STABILITY OF PERTURBED DIFFERENTIAL SYSTEMS
VIA $t_\infty$-SIMILARITY

YOON HOE GOO

ABSTRACT. In this paper, we investigate $h$-stability of the nonlinear perturbed differential systems using the the notion of $t_\infty$-similarity.

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1. Introduction

As is traditional in a perturbation theory of nonlinear differential equations, the behavior of solutions of a perturbed equation is determined in terms of the behavior of solutions of an unperturbed equation. Using the variation of constants formula and the integral inequalities, we study the qualitative behavior of the solutions of perturbed nonlinear system of differential equations:

Pinto [13] introduced $h$-stability($hS$) which is an important extension of the notions of exponential asymptotic stability and uniform Lipschitz stability. Also, he obtained some properties about asymptotic behavior of solutions of perturbed $h$-systems, some general results about asymptotic integration and gave some important examples in [14].


In this paper, we investigate $h$—stability of the nonlinear perturbed differential systems using the the notion of $t_\infty$-similarity.
2. Preliminaries

We consider the nonlinear differential system

\[ x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1} \]

where \( f \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n], \ \mathbb{R}^+ = [0, \infty) \) and \( \mathbb{R}^n \) is the Euclidean \( n \)-space. We assume that the Jacobian matrix \( f_x = \frac{\partial f}{\partial x} \) exists and is continuous on \( \mathbb{R}^+ \times \mathbb{R}^n \) and \( f(t, 0) = 0 \).

Let \( x(t) = x(t, t_0, x_0) \) denote the unique solution of (1) through \( (t_0, x_0) \) in \( \mathbb{R}^+ \times \mathbb{R}^n \) such that \( x(t_0, t_0, x_0) = x_0 \). Also, we consider the associated variational systems around the zero solution of (1) and around \( x(t) \), respectively,

\[ v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0 \tag{2} \]

and

\[ z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0. \tag{3} \]

The fundamental matrix solution \( \Phi(t, t_0, x_0) \) of (3) is given by

\[ \Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0), \]

and \( \Phi(t, t_0, 0) \) is the fundamental matrix solution of (2). The symbol \( \| \cdot \| \) denotes arbitrary vector norm in \( \mathbb{R}^n \).

We recall some notions of \( h \)-stability [13] and the notion of \( t_\infty \)-similarity [5].

**Definition 2.1.** The system (1) (the zero solution \( x = 0 \) of (1)) is called an \( h \)-system if there exist a constant \( c \geq 1 \), and a positive continuous function \( h \) on \( \mathbb{R}^+ \) such that

\[ |x(t)| \leq c|x_0| h(t) h(t_0)^{-1} \]

for \( t \geq t_0 \geq 0 \) and \( |x_0| \) small enough.

**Definition 2.2.** The system (1) (the zero solution \( x = 0 \) of (1)) is called \( h \)-stable (\( hS \)) if there exist \( \delta > 0 \) such that (1) is an \( h \)-system for \( |x_0| \leq \delta \) and \( h \) is bounded.

Let \( \mathcal{M} \) denote the set of all \( n \times n \) continuous matrices \( A(t) \) defined on \( \mathbb{R}^+ \) and \( \mathcal{N} \) be the subset of \( \mathcal{M} \) consisting of those nonsingular matrices \( S(t) \) that are of class \( C^1 \) with the property that \( S(t) \) and \( S^{-1}(t) \) are bounded. The notion of \( t_\infty \)-similarity in \( \mathcal{M} \) was introduced by Conti [5].

**Definition 2.3.** A matrix \( A(t) \in \mathcal{M} \) is \( t_\infty \)-similar to a matrix \( B(t) \in \mathcal{M} \) if there exists an \( n \times n \) matrix \( F(t) \) absolutely integrable over \( \mathbb{R}^+ \), i.e.,

\[ \int_0^\infty |F(t)|dt < \infty \]
such that
\[ \dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \] (4)
for some \( S(t) \in \mathcal{N} \).

The notion of \( t_\infty \)-similarity is an equivalence relation in the set of all \( n \times n \) continuous matrices on \( \mathbb{R}^+ \), and it preserves some stability concepts [4,11]. We give some related properties that we need in the sequel.

**Lemma 2.1** ([15]). The linear system
\[ x' = A(t)x, \ x(t_0) = x_0, \] (5)
where \( A(t) \) is an \( n \times n \) continuous matrix, is an \( h \)-system (respectively \( h \)-stable) if and only if there exist \( c \geq 1 \) and a positive and continuous (respectively bounded) function \( h \) defined on \( \mathbb{R}^+ \) such that
\[ |\phi(t,t_0)| \leq c h(t) h(t_0)^{-1} \] (6)
for \( t \geq t_0 \geq 0 \), where \( \phi(t,t_0) \) is a fundamental matrix of (5).

We need Alekseev formula to compare between the solutions of (1) and the solutions of perturbed nonlinear system
\[ y' = f(t,y) + g(t,y), \ y(t_0) = y_0, \] (7)
where \( g \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n] \). Let \( y(t) = y(t,t_0,y_0) \) denote the solution of (7) passing through the point \((t_0,y_0)\) in \( \mathbb{R}^+ \times \mathbb{R}^n \).

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

**Lemma 2.2.** If \( y_0 \in \mathbb{R}^n \), for all \( t \) such that \( x(t,t_0,y_0) \in \mathbb{R}^n \),
\[ y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^{t} \Phi(t,s,y(s))g(s,y(s))ds. \]

**Theorem 2.3** ([3]). If the zero solution of (1) is \( hS \), then the zero solution of (2) is \( hS \).

**Theorem 2.4** ([4]). Suppose that \( f_x(t,0) \) is \( t_\infty \)-similar to \( f_x(t,x(t_0,x_0)) \) for \( t \geq t_0 \geq 0 \) and \( |x_0| \leq \delta \) for some constant \( \delta > 0 \). Then the solution \( v = 0 \) of (2) is \( hS \) if and only if the solution \( z = 0 \) of (3) is \( hS \).

## 3. Main results

In this section, we investigate \( hS \) for the nonlinear perturbed differential systems. Now, we examine the properties of \( hS \) for the perturbed system of (1)
\[ y' = f(t,y) + \int_{t_0}^{t} g(s,y(s))ds, \ y(t_0) = y_0, \] (8)
where \( g \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n] \) and \( g(t,0) = 0 \).
**Theorem 3.1.** Suppose that \( f_x(t, 0) \) is \( t_\infty \)-similar to \( f_x(t, x(t, t_0, x_0)) \) for \( t \geq t_0 \geq 0 \) and \( |x_0| \leq \delta \) for some constant \( \delta > 0 \), the solution \( x = 0 \) of (1) is hS with the increasing function \( h \), and \( g \) in (8) satisfies

\[
\left| \int_{t_0}^{s} g(\tau, y(\tau)) d\tau \right| \leq a(s) |y(s)| + b(s) \int_{t_0}^{s} c(\tau) |y(\tau)| d\tau, \quad t \geq t_0 \geq 0,
\]

where \( a, b, c \in C(\mathbb{R}^+, \mathbb{R}^+) \) and \( \int_{t_0}^{\infty} [a(s) + b(s) \int_{t_0}^{s} c(\tau) d\tau] ds < \infty \). Then, the solution \( y = 0 \) of (8) is hS.

**Proof.** Let \( x(t) = x(t, t_0, x_0) \) and \( y(t) = y(t, t_0, x_0) \). By Theorem 2.3, since the solution \( x = 0 \) of (1) is hS, the solution \( v = 0 \) of (2) is hS. Therefore, by Theorem 2.4, the solution \( z = 0 \) of (3) is hS. By Lemma 2.1, Lemma 2.2 and the increasing property of the function \( h \), we have

\[
|y(t)| \leq |x(t)| \left( 1 + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} g(\tau, y(\tau)) d\tau \right) ds
\]

\[
\leq c_1 |y_0| h(t) h(t_0)^{-1}
\]

\[
+ \int_{t_0}^{t} c_2 h(t) h(s)^{-1} |a(s)| |y(s)| + b(s) \int_{t_0}^{s} c(\tau) |y(\tau)| d\tau \right) ds.
\]

Set \( u(t) = |y(t)| h(t)^{-1} \). Then, by Gronwall’s inequality, we obtain

\[
|y(t)| \leq c_1 |y_0| h(t) h(t_0)^{-1} e^{c_2 \int_{t_0}^{t} [a(s) + b(s) \int_{t_0}^{s} c(\tau) d\tau] ds}
\]

\[
\leq c_1 c_2 e^{-c_2 \int_{t_0}^{t} [a(s) + b(s) \int_{t_0}^{s} c(\tau) d\tau] ds}
\]

It follows that \( y = 0 \) of (8) is hS. Hence, the proof is complete. \( \Box \)

**Remark 3.1.** In the linear case, we can obtain that if the zero solution \( x = 0 \) of (5) is hS, then the perturbed system

\[
y' = A(t) y + \int_{t_0}^{t} g(s, y(s)) ds, \quad y(t_0) = y_0,
\]

is also hS under the same hypotheses in Theorem 3.1 except the condition of \( t_\infty \)-similarity.

**Remark 3.2.** Letting \( b(s) = 0 \) in Theorem 3.1, we obtain the same result as that of Theorem 3.3 in [8].

To investigate an h-system of (8), we need the following lemma.

**Lemma 3.2.** Let \( u, p, q, w, r \in C(\mathbb{R}^+, \mathbb{R}^+) \) and suppose that, for some \( c \geq 0 \), we have

\[
u(t) \leq c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} [q(\tau) u(\tau) + w(\tau) \int_{t_0}^{\tau} r(a) u(a) da] d\tau ds, \quad t \geq t_0. \quad (9)
\]

Then

\[
u(t) \leq c \exp \left( \int_{t_0}^{t} p(s) \int_{t_0}^{s} [q(\tau) + w(\tau) \int_{t_0}^{\tau} r(a) da] d\tau ds \right), \quad t \geq t_0. \quad (10)
\]
Proof. Setting \(v(t) = c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} [q(\tau) u(\tau) + w(\tau) \int_{t_0}^{\tau} r(a) u(a) da] dr ds\), we have \(v(t_0) = c\) and

\[
v'(t) = p(t) \int_{t_0}^{t} [q(s) u(s) + w(s) \int_{t_0}^{s} r(a) u(a) da] ds
\]

\[
\leq p(t) \int_{t_0}^{t} [q(s) + w(s) \int_{t_0}^{s} r(a) da] v(s) ds
\]

\[
\leq [p(t) \int_{t_0}^{t} [q(s) + w(s) \int_{t_0}^{s} r(a) da] ds] v(t), \quad t \geq t_0,
\]

since \(v(t)\) is nondecreasing and \(u(t) \leq v(t)\). It follows from the Gronwall inequality that (11) yields the estimate (10). \(\square\)

**Theorem 3.3.** Suppose that \(f_x(t, 0)\) is \(t_{\infty}\)-similar to \(f_x(t, x(t, t_0, x_0))\) for \(t \geq t_0 \geq 0\) and \(|x_0| \leq \delta\) for some constant \(\delta > 0\). If the solution \(x = 0\) of (1) is an \(h\)-system with a positive continuous function \(h\) and \(g\) in (8) satisfies

\[
|g(t, y)| \leq \lambda(t)|y| + \beta(t) \int_{t_0}^{t} \gamma(s)|y(s)| ds, \quad t \geq t_0, \quad y \in \mathbb{R}^n,
\]

where \(\lambda, \beta, \gamma : \mathbb{R}^+ \to \mathbb{R}^+\) is continuous with

\[
\int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} [h(\tau) \lambda(\tau) + \beta(\tau) \int_{t_0}^{\tau} h(\tau) \gamma(\tau) dr] dr ds < \infty,
\]

for all \(t_0 \geq 0\), then the solution \(y = 0\) of (8) is an \(h\)-system.

Proof. Let \(x(t) = x(t, t_0, x_0)\) and \(y(t) = y(t, t_0, x_0)\). By Theorem 2.3, since the solution \(x = 0\) of (1) is an \(h\)-system, the solution \(v = 0\) of (2) is an \(h\)-system. Therefore, by Theorem 2.4, the solution \(z = 0\) of (3) is an \(h\)-system. By Lemma 2.1 and Lemma 2.2, we have

\[
\left|g(t)\right| \leq \left|x(t)\right| + \int_{t_0}^{t} \left|\Phi(t, s, y(s))\right| \int_{t_0}^{s} \left|g(\tau, y(\tau))\right| d\tau ds
\]

\[
\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) \int_{t_0}^{s} h(\tau) \lambda(\tau) \left|g(\tau)\right| h(\tau) d\tau ds + \beta(\tau) \int_{t_0}^{\tau} h(\tau) \gamma(\tau) \left|g(\tau)\right| h(\tau) d\tau ds.
\]

Setting \(u(t) = |g(t)| h(t)^{-1}\) and using Lemma 3.2, we obtain

\[
\left|g(t)\right| \leq c_1 |y_0| h(t) h(t_0)^{-1} e^{c_2 \int_{t_0}^{t} \frac{1}{h(\tau)} \int_{t_0}^{\tau} h(\tau) \lambda(\tau) + \beta(\tau) \int_{t_0}^{\tau} h(\tau) \gamma(\tau) dr} dr ds
\]

\[
\leq c |y_0| h(t) h(t_0)^{-1}, \quad t \geq t_0,
\]

where \(c = c_1 e^{c_2 \int_{t_0}^{\infty} \frac{1}{h(\tau)} \int_{t_0}^{\tau} h(\tau) \lambda(\tau) + \beta(\tau) \int_{t_0}^{\tau} h(\tau) \gamma(\tau) dr} dr ds\). It follows that \(y = 0\) of (8) is an \(h\)-system. Hence, the proof is complete. \(\square\)

**Remark 3.3.** Letting \(\beta(t) = 0\) in Theorem 3.3, we have the similar result as that of Theorem 2.5 in [7].
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Yoon Hoe Goo received the BS from Cheongju University and Ph.D at Chungnam National University under the direction of Chin-Ku Chu. Since 1993 he has been at Hanseo University as a professor. His research interests focus on topological dynamical systems and differential equations.

Department of Mathematics, Hanseo University, Seasan 356-706, Korea.

e-mail: yhgoo@hanseo.ac.kr