GLOBAL DYNAMICS OF A NON-AUTONOMOUS RATIONAL DIFFERENCE EQUATION

ÖZKAN ÖCALAN

Abstract. In this paper, we investigate the boundedness character, the periodic character and the global behavior of positive solutions of the difference equation
\[ x_{n+1} = p_n + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \]
where \( \{p_n\} \) is a two periodic sequence of nonnegative real numbers and the initial conditions \( x_{-1}, x_0 \) are arbitrary positive real numbers.

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1. Introduction

Recently, there has been an increasing interest in the study of the qualitative analyses of rational difference equations. For example, see [1 – 8] and the references cited therein.

This work studies the boundedness character and the global asymptotic stability for the positive solutions of the difference equation
\[ x_{n+1} = p_n + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \]  
(1.1)
where \( \{p_n\} \) is a two periodic sequence of nonnegative real numbers and the initial conditions \( x_{-1}, x_0 \) are arbitrary positive numbers.

As far as we can examine, this is the first work devote to the investigation of the type Eq.(1.1).

Now, we assume \( p_{2n} = \alpha \) and \( p_{2n+1} = \beta \) in Eq.(1.1). Then we have
\[ x_{2n+1} = \alpha + \frac{x_{2n}}{x_{2n-1}}, \quad n = 0, 1, \ldots \]  
(1.2)
and
\[ x_{2n+2} = \beta + \frac{x_{2n+1}}{x_{2n}}, \quad n = 0, 1, \ldots \]  
(1.3)
The autonomous case of Eq. (1.1) is
\[ x_{n+1} = p + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots \]  
(1.4)
where \( p > 0 \) and the initial conditions \( x_0, x_1 \) are arbitrary positive numbers. We now consider the local asymptotic stability of the unique equilibrium \( x = p + 1 \) of Eq. (1.4).

The linearized equation for Eq. (1.4) about the positive equilibrium \( x = p + 1 \) is
\[ x_{n+1} - \frac{1}{p+1} x_n + \frac{1}{p+1} x_{n-1} = 0. \]
The following theorem is given in [1].

**Theorem A.** Consider Eq. (1.4) and assume that \( x_0, x_1, p \in (0, \infty) \). Then the unique positive equilibrium \( x = p + 1 \) of Eq. (1.4) is globally asymptotically stable.

2. **Boundedness Character of Eq. (1.1)**

In this section, we investigate the boundedness character of Eq. (1.1). So, we have the following result.

**Theorem 2.1.** Suppose that \( \alpha > 1 \) and \( \beta > 1 \) with \( \alpha \neq \beta \), then every positive solution of Eq. (1.1) is bounded.

**Proof.** It is clear from Eq. (1.2) and (1.3) that
\[ x_{2n} > \alpha \quad \text{and} \quad x_{2n-1} > \alpha, \quad \text{for every } n \geq 1. \]
(2.1)
Then, from (1.2) and (2.1) we obtain
\[ x_{2n+1} = \alpha + \frac{x_{2n}}{x_{2n-1}} < \alpha + \frac{x_{2n}}{\alpha} \]  
(2.3)
and from (1.3) and (2.1) we obtain
\[ x_{2n} = \beta + \frac{x_{2n-1}}{x_{2n-2}} < \beta + \frac{x_{2n-1}}{\beta}. \]  
(2.4)
From (2.3), (2.4) using induction we get
\[
x_{2n+1} < \alpha + \frac{\beta}{\alpha} + \frac{1}{\beta} \left( 1 + \frac{1}{\alpha} + \frac{1}{\alpha^2 \beta^2} + \cdots \right) + \frac{1}{\alpha^2} \left( 1 + \frac{1}{\alpha} + \frac{1}{\alpha^2 \beta^2} + \cdots \right) + x_{-1}
\]
\[
= \alpha + \frac{\beta}{\alpha} + \frac{1}{\beta} \left( \frac{\alpha}{\alpha \beta - 1} \right) + \frac{1}{\alpha^2} \left( \frac{\alpha}{\alpha \beta - 1} \right) + x_{-1}
\]
\[
= \alpha + \frac{\beta}{\alpha} + \frac{\alpha}{\alpha \beta - 1} + \frac{1}{\alpha} \left( \frac{\beta}{\alpha \beta - 1} \right) + x_{-1}
\]
\[
= \left( \alpha + \frac{\beta}{\alpha} \right) \frac{\alpha \beta}{(\alpha \beta - 1)} + x_{-1}.
\]
The result now follows.

### 3. Stability and Periodicity for Eq. (1.1)

In this section, we investigate the periodicity and stability character of positive solutions of Eq. (1.1). Now, we have the following result.

**Proposition 3.1.** Consider Eq. (1.1) when the case $\alpha \neq \beta$ and assume that $\alpha, \beta \in (0, \infty)$. Then there exist prime two periodic solutions of Eq. (1.1).

**Proof.** In order Eq. (1.1) to possess a periodic solution $\{x_n\}$ of prime period 2, we must find positive numbers $x_{-1}, x_0$ such that

$$
x_{-1} = x_1 = \alpha + \frac{x_0}{x_{-1}}, \quad x_0 = x_2 = \beta + \frac{x_{-1}}{x_0}.
$$

(3.1)

Let $x_{-1} = x$, $x_0 = y$, then from (3.1) we obtain the system of equations

$$
x = \alpha + \frac{y}{x}, \quad y = \beta + \frac{x}{y}.
$$

(3.2)

We prove that (3.2) has a solution $(\bar{x}, \bar{y})$, $\bar{x} > 0$, $\bar{y} > 0$. From the first relation of (3.2) we have

$$
y = (x - \alpha)x.
$$

(3.3)

From (3.3) and the second relation of (3.2) we obtain

$$
x(x - \alpha) = \beta + \frac{x}{x(x - \alpha)} = \beta + \frac{1}{x - \alpha} \quad \text{and} \quad x(x - \alpha)^2 - \beta(x - \alpha) - 1 = 0.
$$

Now we consider the function

$$
f(x) = x(x - \alpha)^2 - \beta(x - \alpha) - 1, \quad x > \alpha.
$$

(3.4)

Then from (3.4) we get

$$
\lim_{x \to \infty} f(x) = -1, \quad \lim_{x \to \infty} f(x) = \infty.
$$

(3.5)

Hence Eq. (3.4) has a solution $\bar{x} > \alpha$. Then if $\bar{y} = (\bar{x} - \alpha)\bar{x}$, we have that the solution $\bar{x}$ of Eq. (1.1) with initial values $x_{-1} = \bar{x}$, $x_0 = \bar{y}$ is a periodic solution of period two. \qed
Theorem 3.2. Consider Eq. (1.1) when the case $\alpha \neq \beta$ and assume that $\alpha, \beta \in (0, \infty)$. Suppose that
\[
\frac{\alpha}{\beta^2} + \frac{1}{\alpha \beta} + \frac{1}{\alpha^3} < 1.
\] (3.6)
Then the two periodic solutions of Eq. (1.1) are locally asymptotically stable.

Proof. From equations (1.3), (1.4) and Proposition 3.1 there exist $\pi, \gamma$ such that
\[
\pi = \alpha + \frac{\gamma}{\beta}, \quad \gamma = \beta + \frac{\pi}{\gamma}.
\] (3.7)
We set $x_{2n-1} = u_n$, $x_{2n} = v_n$ in equations (1.3), (1.4) and so we have
\[
u_{n+1} = \alpha + \frac{v_n}{u_n}, \quad v_{n+1} = \beta + \frac{u_{n+1}}{v_n} = \beta + \frac{v_n}{u_n} = \beta + \frac{\alpha u_n + v_n}{u_n v_n}.
\] (3.8)
Then $(\pi, \gamma)$ is the positive equilibrium of Eq. (3.8), and the linearised system of Eq. (3.8) about $(\pi, \gamma)$ is the system
\[
z_{n+1} = Bz_n, \quad \text{where} \quad B = \left(\begin{array}{cc}
\frac{\alpha}{\beta^2} & \frac{1}{\beta^2} \\
\frac{\alpha}{\gamma^2} & \frac{1}{\gamma^2}
\end{array}\right), \quad z_n = \left(\begin{array}{c}
u_n \\
v_n
\end{array}\right).
\] The characteristic equation of $B$ is
\[
\lambda^2 + \lambda \left(\frac{\alpha}{\beta^2} + \frac{\gamma}{\beta^2}\right) + \frac{\alpha}{\beta^3} + \frac{1}{\beta^3} = 0.
\] (3.9)
Using Eq. (3.6), from Eq. (3.7), since $\pi > \alpha$, $\gamma > \beta$ we have
\[
\frac{\alpha}{\beta^2} + \frac{\gamma}{\beta^2} + \frac{\alpha}{\gamma^2} + \frac{1}{\gamma^2} < \frac{\alpha}{\beta^2} + \frac{1}{\alpha \beta} + \frac{1}{\alpha^3} + 1 - \frac{\alpha}{\gamma} < 1
\]
and we obtain
\[
\frac{\alpha}{\beta^2} + \frac{1}{\alpha \beta} + \frac{1}{\alpha^3} < \frac{\alpha}{\gamma} < 1.
\] (3.10)
Then, from (3.10) and Theorem 1.3.7 of Kocic and Ladas in [4], all the roots of Eq. (3.9) are modulus less than 1. Therefore, from Proposition 3.1, system (3.8) is asymptotically stable. The proof is complete. \[\square\]

Theorem 3.3. Consider Eq. (1.1) when the case $\alpha \neq \beta$. Assume that $\alpha > 1$, $\beta > 1$. Then every positive solution of Eq. (1.1) converges to a two-periodic solution of Eq. (1.1).

Proof. Since $\alpha > 1$, $\beta > 1$, we know by Theorem 2.1 that every positive solution of Eq. (1.1) is bounded, it follows that there are finite
\[
s = \liminf_{n \to \infty} x_{2n+1} \quad \text{and} \quad S = \limsup_{n \to \infty} x_{2n+1},
\]
\[
l = \liminf_{n \to \infty} x_{2n} \quad \text{and} \quad L = \limsup_{n \to \infty} x_{2n}
\]
exist. Then it is easy to see from Eq. (1.2) and (1.3) that
\[
s \geq \alpha + \frac{l}{S} \quad \text{and} \quad S \leq \alpha + \frac{L}{s}.
\]
and 
\[ l \geq \beta + \frac{s}{L} \quad \text{and} \quad L \leq \beta + \frac{S}{L}. \]

Thus, we have 
\[ sS \geq \alpha S + l \quad \text{and} \quad Ss \leq \alpha s + L \]

and 
\[ Ll \geq \beta L + s \quad \text{and} \quad Ll \leq \beta l + S. \]

This implies that 
\[ \alpha S + l \leq Ss \leq \alpha s + L \]

and 
\[ \beta L + l \leq Ll \leq \beta l + S. \]

Then, we get 
\[ \alpha (S - s) \leq (L - l) \] (3.11)

and 
\[ \beta (L - l) \leq (S - s). \] (3.12)

Now, we shall prove that \( s = S \) and \( l = L \). It is clear that if \( l = L \), then by (3.11) it must be \( s = S \). Similarly, if \( s = S \), then by (3.12) it must be \( l = L \).

Hence we assume that \( s < S \) and \( l < L \). From (3.11) and (3.12) we have 
\[ \alpha (S - s) + \beta (L - l) \leq (S - s) + (L - l), \]
then we obtain a contradiction. So, we get \( s = S \) and \( l = L \).

Moreover, it is obvious that since \( \alpha \neq \beta \), then from Eq. (1.2) and Eq. (1.3) 
\[ \lim_{n \to \infty} x_{2n+1} \neq \lim_{n \to \infty} x_{2n}. \]

Then it is clear that every positive solution of Eq. (1.1) converges to a two-periodic solution of Eq. (1.1). The proof is complete. \( \square \)

Finally, using Proposition 3.1, Theorems 3.1 and 3.2, we have the following Theorem:

**Theorem 3.4.** Consider Eq. (1.1) when the case \( \alpha \neq \beta \). Assume that \( \alpha > 1, \beta > 1 \) and that (3.6) holds. Then two-period solutions of Eq. (1.1) are globally asymptotically stable.
REFERENCES


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