THE VERTEX AND EDGE PI INDICES OF GENERALIZED HIERARCHICAL PRODUCT OF GRAPHS

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1. Introduction

Throughout this paper all graphs considered are finite, simple and connected. The distance \(d_G(u, v)\) between the vertices \(u\) and \(v\) of a graph \(G\) is equal to the length of a shortest path that connects \(u\) and \(v\). Suppose \(G\) is a graph with vertex and edge sets \(V = V(G)\) and \(E = E(G)\), respectively. Suppose \(e = ab \in E(G)\). The number of edges of \(G\) whose distance to the vertex \(u\) is smaller than the distance to the vertex \(v\) is denoted by \(m^G_{uv}(e)\). The edge PI index of \(G\), \(\text{PI}_e(G)\), of a graph \(G\) is defined as \(\text{PI}_e(G) = \sum_{e=uv \in E(G)} (m^G_{gu}(e) + m^G_{gv}(e))\) [4, 5]. In a similar way, the quantities \(n^G_a(e)\) is defined as the number of vertices closer to \(a\) than to \(b\). In other words, \(n^G_a(e) = |\{u \in V(G)|d(u, a) < d(u, b)\}|. \) The vertex PI index of \(G\), \(\text{PI}_v(G)\), is defined as the summation of \(n^G_u(uv) + n^G_v(uv)\) over all edges of \(G\) [6, 7].

The edges \(e = uv\) and \(f = xy\) of \(G\) are said to be equidistant edges if \(\min\{d_G(u, x), d_G(u, y)\} = \min\{d_G(v, x), d_G(v, y)\}\). For \(e = uv\) in \(G\), the number of equidistant vertices of \(e\) is denoted by \(N_G(e)\) and the number of equidistant

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edges of $e$ is denoted by $M_G(e)$. Then the above definitions are equivalent to

\[
\text{PI}_v(G) = |V(G)||E(G)| - \sum_{e \in E(G)} N_G(e), \quad \text{PI}_e(G) = |E(G)|^2 - \sum_{e \in E(G)} M_G(e).
\]

Suppose $G$ and $H$ are graphs and $U \subseteq V(G)$. The generalized hierarchical product, denoted by $G(U) \cap H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices $(g, h)$ and $(g', h')$ are adjacent if and only if $g = g' \in U$ and $hh' \in E(H)$ or, $gg' \in E(G)$ and $h = h'$. This graph operation introduced recently by Barrière et al. [2, 3] and found some applications in computer science.

Most of our notation is standard and taken mainly from [1, 9]. The path graph with $n$ vertices is denoted by $P_n$.

## 2. Main results

Let $G = (V, E)$ be a graph and $U \subseteq V$. We need some notation than taken from [8]. We encourage the interested readers to consult this paper and references therein for more information on this topic. Following Pattabiraman and Paulraja [8], an $u-v$ path through $U$ in $G(U)$ is an $u-v$ path in $G$ containing some vertex $w \in U$ (vertex $w$ could be the vertex $u$ or $v$). Let $d_{G(U)}(u, v)$ denote the length of a shortest $u-v$ path through $U$ in $G$. Notice that, if one of the vertices $u$ and $v$ belong to $U$, then $d_{G(U)}(u, v) = d_G(u, v)$. A vertex $x \in V(G(U))$ is said to be equidistant from $e = uv \in E(G(U))$ through $U$ in $G(U)$, if $d_{G(U)}(u, x) = d_{G(U)}(v, x)$. For an edge $e$ in $G(U)$, let $N_{G(U)}(e)$ denote the number of equidistant vertices of $e$ through $U$ in $G(U)$. Then $\text{PI}_v(G(U))$ can be defined as follows:

\[
\text{PI}_v(G(U)) = \sum_{e \in E(G(U))} (|V(G(U))| - N_{G(U)}(e)).
\]

For $e \in E(G)$ and $S \subseteq V(G)$, let $N_{S}(e)$ denote the number of equidistant vertices of $e$ (in $G$) contained in $S$. The edges $e = uv$ and $f = xy$ of $G(U)$ are said to be equidistant edges through $U$ in $G(U)$ if $\min\{d_{G(U)}(u, x), d_{G(U)}(u, y)\} = \min\{d_{G(U)}(v, x), d_{G(U)}(v, y)\}$. Let $M_{G(U)}(e)$ denote the number of equidistant edges of $e$ through $U$ in $G(U)$. Then $\text{PI}_e(G(U))$ is defined as follows:

\[
\text{PI}_e(G(U)) = \sum_{e \in E(G(U))} (|E(G(U))| - M_{G(U)}(e)).
\]

Let $G_i = (V_i, E_i)$, $1 \leq i \leq N$, be a graph with vertex set $V_i$ having a distinguished or root vertex 0. Following Barrière et al. [2, 3], the hierarchical product $H = G_N \cap \ldots \cap G_2 \cap G_1$ is the graph with vertices the $N$-tuples $x_N \ldots x_3x_2x_1$, $x_i \in V_i$, and edges defined by the adjacencies:

\[
\begin{align*}
    x_N \ldots x_3x_2x_1 & \sim \\
    x_N \ldots x_3x_2y_1 & \text{ if } y_1 \sim x_1 \text{ in } G_1, \\
    x_N \ldots x_3y_2x_1 & \text{ if } y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 = 0, \\
    x_N \ldots y_3x_2x_1 & \text{ if } y_3 \sim x_3 \text{ in } G_3 \text{ and } x_1 = x_2 = 0, \\
    & \vdots \vdots \\
    y_N \ldots x_3x_2x_1 & \text{ if } y_N \sim x_N \text{ in } G_N \text{ and } x_1 = x_2 = \ldots = x_{N-1} = 0.
\end{align*}
\]
A path graph with \( n \) vertices, is denoted by \( P_n \) and a caterpillar is a tree in which all the vertices are within distance 1 of a central path. By definition of hierarchical product, it is clear that if \( P_m \) is a path graph and \( S_n \) is a rooted star graph with root vertex \( r \) such that \( \text{deg}(r) > 1 \) then \( P_m \cap S_n \) is a caterpillar with order \( mn \) and generally, the hierarchical product of an arbitrary sequence of acyclic graphs is again an acyclic graph. Therefore, we can write:

**Lemma 2.1.** If \( G_1, G_2, \ldots, G_n \) are trees with orders \( m_1, \ldots, m_n \), respectively, then

\[
\text{PL}_v(G_1 \cap \ldots \cap G_n) = \prod_{i=1}^{n} m_i - 1 \prod_{i=1}^{n} m_i,
\]

\[
\text{PL}_v(G_1 \cap \ldots \cap G_n) = \prod_{i=1}^{n} m_i - 1(\prod_{i=1}^{n} m_i - 2).
\]

Let \( G_1, G_2, \ldots, G_n \) be connected rooted graphs with root vertices \( r_1, \ldots, r_n \), respectively and \( e = (a_{n-1}, u, r_1, \ldots, r_1)(a_{n-1}, v, r_1, \ldots, r_1) \) is an edge of \( H \) such that \( uv \in E(G_1) \). In order to simplify our notation, we will denote \( n(a_{n-1}, a_{n+1}, r_1, \ldots, r_1) \) by \( m_1(e) \), \( n(a_{n-1}, a_{n+1}, v, r_1, \ldots, r_1) \) by \( m_2(e) \), \( m(a_{n-1}, a_{n+1}, r_1, \ldots, r_1) \) by \( m_1(e) \) and \( m(a_{n-1}, a_{n+1}, v, r_1, \ldots, r_1) \) by \( m_2(e) \).

In what follows, let \( \prod_i f_i = 1 \) and \( \sum_i f_i = 0 \) for each \( i, j \in \{0, 1, 2, \ldots\} \), that \( i \neq j = 1 \). Furthermore, let \( \prod_i f_i = \sum_i f_i = 0 \) for every \( i, j \in \{0, 1, 2, \ldots\} \), such that \( i \neq j > 1 \). Also, for a sequence of graphs, \( G_1, G_2, \ldots, G_n \), we set \( |V_{i,j}| = \prod_{k=i}^{j} |V(G_k)| \) and \( |V_{i,j}| = \prod_{k=i}^{j} |V(G_k)| \).

The main results of [8] are Theorems 2.2 and 3.1. We claim that these results are incorrect. We first explain the reason that makes Theorem 2.2 to be incorrect. In [8, Eq. 2.3], the authors claim that for each edge \( e' = (u, v) \in G(U) \cap H \) such that \( v \in V(H) \) and \( e = u, \ u \in E(G) \), we have \( N_{G(U)}(e) = |V(H)|N_{G(U)}(e) \). In Figure 2, a counterexample for this argument is presented. Notice that if \( U = \{r\}, e' = (y, 1)(z, 1) \) then \( N_{G(U)}(e') = 6 \), but \( |V(H)|N_{G(U)}(e) = 2 \), which is impossible. In Figure 3, a family of enough large counterexamples are presented. In this figure, \( H = P_m, U = \{x\} \) and \( |V(G)| = 2n+1 \). Then \( \text{PL}_v(G(U) \cap H) = 2mn(2mn+2m+n-2)+m(m-1) \). But, [8, Theorem 2.2] implies that \( \text{PL}_v(G(U) \cap H) = 2mn(3mn+2m-1)+m(m-1) \). Then \( \{2mn(2mn+2m+n-2)+m(m-1) \} = 2mn(3mn+2m-1)+m(m-1) \), leads to another contradiction.

In the following theorem a correct form of [8, Theorem 2.2] is presented.

**Theorem 2.2.** Suppose \( G_1, G_2, \ldots, G_n \) are connected rooted graphs with root vertices \( r_1, \ldots, r_n \), respectively. Then

\[
\text{PL}_v(G_1 \cap \ldots \cap G_n) = \sum_{i=1}^{n} |V_{i,n}| \text{PL}_v(G_i) + \sum_{i=1}^{n-1} |V_{i+1,n}|(|E(G_i)| - N_{r_i}).
\]
\[
\times \sum_{j=i+1}^{n} ([V(G_j)] - 1)|V_{1,j-1}|,
\]

where \(N_{r_i} = \{|uv \in E(G_i) \mid d_{G_i}(u, r_i) = d_{G_i}(v, r_i)|\} \).

**Proof.** Let \( H = G_n \cap \cdots \cap G_2 \cap G_1 \) and \( e = (a_n, \ldots, a_{i+1}, u, r_{i-1}, \ldots, r_1)(a_n, \ldots, a_{i+1}, v, r_{i-1}, \ldots, r_1) \) be an edge of \( H \) such that \( uv \in E(G_i) \), and \( a_j \in V(G_j) \). It follows from the edge structure of \( H \) that, if \( d_{G_i}(u, r_i) \neq d_{G_i}(v, r_i) \) then

\[
n_1^H(e) + n_2^H(e) = (n_{G_i}^v(uv) + n_{G_i}^v(uv)) \prod_{j=1}^{i-1} |V(G_j)| + \sum_{j=i+1}^{n} ([V(G_j)] - 1) \prod_{k=1}^{j-1} |V(G_k)|
\]

and if \( d_{G_i}(u, r_i) = d_{G_i}(v, r_i) \) then

\[
n_1^H(e) + n_2^H(e) = (n_{G_i}^v(uv) + n_{G_i}^v(uv)) \prod_{j=1}^{i-1} |V(G_j)|.
\]

Thus, the summation of \([n_1^H(uv) + n_2^H(uv)]\) over all edges of copies of \( G_i \), is equal to:

\[
(\prod_{j=1,j \neq i}^{n} |V(G_j)|) \text{PL}_v(G_i) + (|E(G_i)| - N_{r_i}) (\prod_{j=1,j \neq i}^{n} |V(G_j)|) \sum_{j=i+1}^{n} ([V(G_j)] - 1) \prod_{k=1}^{j-1} |V(G_k)|.
\]

Therefore,

\[
\text{PL}_v(H) = \sum_{i=1}^{n} \left[ (\prod_{j=1,j \neq i}^{n} |V(G_j)|) \text{PL}_v(G_i) + (|E(G_i)| - N_{r_i}) (\prod_{j=1,j \neq i}^{n} |V(G_j)|) \sum_{j=i+1}^{n} ([V(G_j)] - 1) \prod_{k=1}^{j-1} |V(G_k)| \right]
\]

\[
= \sum_{i=1}^{n} (\prod_{j=1,j \neq i}^{n} |V(G_j)|) \text{PL}_v(G_i) + \sum_{i=1}^{n-1} \left( (\prod_{j=i+1}^{n} |V(G_j)|) (|E(G_i)| - N_{r_i}) \sum_{j=i+1}^{n} ([V(G_j)] - 1) \prod_{k=1}^{j-1} |V(G_k)| \right),
\]

which proves the theorem. \(\square\)

**Corollary 2.3.** Suppose \( G_1, G_2, \ldots, G_n \) are connected rooted graphs with root vertices \( r_1, \ldots, r_n \), respectively. We also assume that \( r_i, 1 \leq i \leq n, \) lies on no odd cycle of \( G_i \). Then

\[
\text{PL}_v(G_n \cap \cdots \cap G_2 \cap G_1) = \sum_{i=1}^{n} |V_{1,n}^i| \text{PL}_v(G_i) + \sum_{i=1}^{n-1} |V_{i+1,n}||E(G_i)|
\]

\[
\times \sum_{j=i+1}^{n} ([V(G_j)] - 1)|V_{i,j-1}|.
\]
We now prove that the [8, Theorem 3.1] is incorrect. We first explain the reason that makes this Theorem to be incorrect. In [8, Eq. 3.8 and 3.9], the authors claim that for each edge \( e' = (u_\alpha, v_\alpha)(u_\beta, v_\beta) \in G(U) \cap H \) such that \( v_\alpha \in V(H) \) and \( e = u_\alpha u_\beta \in E(G) \), we have \( M_{G(U) \cap H}(e') = |V(H)|M_{G(U)}(e) + |E(H)|N_{G(U)}(e) \). In Figure 4, a counterexample for this argument is presented. Notice that if \( U = \{x, y, z\} \) and \( e' \) is corresponding edge of \( e \) in \( G(U) \cap H \) then \( M_{G(U) \cap H}(e') = 7 \), but \( |V(H)|M_{G(U)}(e) + |E(H)|N_{G(U)}(e) = 9 \), which is impossible. On the other hand, by [8, Theorem 3.1] \( PL(G(U) \cap H) = 168 \), that is incorrect. The correct value of \( PL \) is 164.

In the following theorem a correct form of [8, Theorem 3.1] is presented.

**Theorem 2.4.** Suppose \( G_1, G_2, \ldots, G_n \) are connected rooted graphs with root vertices \( r_1, \ldots, r_n \), respectively. Then

\[
PL(G_n \cap \ldots \cap G_2 \cap G_1) = \sum_{i=1}^{n} |V_{i+1,n}|PL(G_i) + \sum_{i=1}^{n} |V_{i+1,n}| \left( \sum_{j=1}^{i-1} |E(G_j)||V_{j+1,i-1}| \right) PL(G_i)
\]

\[
+ \sum_{i=1}^{n} \left( |E(G_i)| - N_{r_i} \right) |V_{i+1,n}| \sum_{j=i+1}^{n} \left( |(V(G_j)| - 1) \right)
\]

\[
\times \sum_{k=1}^{j-1} |E(G_k)||V_{k+1,j-1}| + |E(G_j)| \right),
\]

where \( N_{r_i} = |\{uv \in E(G_i) \mid d_{G_i}(u, r_i) = d_{G_i}(v, r_i)\}|\).

**Proof.** Let \( H = G_n \cap \ldots \cap G_2 \cap G_1 \). By the edge structure of \( H \), it is not difficult to see that, for every edge \( e = (a_n, \ldots, a_{i+1}, u, r_{i-1}, \ldots, r_1)(a_n, \ldots, a_{i+1}, v, r_{i-1}, \ldots, r_1) \) of \( H \) such that \( uv \in E(G_i) \) and \( a_j \in V(G_j) \) (for \( j = i + 1, i + 2, \ldots, n \)), if \( d_{G_i}(u, r_i) \neq d_{G_i}(v, r_i) \) then

\[
m_i^H(e) + m_i^H(e) = m_{i-1}^G(uv) + m_{i-2}^G(uv) + (m_{i-1}^G(uv) + m_{i-2}^G(uv)) \sum_{j=1}^{i-1} |E(G_j)|
\]

\[
\times \prod_{k=j+1}^{i-1} |V(G_k)| + \sum_{j=i+1}^{n} \left( |(V(G_j)| - 1) \sum_{k=1}^{j-1} |E(G_k)||V(G_i)| + |E(G_j)| \right)
\]

and if \( d_{G_i}(u, r_i) = d_{G_i}(v, r_i) \) then

\[
m_i^H(e) + m_i^H(e) = m_{i-1}^G(uv) + m_{i-2}^G(uv) + (m_{i-1}^G(uv) + m_{i-2}^G(uv)) \sum_{j=1}^{i-1} |E(G_j)| \prod_{k=j+1}^{i-1} |V(G_k)|.
\]

Thus, the summation of \( [m_i^H(uv) + m_i^H(uv)] \) over all edges of copies of \( G_i \), is equal to:

\[
( \prod_{j=i+1}^{n} |V(G_j)|)PL(G_i) + ( \prod_{j=i+1}^{n} |V(G_j)|)(\sum_{j=i+1}^{i-1} |E(G_j)| \prod_{k=j+1}^{i-1} |V(G_k)|)PL(G_i)
\]
Suppose Corollary 2.5. 

vertices \( r \)

as desired.

and therefore

\[
\prod_{i=1}^{n} |V(G_j)| \prod_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|
\]

and therefore

\[
\prod_{i=1}^{n} |V(G_j)| \prod_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|
\]

\[
= \sum_{i=1}^{n} \left( \prod_{j=i+1}^{n} |V(G_j)| \right) \prod_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|
\]

as desired. \( \square \)

**Corollary 2.5.** Suppose \( G_1, G_2, \ldots, G_n \) are connected rooted graphs with root vertices \( r_1, \ldots, r_n \), respectively. We also assume that \( r_i \) lies on no odd cycle of \( G_i \), \( i = 1, 2, \ldots, n \). Then

\[
\prod_{i=1}^{n} |V(G_j)| \prod_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|
\]

\[
\times \prod_{i=1}^{n} |V(G_j)| \prod_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|
\]

\[
= \sum_{i=1}^{n} \left( |E(G_i)| \prod_{j=i+1}^{n} |V(G_j)| \right) \prod_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|
\]

as desired. \( \square \)
Example 2.6. Consider a rooted cycle graph $C_m$ with root vertex $r$. By definition of this graph, Figure 1, it is clear that

$$N_r = \begin{cases} 1 & 2 \mid m \\ 0 & 2 \mid m \end{cases} \quad \text{PL}_v(C_m) = \begin{cases} m(m-1) & 2 \mid m \\ m^2 & 2 \mid m \end{cases} \quad \text{PL}_e(C_m) = \begin{cases} m(m-1) & 2 \mid m \\ m(m-2) & 2 \mid m \end{cases}$$

So, by Theorems 2.2 and 2.4, we calculate that

1. $\text{PL}_v\left(C_m \cap \cdots \cap C_m \right) = \begin{cases} m^{2n} - m^n & 2 \mid m \\ mm^{n+1} + \frac{m}{m-1} (m^{2n} - nm^{n+1} + (n-1)m^n) & 2 \mid m \end{cases}$

2. $\text{PL}_e\left(C_m \cap \cdots \cap C_m \right) = \begin{cases} \frac{m^{2n+1}}{m-1} - \frac{m^{n+2}}{(m-1)^2} + m^{n+1} \left(1 + \frac{1}{(m-1)^2}\right) + \frac{m}{m-1} & 2 \mid m \\ \frac{1}{(m-1)^2} (m^{2n+2} - 2m^{n+1}(2m-1) + m(3m-2)) & 2 \mid m \end{cases}$
Figure 3. The Hierarchical Product of $G(U)$ and $H$

Figure 4. The Generalized Hierarchical Product of $G(U)$ and $H$

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