A NOTE ON THE WEIGHTED
LEBESGUE-RADON-NIKODYM THEOREM WITH RESPECT
TO p-ADIC INVARIANT INTEGRAL ON \( \mathbb{Z}_p \)

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Abstract. In this paper, we give the weighted Lebesgue-Radon-Nikodym theorem with respect to \( p \)-adic invariant integral on \( \mathbb{Z}_p \).

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1. Introduction

Let \( p \) be a fixed odd prime number. Throughout this paper, the symbols \( \mathbb{Z}_p \), \( \mathbb{Q}_p \), and \( \mathbb{C}_p \) denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers, and the completion of the algebraic closure of \( \mathbb{Q}_p \), respectively. The \( p \)-adic norm \(|x|_p\) is defined by \(|x|_p = p^{-r}\) for \( x = p^r s/t \) with \( s, t \in \mathbb{Z} \) with \((p, s) = (p, t) = 1\) and \( r \in \mathbb{Q} \) (see [1-8]).

Let \( C(\mathbb{Z}_p) \) be the space of continuous functions on \( \mathbb{Z}_p \). The fermionic invariant measure on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
\mu_{-1}(a + p^n\mathbb{Z}_p) = (-1)^a, \tag{1}
\]

where

\[
a + p^n\mathbb{Z}_p = \{ x \in \mathbb{Z}_p | x \equiv a \pmod{p^n} \},
\]

and \( a \in \mathbb{Z} \) with \( 0 \leq a < p^n \) (see [3,6,7]). From (1), the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} f(x)(-1)^x, \tag{2}
\]
where \( f \in C(\mathbb{Z}_p) \) (see [3,6,7,8]).

Let us assume that \( w \in \mathbb{C}_p \) with \(|1 - w|_p < 1\). By (1), we get

\[
\int_{\mathbb{Z}_p} e^{xt}w^x d\mu_{-1}(x) = \frac{2}{we^{t} - 1} = \sum_{x=0}^{\infty} E_{n,w}\frac{t^n}{n!}, \quad \text{(see [7])},
\]

where \( E_{n,w} \) is weighted Euler numbers. The weighted Euler polynomials are also defined by

\[
\int_{\mathbb{Z}_p} e^{(x+y)t}w^y d\mu_{-1}(y) = \frac{2}{we^{t} - 1}e^{xt} = \sum_{n=0}^{\infty} E_{n,w}(x)\frac{t^n}{n!}.
\]

By (3) and (4), we get

\[
E_{n,w}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l}E_{l,w} = (x + E_w)^n,
\]

with the usual convention about replacing \((E_w)^n\) by \( E_{n,w} \) (see [7]).

The idea for generalizing the fermionic integral is replacing the fermionic Haar measure with weakly (strongly) fermionic measure \( \mathbb{Z}_p \) satisfying

\[
|\mu_{-1}(a + p^n\mathbb{Z}_p) - \mu_{-1}(a + p^{n+1}\mathbb{Z}_p)|_p \leq \delta_n, \quad \text{see [3]},
\]

where \( \delta_n \to 0 \), \( a \) is an element of \( \mathbb{Z}_p \), and \( \delta_n \) is independent of \( a \) (for strongly fermionic measure, \( \delta_n \) is replaced by \( Cp^{-n} \), where \( C \) is a positive constant).

Let \( f(x) \) be a function defined on \( \mathbb{Z}_p \). The fermionic integral of \( f \) with respect to a weakly fermionic measure \( \mu_{-1} \) is

\[
\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{n\to\infty} \sum_{x=0}^{p^n-1} f(x)\mu_{-1}(x + p^n\mathbb{Z}_p),
\]

if the limit exists.

If \( \mu_{-1} \) is a weakly fermionic measure on \( \mathbb{Z}_p \), then we can define Radon-Nikodym derivative of \( \mu_{-1} \) with respect to the Haar measure on \( \mathbb{Z}_p \) as follows:

\[
f_{\mu_{-1}}(x) = \lim_{n\to\infty} \mu_{-1}(x + p^n\mathbb{Z}_p), \quad \text{see [3]},
\]

Note that \( f_{\mu_{-1}} \) is only a continuous function on \( \mathbb{Z}_p \). Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), let us define \( \mu_{-1,f} \) as follows:

\[
\mu_{-1,f}(x + p^n\mathbb{Z}_p) = \int_{x + p^n\mathbb{Z}_p} f(x)d\mu_{-1}(x), \quad \text{see [3]},
\]

By (3) and (4), we get

\[
E_{n,w}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l}E_{l,w} = (x + E_w)^n,
\]

with the usual convention about replacing \((E_w)^n\) by \( E_{n,w} \) (see [7]).
where the integral is the fermionic $p$-adic invariant integral. From (7), we can easily note that $\mu_{-1,f}$ is a strongly fermionic measure on $\mathbb{Z}_p$. Since
\[
\left| \mu_{-1,f}(x + p^n\mathbb{Z}_p) - \mu_{-1,f}(x + p^{n+1}\mathbb{Z}_p) \right|_p = \left| \sum_{x=0}^{p^n-1} f(x)(-1)^x - \sum_{x=0}^{p^n} f(x)(-1)^x \right|_p = \left| \frac{f(p^n)}{p^n} \right|_p |p^n|_p \leq Cp^{-n},
\]
where $C$ is a positive constant.

The purpose of this paper is to derive the weighted Lebesgue-Radon-Nikodym’s type theorem with respect to the fermionic $p$-adic invariant measure on $\mathbb{Z}_p$.

2. The weighted Lebesgue-Radon-Nikodym theorem

In this section, we assume that the weighted function $w(x)$ is defined by $w(x) = w^x$ where $w \in \mathbb{C}_p$ with $|1 - w|_p < 1$. For any positive integer $a$ and $n$ with $a < p^n$ and $f \in UD(\mathbb{Z}_p)$, we define the strongly weighted fermionic measure on $\mathbb{Z}_p$ as follows:
\[
\mu_{f,-w}(a + p^n\mathbb{Z}_p) = \int_{a + p^n\mathbb{Z}_p} f(x)w^xd\mu_{-1}(x), \quad (8)
\]
where the integral is the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$. From (8), we note that
\[
\mu_{f,-w}(a + p^n\mathbb{Z}_p) = \lim_{m \to \infty} \sum_{x=0}^{p^m-1} f(a + p^n x)(-1)^{a+p^n x}w^{a+p^n x} = (-1)^a w^a \lim_{m \to \infty} \sum_{x=0}^{p^m-n-1} f(a + p^n x)(-1)^x w^{p^n x} \quad (9)
\]
By (9), we get
\[
\mu_{f,-w}(a + p^n\mathbb{Z}_p) = (-1)^a \int_{\mathbb{Z}_p} f(a + p^n x)w^{a+p^n x}d\mu_{-1}(x). \quad (10)
\]
Thus, by (10), we have
\[
\mu_{\alpha f + \beta g,-w} = \alpha \mu_{f,-w} + \beta \mu_{g,-w}, \quad (11)
\]
where $f, g \in UD(\mathbb{Z}_p)$ and $\alpha, \beta$ are positive constants. By (8), (9), (10) and (11), we get
\[
\left| \mu_{f,-w}(a + p^n\mathbb{Z}_p) \right|_p \leq |f_w|_\infty, \quad (12)
\]
where $|f_w|_\infty = \sup_{x \in \mathbb{Z}_p} |f(x)w^x|_p$.
Let \( P(x) \in \mathbb{C}_p[[x]] \) be an arbitrary polynomial. Now we show \( \mu_{P,-w} \) is a strongly weighted fermionic \( p \)-adic invariant measure on \( \mathbb{Z}_p \). Without a loss of generality, it is enough to prove the statement for \( P(x) = x^k \).

For \( a \in \mathbb{Z} \) with \( 0 \leq a < p^n \), we have

\[
\mu_{P,-w}(a + p^n \mathbb{Z}_p) = \lim_{m \to \infty} (-1)^a p^{m-n-1} \sum_{i=0}^{p^m-1} (a + ip^n)^k w^{a+ip^n} (-1)^i.
\]

From binomial theorem, we note that

\[
(a + ip^n)^k = \sum_{l=0}^{k} a^{k-l} \binom{k}{l} (ip^n)^l = a^k + \binom{k}{1} a^{k-1} p^n i + \cdots + p^n i^k.
\]

and

\[
w^{a+ip^n} = w^a \sum_{l=0}^{ip^n} \binom{ip^n}{l} (w-1)^l \equiv w^a \pmod{p^n}.
\]

Thus, by (13) and (14), we get

\[
\mu_{P,-w}(a + p^n \mathbb{Z}_p) \equiv \left( -1 \right)^a a^k \pmod{p^n}
\]

For \( x \in \mathbb{Z}_p \), let \( x \equiv x_n \pmod{p^n} \) and \( x \equiv x_{n+1} \pmod{p^{n+1}} \), where \( x_n, x_{n+1} \in \mathbb{Z} \) with \( 0 \leq x_n < p^n \) and \( 0 \leq x_{n+1} < p^{n+1} \).

Then we have

\[
\left| \mu_{P,-w}(a + p^n \mathbb{Z}_p) - \mu_{P,-w}(a + p^{n+1} \mathbb{Z}_p) \right| \leq Cp^{-n},
\]

where \( C \) is a positive constant and \( n \gg 0 \).

Let

\[
f_{\mu_{P,-w}}(a) = \lim_{n \to \infty} \mu_{P,-w}(a + p^n \mathbb{Z}_p).
\]

Then, by (15) and (16), we see that

\[
f_{\mu_{P,-w}}(a) = \left( -1 \right)^a a^k = \left( -1 \right)^a a^k P(a).
\]

Since \( f_{\mu_{P,-w}}(x) \) is a continuous function on \( \mathbb{Z}_p \). For \( x \in \mathbb{Z}_p \), we have

\[
f_{\mu_{P,-w}}(x) = (-1)^x w^x x^k, (k \in \mathbb{Z}_+).
\]

Let \( g \in UD(\mathbb{Z}_p) \). Then, by (16), (17) and (18), we get

\[
\int_{\mathbb{Z}_p} g(x) d\mu_{P,-w}(x) = \lim_{n \to \infty} \sum_{x=0}^{p^n-1} g(x) \mu_{P,-w}(x + p^n \mathbb{Z}_p)
\]

\[
= \lim_{n \to \infty} \sum_{x=0}^{p^n-1} g(x) w^x x^k (-1)^x
\]

\[
= \int_{\mathbb{Z}_p} g(x) w^x x^k d\mu_{-1}(x).
\]

Therefore, by (19), we obtain the following theorem.
Theorem 1. Let \( P(x) \in \mathbb{C}_p[[x]] \) be an arbitrary polynomial. Then \( \mu_{P,-w} \) is a strongly weighted fermionic \( p \)-adic invariant measure on \( \mathbb{Z}_p \). That is, \[ f_{\mu_{P,-w}} = (-1)^x w^x P(x) \quad \text{for all} \quad x \in \mathbb{Z}_p. \]

Furthermore, for any \( g \in UD(\mathbb{Z}_p) \),
\[
\int_{\mathbb{Z}_p} g(x) d\mu_{P,-w}(x) = \int_{\mathbb{Z}_p} g(x) P(x) w^x d\mu_{-1}(x),
\]
where the second integral is fermionic strongly weighted fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \).

Let \( f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \) be the Mahler expansion for \( f \in UD(\mathbb{Z}_p) \). Then we note that \( \lim_{n \to \infty} n|a_n|_p = 0 \). Now, we get \( f_m(x) = \sum_{i=0}^{m} a_i \binom{x}{i} \in \mathbb{C}_p[[x]] \). Thus, we have
\[
\|f - f_m\|_\infty \leq \sup_{n \geq m} n|a_n|_p. \tag{20}
\]
The function \( f(x) \) can be rewritten as \( f = f_m + f - f_m \). Thus, by (11) and (20), we get
\[
|\mu_{f,-w}(a + p^n \mathbb{Z}_p) - \mu_{f,-w}(a + p^{n+1} \mathbb{Z}_p)|_p \\
\leq \max \left\{ |\mu_{f,-w}(a + p^n \mathbb{Z}_p) - \mu_{f_m,-w}(a + p^{n+1} \mathbb{Z}_p)|_p, \right. \\
|\mu_{f-f_m,-w}(a + p^n \mathbb{Z}_p) - \mu_{f-f_m,-w}(a + p^{n+1} \mathbb{Z}_p)|_p \right\}. \tag{21}
\]
From Theorem 1 and (21), we note that
\[
|\mu_{f-f_m,-w}(a + p^n \mathbb{Z}_p)|_p \leq C^* \|f - f_m\|_\infty \leq C_1 p^{-n}, \tag{22}
\]
where \( C^* \) and \( C_1 \) are positive constants. For \( m \gg 0 \), we have \( \|f\|_\infty = \|f_m\|_\infty \).
So, we see that
\[
|\mu_{f_m,-w}(a + p^n \mathbb{Z}_p) - \mu_{f_m,-w}(a + p^{n+1} \mathbb{Z}_p)|_p \\
= |f_m(p^n) w^{p^n}|_p = |f_m(p^n) w^{p^n} - p^n|_p \\
\leq \|f_m w^x\|_{\infty p^{-n}} \leq C_2 p^{-n},
\]
where \( C_2 \) is a positive constant. By (22), we get
\[
\|(-1)^a f(a) w^a - \mu_{f,-w}(a + p^n \mathbb{Z}_p)|_p \\
\leq \max \left\{ |w^a f(a)|_p, |f_m(a) - \mu_{f_m,-w}(a + p^n \mathbb{Z}_p)|_p, \\
|\mu_{f-f_m,-w}(a + p^n \mathbb{Z}_p)|_p \right\} \\
\leq \max \left\{ |f(a)|_p, |f_m(a)|_p, |\mu_{f_m,-w}(a + p^n \mathbb{Z}_p)|_p, \|f - f_m\|_\infty \right\}
\]
Let us assume that fix $\epsilon > 0$, and fix $m$ such that $\|f - f_m\| < \epsilon$. Then we have
\[
\left| (-w)^n f(a) - \mu_{f, -w}(a + p^nZ_p) \right|_p \leq \epsilon \quad \text{for} \quad n \gg 0.
\] (24)

Thus, by (24), we have
\[
f_{\mu_{f, -w}}(a) = \lim_{n \to \infty} \mu_{f, -w}(a + p^nZ_p) = (-1)^n w^n f(a)
\] (25)

Let $m$ be the sufficiently large number such that $\|f - f_m\|_{\infty} \leq p^{-n}$. Then we get
\[
\mu_{f, -w}(a + p^nZ_p) = \mu_{f_m, -w}(a + p^nZ_p) + \mu_{f - f_m, -w}(a + p^nZ_p)
\] \[= (-1)^n w^n f(a) \quad (\text{mod } p^n).
\]

For $g \in UD(Z_p)$, we have
\[
\int_{Z_p} g(x)d\mu_{f, -w}(x) = \int_{Z_p} f(x)g(x)w^n \, d\mu_{-1}(x).
\]

Let $f$ be the function from $UD(Z_p)$ to $Lip(Z_p)$. We easily see that $w^n\mu_{-1}(x + p^nZ_p)$ is a strongly weighted $p$-adic invariant measure on $Z_p$ and
\[
\left| (f_w)_{\mu_{-1}}(a) - w^n \mu_{-1}(a + p^nZ_p) \right|_p \leq C_3 p^{-n},
\]
where $f_w(x) = f(x)w^n$ and $C_3$ is a positive constant and $n \in \mathbb{Z}_+$.

If $\mu_{1, -w}$ is associated with strongly weighted fermionic invariant measure on $Z_p$, then we have
\[
\left| \mu_{1, -w}(a + p^nZ_p) - (f_w)_{\mu_{-1}}(a) \right|_p \leq C_4 p^{-n},
\]
where $n > 0$ and $C_4$ is a positive constant.

For $n \gg 0$, we have
\[
\left| w^n \mu_{-1}(a + p^nZ_p) - \mu_{1, -w}(a + p^nZ_p) \right|_p
\] \[\leq \left| w^n \mu_{-1}(a + p^nZ_p) - (f_w)_{\mu_{-1}}(a) \right|_p + \left| (f_w)_{\mu_{-1}}(a) - \mu_{1, -w}(a + p^nZ_p) \right|_p
\] (26)
\[\leq K,
\]
where $K$ is a positive constant. Hence, $w\mu_{-1} - \mu_{1, -w}$ is a weighted measure on $Z_p$. Therefore, we obtain the following theorem.

**Theorem 2.** Let $w\mu_{-1}$ be a strongly weighted $p$-adic invariant measure on $Z_p$, and assume that the fermionic weighted Radon-Nikodym derivative $(f_w)_{\mu_{-1}}$ on $Z_p$ is uniformly differentiable function. Suppose that $\mu_{1, -w}$ is the strongly weighted fermionic $p$-adic invariant measure associated with $(f_w)_{\mu_{-1}}$. Then there exists a weighted measure $\mu_{2, -w}$ on $Z_p$ such that
\[
w^2 \mu_{-1}(x + p^nZ_p) = \mu_{1, -w}(x + p^nZ_p) + \mu_{2, -w}(x + p^nZ_p).
\]
REFERENCES


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