||x''||_{\infty}\). And then \(|x||_{\infty} \leq A + \frac{1}{2}||x''||_{\infty}\).

Also,

\[
x''(t) = \phi_q(\int_{0}^{t} (t-s)\lambda f(s,x(s),x'(s),x''(s))ds).
\]

(I) For \(1 < p < 2\), from (H2) and Proposition 2.1, one gets

\[
||x'||_{\infty} = \sup_{t \in [0,1]} |\phi_q(\int_{0}^{t} (t-s)\lambda f(s,x(s),x'(s),x''(s))ds)|
\]

\[
\leq \phi_q(\int_{0}^{1} (\alpha(t)|x(t)|^{p-1} + \beta(t)|x'(t)|^{p-1} + \gamma(t)|x''(t)|^{p-1} + \rho(t))dt)
\]

\[
\leq \phi_q(\alpha||x||_{\infty}^{p-1} + \beta||x'||_{\infty}^{p-1} + \gamma||x''||_{\infty}^{p-1} + \rho_1)
\]

\[
\leq \phi_q(\alpha(1 + 2^{p-1}\gamma_1)(\frac{||x'||_{\infty}^{p-1} + (\alpha_1 A^{p-1} + \rho_1)}{2})^{p-1} + (\alpha_1 A^{p-1} + \rho_1))
\]

\[
\leq 2^{q-3}(\alpha_1 + \beta_1 + 2^{p-1}\gamma_1)^{q-1}||x''||_{\infty}^{q-1} + 2^{q-2}(\alpha_1 A^{p-1} + \rho_1)^{q-1}.
\]

Noticing (H4), one arrives at

\[
||x'||_{\infty} \leq \frac{2^{q-2}(\alpha_1 A^{p-1} + \rho_1)^{q-1}}{1 - 2^{q-3}(\alpha_1 + \beta_1 + 2^{p-1}\gamma_1)^{q-1}} = L_1.
\]

(4.6)

which yields \(|x'||_{\infty} \leq \frac{1}{2}L_1\) and \(|x||_{\infty} \leq A + \frac{1}{2}L_1\). Let \(L_2 = \max\{L_1, A + \frac{1}{2}L_1\}\).
(II) For \( p \geq 2 \), similarly, we have

\[
||x''||_{\infty} = \sup_{t \in [0,1]} |\phi_q(\int_0^t (t-s)\lambda f(s, x(s), x'(s), x''(s))ds)|
\]

\[
\leq \phi_q[\alpha_1 ||x||_{\infty}^{p-1} + \beta_1 ||x'||_{\infty}^{p-1} + \gamma_1 ||x''||_{\infty}^{p-1} + \rho_1]
\]

\[
\leq \phi_q[2^{p-2}\alpha_1 A^{p-1} + \frac{1}{2p-1} ||x''||_{\infty}^{p-1} + \frac{1}{2p-1} \beta_1 ||x''||_{\infty}^{p-1} + ||x''||_{\infty}^{p-1} + \rho_1]
\]

\[
= \phi_q[(2^{p-2}\alpha_1 + \beta_1 + 2^{p-1}\gamma_1)(p-1) + (2^{p-2}\alpha_1 A^{p-1} + \rho_1)]
\]

\[
\leq \frac{1}{2} (2^{p-2}\alpha_1 + \beta_1 + 2^{p-1}\gamma_1) ||x''||_{\infty}^{q-1} + (2^{p-2}\alpha_1 A^{p-1} + \rho_1)^{q-1}.
\]

From (H4), we have

\[
||x''||_{\infty} \leq \frac{(2^{p-2}\alpha_1 A^{p-1} + \rho_1)^{q-1}}{1 - \frac{1}{2} (2^{p-2}\alpha_1 + \beta_1 + 2^{p-1}\gamma_1)^{q-1} ||x''||_{\infty}^{q-1}} := M_1,
\]

which leads to \( ||x''||_{\infty} \leq \frac{1}{2} M_1 \) and \( ||x||_{\infty} \leq A + \frac{1}{2} M_1 \).

Let \( M_2 = \max\{M_1, A + \frac{3}{2} M_1\} \).

Thus, \( ||x||_X \leq \max\{L_2, M_2\} \), i.e. \( U_1 \) is bounded. □

**Lemma 4.2.** If \( U_2 = \{x \in \ker M : -\lambda x + (1 - \lambda)JQx = 0, \lambda \in [0,1]\} \), where \( J : \text{Im}Q \to \ker M \) is a homomorphism, then \( U_2 \) is bounded.

**Proof.** Define \( J : \text{Im}Q \to \ker M \) by \( J(b) = b \). Then for \( \forall b \in U_2 \),

\[
\lambda b = 2(1 - \lambda)\phi_p \left( \frac{2^{2q}q(2q - 1)}{\sum_{i=1}^n \mu_i(2q - (2\xi_i)^{q-1})} \right) \phi_p \times \left( \sum_{i=1}^n \mu_i \int_0^{\xi_i} \frac{1}{2} \phi_q \left( \int_0^s (s - \tau) f(\tau, b, 0, 0) d\tau \right) ds dt \right).
\]

If \( \lambda = 1 \), then \( b = 0 \). In the case \( \lambda \in [0,1] \), if \( |b| > B \), then by (4.2), we have

\[
0 \leq \lambda b^2 = 2(1 - \lambda) b \phi_p \left( \frac{2^{2q}q(2q - 1)}{\sum_{i=1}^n \mu_i(2q - (2\xi_i)^{q-1})} \right) \phi_p \times \left( \sum_{i=1}^n \mu_i \int_0^{\xi_i} \frac{1}{2} \phi_q \left( \int_0^s (s - \tau) f(\tau, b, 0, 0) d\tau \right) ds dt \right) < 0,
\]

which is a contradiction. Thus, \( ||x||_X = |b| \leq B \) for \( \forall x \in U_2 \), that is, \( U_2 \) is bounded.

□

**Proof of Theorem 4.1.** Let \( U = \{x \in \text{dom}M : ||x||_X < \max\{L_2, M_2, B\} + 1\} \), then \( U \supset U_1 \cup U_2 \) be a bounded and open set, then from Lemmas 4.1 and
4.2, we have
(i) $Mx \neq N_x$ for $\forall (x, \lambda) \in [\text{dom} M \cap \partial U] \times (0, 1)$;
(ii) Let $H(x, \lambda) = -\lambda x + (1 - \lambda)JQN x$, $J$ is defined as in Lemma 4.2. and we can see that $H(x, \lambda) \neq 0$, $\forall x \in \text{dom} M \cap \partial U$. Therefore,

$$\deg \{JQN | \partial U \cap \text{ker} M, \ U \cap \text{ker} M, 0\} = \deg \{H(\cdot, 0), \ U \cap \text{ker} M, 0\} = \deg \{\frac{1}{\lambda}H(\cdot, \lambda), \ U \cap \text{ker} M, 0\} = \deg \{-I, \ U \cap \text{ker} M, 0\} = 0.$$

Theorem 2.1 yields that $Mx = Nx$ has at least one symmetric solution $x^* \in \text{dom} M \cap U$. Observe that $x^*(t)$ is not a constant. Otherwise, suppose $x^* \equiv 0$, then from (1.1) we have $f(t, b, 0, 0) \equiv 0$, which contradicts (A1). The proof is completed.

**Remark 4.1.** When the second part of condition (H3) holds, if we choose $\tilde{U}_2 = \{x \in \text{ker} M : \lambda x + (1 - \lambda)JQN x = 0, \lambda \in [0, 1]\}$ and take homomorphism $\tilde{H}(x, \lambda) = \lambda x + (1 - \lambda)JQN x$. Then by a similar argument, we can complete the proof.

**Example 4.1.** Consider

$$\begin{cases}
(\phi_3(x''(t)))'' = f(t, x(t), x'(t), x''(t)), \ a.e. \ t \in [0, 1], \\
x''(0) = 0, (\phi_3(x''(0)))' = 0 \\
x(0) = 2x(\frac{1}{6}) - x(\frac{1}{4}), \ x(t) = x(1 - t).
\end{cases} \quad (4.7)$$

Corresponding to the BVP (1.1)-(1.3), we have $p = 3$, $q = \frac{3}{2}$, $\mu_1 = 2$, $\mu_2 = -1$, $\xi_1 = \frac{1}{6}$, $\xi_2 = \frac{1}{4}$ and

$$f(t, u, v, w) = 2t(1 - t)e^{t(1 - t)} + \frac{1}{2}(1 - t)u^2 + \left(t - t^2 + \frac{1}{12}\right)v^2 + t^2(1 - t)^2w^2.$$  

We can easily verify that (A1)-(A2) hold. Let $\alpha(t) = \frac{1}{2}t(1 - t)$, $\beta(t) = t - t^2 + \frac{1}{12}$, $\gamma(t) = t^2(1 - t)^2$, $\rho(t) = 2t(1 - t)e^{t(1 - t)}$, then $\alpha_1 = \frac{1}{12}$, $\beta_1 = \frac{1}{4}$, $\gamma_1 = \frac{1}{12}$. Also, we can check that (H1)-(H4) are all satisfied. Thus, BVP (4.7) has a nonconstant symmetric solution, by using Theorem 4.1.

**References**


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