ON THE ORDERING OF ASYMPTOTIC PAIRWISE NEGATIVELY DEPENDENT STRUCTURE OF STOCHASTIC PROCESSES†

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ABSTRACT. In this paper, we introduced a new asymptotic pairwise negatively dependent (APND) structure of stochastic processes. We are also important to know the degree of APND-ness and to compare pairs of stochastic vectors as to their APND-ness. So, we introduced a definitions and some basic properties of APND ordering. Some preservation results of APND ordering are derived. Finally, we shown some examples and applications.

AMS Mathematics Subject Classification : 65H05, 65F10.
Key words and phrases : Hitting time, APND, APND ordering, Convolution, Mixture, Compound distribution

1. Introduction

Lehmann(1966) first introduced the concept of positive(negative) quadrant dependence (PQD(NQD)) together with some other dependence concepts. Since then, a great many papers have been written on the subject and its extensions and numerous multivariate inequalities have been obtained. For reference of available results(see Karlin and Rinott(1980), Shaked(1982b), Ebrahimi and Ghosh(1981), Sampson(1983), Barlow and Proschan(1973), Tong(1980), Joag-Deo et al(1983), Ebrahimi(1982), Bozorgnia et al(1996) and Amini et al (2004)). Concepts of these dependence have subsequently been extended to stochastic processes in different directions by many authors( see Ascher et al (1984), Baek et al(2002), Cox et al(1980), Deheuvels(1983), Ebrahimi(1994), Ebrahimi et al (1988), Pollard(1984)). Certain kinds of dependence concepts, when they are

†This paper was supported by Wonkwang University Research Grant in 2017.
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imposed on processes, are reflected as analogous properties of corresponding hitting times. These results are of value as they help us to understand in what ways the hitting times for dependence structures of hitting times can be inherited from the corresponding processes. Furthermore, these result sometimes can tell us how to control the reliability of a system by controlling its characteristics.

But, since these results also are qualitative form of dependence, it would seem difficult or impossible to compare different pairs of random variables as to their "degree of asymptotic pairwise negatively quadrant dependent (APND)-ness". Fortunately, in this paper we develop a partial ordering which permits us to compare pairs of APND bivariate stochastic vectors of interest as to their degree of APND-ness. Furthermore, this is the first study in which a partial ordering is developed for the degree of qualitative dependence. The definitions and some basic properties of ordering are introduced in Section 2. Some preservation results of APND ordering are derived in Section 3. It is shown that the APND ordering is preserved under combination, mixture, transformations of stochastic vectors by increasing(decreasing) functions, and limit in distributions. Finally, in section 4, we show some examples and applications.

2. Preliminaries

In this section, we present concepts, definitions, notations and basic facts used throughout the paper. Suppose that we are given a n-dimensional \((n \geq 2)\) stochastic vector process \(\{(X_i(t) = X_n(t))|t \in \Lambda\}\) where the index set \(\Lambda\) will always be a subset of \(R^+_n = [0, \infty)\). The state space of \(X(t)\) is the cartesian product \(E = E_1 \times \cdots \times E_n\), which will be a subset of \(n\)-dimensional Euclidean space \(R^n\). For any states \(a_i \in E_i, i = 1, 2, \ldots, n\), define the random times
\[
T_i(a_i) = \inf\{t|X_i(t) \leq a_i, 0 \leq t \leq \infty\}.
\]
In other words, \(T_i(a_i)\) is the hitting time that the \(i\)th component process \(X_i(t)\) reaches or goes below \(a_i\). The stochastic process \(\{T_i(a), a \in E_i\}\) will be referred to as the hitting time process of \(X_i(t)\).

Let \(\beta = \beta(F, G)\) denote the class of bivariate distribution functions(df’s) \(H\) on \(R^2\) having specified marginal df’s \(F\) and \(G\), and \(F\) and \(G\) being nondegerate. Use the notation \(H(t_i, t_j) = P(T_i(a_i) > t_i, T_j(a_j) > t_j)\) and \(P(T_i(a_i) \leq t_i, T_j(a_j) \leq t_j)\).

**Definition 2.1.** The stochastic vector process \(\{(X_i(t), X_j(t))|t \in \Lambda\}\) is asymptotic negatively dependent type 1 if for all \(t_i, t_j \in \Lambda\), where \(q(n) = o(n^{-\omega}), \omega > \)
The stochastic vector process is given by $Y(t) = (X(t), Z(t))$, where $X(t)$ and $Z(t)$ are random variables.

Firstly, to show that $X(t)$ is stochastically increasing, we need to prove that

$$P(T_i(a_i) > t_1, T_j(a_j) > t_2) \leq (1 + q(|j - i|))P(T_i(a_i) > t_1)P(T_j(a_j) > t_2).$$

**Definition 2.2.** The stochastic vector process $\{(X_i(t), X_j(t))|t \in \Lambda\}$ is asymptotic negatively dependent type 2 if for all $t_i, t_j \in \Lambda$, where $q(n) = o(n^{-\omega}), \omega > 0$,

$$P(T_i(a_i) \leq t_1, T_j(a_j) \geq t_2) \leq (1 + q(|i - j|))P(T_i(a_i) \leq t_1)P(T_j(a_j) \geq t_2).$$

Similarly, the stochastic vector process $\{(X_i(t), X_j(t))|t \in \Lambda\}$ is asymptotic positively dependent type 1 and asymptotic positively dependent type 2 if Definition 2.1 and Definition 2.2 hold with the inequalities sign reserved. The stochastic processes $X_1(t), X_2(t), \cdots$ are said to be asymptotic pairwise negatively dependent (APND) if stochastic vector process $\{(X_i(t), X_j(t))|t \in \Lambda\}$ is asymptotic negatively dependent type 1 and type 2 for every $i \neq j, i, j \geq 1$. If for $i, j, q|i - j| = 0$, then the random variables (r.v.’s) are negatively dependent (Lehmann (1966)). A special subclass of APND r.v.’s is the pairwise negatively dependent r.v.’s studied by Nili Sani et al. (2005).

Let $\beta^+$ denote the subclass of $\beta$ where $H$ is APND. Suppose $H_1$ and $H_2$ both belong to $\beta^+$.

**Definition 2.3.** The bivariate distribution $H_2$ is said to be more asymptotic pairwise negatively dependent than $H_1$ if

$$H_2(t_1, t_2) \leq H_2(t_i, t_j) \text{ for all } t_i, t_j \in \Lambda, i \neq j. \quad (1)$$

We write $H_2 >_{APND} H_1$.

**Remark 2.1.** Note that an equivalent form of (1) is $H_2(t_i, t_j) \leq H_2(t_1, t_j)$ for all $t_i, t_j \in \Lambda, i \neq j$.

**Definition 2.4.** A stochastic vector process $Y(t)$ is stochastically increasing (decreasing) $(SI(SD))$ in the stochastic vector process $X(t)$ if $E(f(S(a)) | T(a) = T(t), t \in \Lambda)$ is increasing (decreasing) in $T$ for all real valued of increasing (decreasing) function $f$.

The next Lemma shows that ordering of stochastic vector process is preserved under convolution.

**Lemma 2.5.** Suppose that $X(t) = (X_i(t), X_j(t))$ and $Y(t) = (Y_i(t), Y_j(t))$ are stochastic vector process with increasing sample paths and have distributions $H_1$ and $H_2$ respectively, where $H_1$ and $H_2$ belongs to $\beta^+$ such that $H_1 >_{APND} H_2$ and coefficients $q_1, q_j$ for $i \neq j, i, j \geq 1$, and $Z = (Z_i, Z_j)$ with an arbitrary APND distribution $H$ and independent and increasing sample paths of both $X(t)$ and $X(t)$. Then $X(t) + Z >_{APND} Y(t) + Z$.

**Proof.** Firstly, to show that $X(t) + Z$ are APND, the proof will be given for $i = 1, 2$.

Let $W_i(a_i) = \inf \{s|X_i(s) + Z \leq a_i\}$, $i = 1, 2$. Then,

$$P(W_1(a_1) > t_1, W_2(a_2) > t_2)$$
Suppose are First, using Lemma 2.1, we can know that for coefficients are
\( t \) shocks received by time APND The following theorem is very important in recognizing
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Secondly, we have to show that \( X(t) + Z > APND Y(t) + Z \).
Let \( W_i(a_i) = \inf \{ s \mid X_i(s) + Z_i \leq a_i \} \) and \( V_i(a_i) = \inf \{ s \mid Y_i(s) + Z_i \leq a_i \} \), \( i = 1, 2 \). Then,
\[
P(W_1(a_1) > t_1, W_2(a_2) > t_2) = P(\inf \{ s \mid X_1(s) + Z_1 \leq a_1 \} > t_1, \ (\inf \{ s \mid X_2(s) + Z_2 \leq a_2 \}) > t_2] = \int \int P(T_1(a_1 - z_1) > t_1, T_2(a_2 - z_2) > t_2) dH_{Z_1, Z_2}(z_1, z_2) \leq (1 + q_1) \int \int P(T_1(a_1 - z_1) > t_1)P(T_2(a_2 - z_2) > t_2) dH_{Z_1}(z_1) dH_{Z_2}(z_2) = (1 + q_1) P(W_1(a_1) > t_1)P(W_2(a_2) > t_2)
\]
The inequality follows from APND and independent assumptions.
So, \( X(t) + Z \) are APND, similarly we can show that \( Y(t) + Z \) are APND.

3. Closure properties of APND ordering

In this section we show preservation of the APND ordering under combination,
mixture, transformation of stochastic vector processes by increasing (decreas-
ing) function, limits in distributions.

The following theorem is very important in recognizing APND ordering in com-
pound distribution which arise naturally in stochastic vector process.

**Theorem 3.1.** Suppose \( (X_i, Y_i) \) and \( (U_i, V_i) \) are such that \( X_i, Y_i > APND \)
\( U_i, V_i \) for \( i = 1, 2, \ldots \), and let \( N(t) \) be the Poisson process with number of
shocks received by time \( t \). If \( \{(X_i, Y_i)\}_{i=1}^\infty \) and \( \{(U_i, V_i)\}_{i=1}^\infty \)
are independent bivariate processes and independent of \( N(t) \) respectively, then
\( \sum_{i=1}^{N(t)} X_i, \sum_{i=1}^{N(t)} Y_i > APND (\sum_{i=1}^{N(t)} U_i, \sum_{i=1}^{N(t)} V_i) \).

**Proof.** First, using Lemma 2.1, we can know that for coefficients \( q_i, q_j, \ i \neq \)
\( j, \ i, j \geq 1, (\sum_{i=1}^{N(t)} X_i, \sum_{i=1}^{N(t)} Y_i) \) and \( (\sum_{i=1}^{N(t)} U_i, \sum_{i=1}^{N(t)} V_i) \) are APND respectively. Next, the proof will be given for \( i = 1, 2, 3, \ldots \).

\[
P(T_1(a_1) > t_1, T_2(a_2) > t_2) = P(\sum_{i=1}^{N(t)} Z_i \leq a_1, t_1 \leq t < \infty, \sum_{i=1}^{N(t)} Y_i \leq a_2, t_2 \leq t < \infty)
\]
Let \( (a) \) stochastic vector process with increasing sample paths

Suppose that stochastic vector processes with increasing sample paths.

From theorem 3.2, \( (X_i(t), Y_j(t)) \) are ordered according to \( \beta^+ \).

Theorem 3.2. Let \( (a) \) stochastic vector process with increasing sample paths

\( (X_i(t), Y_j(t)) \) be a conditionally APND and coefficients \( q_1, q_2 \) for \( i \neq j \), \( i, j \geq 1 \), and \( (b) \) \( X_i(t) \) be \( SI(SD) \) in \( \lambda \) and \( X_j(t) \) be \( SD(SI) \) in \( \lambda \). Then \( (X_i(t), Y_j(t)) \) is APND.

Proof. Let \( f_1 \) and \( f_2 \) concordant functions. Then for fixed \( t_i, t_j \geq 0 \), \( a_i \in E_i \), \( a_j \in E_j \), \( i \neq j \), \( i, j \geq 1 \),

\[
P(T_i(a_i) > t_i, T_j(a_j) > t_j) = E(I_{f(T_i(a_i))} I_{g(T_j(a_j))}) \\ 
\leq (1 + q_1) E(\beta^+(X_i)\lambda)(E(\beta^+(X_j)\lambda)), \text{ by \( (a) \), \( (b) \)} \\
= (1 + q_1) E(I_{f(T_i(a_i))} I_{g(T_j(a_j))}) \\
= (1 + q_1) P(T_i(a_i) > t_i) P(T_j(a_j) > t_j).
\]

Thus, we obtain that stochastic vector process \( (X_i(t), Y_j(t)) \) are APND.

The next theorem deals with the preservation of the APND ordering under mixture. For \( i \neq j \), \( i, j \geq 1 \), we may define the following class that \( \beta^+_\lambda = \{H_\lambda | H(t_i, \infty) = F(t_i|\lambda), H(\infty, t_j) = G(t_j|\lambda), H_\lambda \text{ is APND}, X_i(t) \text{ is SD(SI)} \}

Consider \( (\beta^+_\lambda, >_{APND}) \). The following theorem shows that if two elements of \( \beta^+_\lambda \) are ordered according to \( >_{APND} \), then after mixing \( \lambda \), the resulting element in \( \beta^+ \) preserve the same order.

Theorem 3.3. Suppose that stochastic vector processes with increasing sample paths \( (X_i(t), Y_j(t)) | \lambda \) and \( (Y_i(t), Y_j(t)) | \lambda \) belong to \( \beta^+_\lambda \) and coefficients \( q_i, q_j \) for \( i \neq j \), \( i, j \geq 1 \), and let \( (X_i(t), Y_j(t)) | \lambda >_{APND} (Y_i(t), Y_j(t)) | \lambda \) for all \( \lambda \). Then \((X_i(t), Y_j(t)), (Y_i(t), Y_j(t)) | \lambda >_{APND} \) \( (X_i(t), Y_j(t)) | \lambda >_{APND} \) \( (Y_i(t), Y_j(t)) \).

Proof. From theorem 3.2, \( (X_i(t), Y_j(t)) \) and \( (Y_i(t), Y_j(t)) \) are APND respectively and for fixed \( t_i, t_j \geq 0 \), \( a_i \in E_i \), \( a_j \in E_j \), \( i \neq j \), \( i, j \geq 1 \), define the following hitting times \( T_i(a_i) = \inf\{t : X_i(t) \leq a_i\} \) and \( S_i(a_i) = \inf\{t : Y_i(t) \leq a_i\} \).

Then,

\[
P(T_i(a_i) > t_i, T_j(a_j) > t_j) = E_\lambda(E(I_{f(T_i(a_i))} I_{g(T_j(a_j))} | \lambda))
\]
Let $\sup_{i} E(I_{f(S_{i}(a_{i}))}I_{g(S_{i}(a_{i}))}\lambda)) = E(I_{f(S_{i}(a_{i}))}I_{g(S_{i}(a_{i}))}) = P(S_{i}(a_{i}) > t_{i}, S_{j}(a_{j}) > t_{j})$.

Next, we show that APND ordering is preserved under transformation of increasing(decreasing) functions.

**Theorem 3.4.** Let $\{ (X_{i}(t), Y_{i}(t))^{H_{j}}, j = 1, 2, \ldots, n \}$ be $n$ independent pairs from a bivariate distribution $H_{j}$, $j = 1, 2$ and increasing sample paths with coefficients $q_{1}, q_{2}, \ldots, q_{n}$. Suppose $H_{1}$ and $H_{2}$ such that $H_{1} >_{APND} H_{2}$. If $f_{1}$ and $f_{2}$ are concordant functions, then $P_{H_{1}}(f_{1}(X_{i}(t)), f_{2}(Y_{i}(t)), i = 1, 2, \ldots, n) >_{APND} P_{H_{2}}(f_{1}(X_{i}(t)), f_{2}(Y_{i}(t)), i = 1, 2, \ldots, n)$.

**Proof.** Let $f_{1}$ and $f_{2}$ concordant functions. To show that $(f_{1}(X_{i}(t)), f_{2}(Y_{i}(t)), i = 1, 2, \ldots, n)$ are APND for general $n$, the proof will be given for $n = 2$.

We introduce the random variables $V_{1} = X_{2}(t_{1}), V_{2} = Y_{2}(t_{2}), U_{1} = \sup_{0 < s \leq t_{1}} f_{1}(X_{1}(s), X_{2}(s))$ and $U_{2} = \sup_{0 \leq s \leq t_{2}} f_{2}(Y_{1}(s), Y_{2}(s))$ for fixed $t_{1}, t_{2} \geq 0$ and simplicity, $t_{1}$, $t_{2}$ have been suppressed in $V_{1}$ and $U_{n}$, for $i = 1, 2$. Define the following any hitting times of $Z_{1}(s) = f_{1}(X_{1}(s), X_{2}(s)), Z_{2}(s) = f_{2}(Y_{1}(s), Y_{2}(s))$ by $W_{1}(a_{1}) = \inf \{ s : Z_{1}(s) \leq a_{1} \}$ and $W_{2}(a_{2}) = \inf \{ s : Z_{2}(s) \leq a_{2} \}$. Note the facts that $U_{1} = \sup_{0 \leq s \leq t} f_{1}(X_{1}(s), V_{1})$ and $U_{2} = \sup_{0 \leq s \leq t} f_{2}(Y_{1}(s), V_{2})$ and that $V_{1}$ and $V_{2}$ are APND by assumptions. Thus,

$$P(W_{1}(a_{1}) > t_{1}, W_{2}(a_{2}) > t_{2}) = \mathbb{E}(P(U_{1} < a_{1}, U_{2} < a_{2}|V_{1}, V_{2}) \leq (1 + q_{1})P(\mathbb{E}(P(U_{1} < a_{1}|V_{1})P(U_{2} < a_{2}|V_{2})) \leq (1 + q_{1})P(U_{1} < a_{1}|V_{1})P(U_{2} < a_{2}|V_{2}) = (1 + q_{1})P(W_{1}(a_{1}) > t_{1})P(W_{2}(a_{2}) > t_{2})$$

So, $(f_{1}(X_{i}(t)), f_{2}(Y_{i}(t)), i = 1, 2, \ldots, n)$ are APND and note that $(X_{i}(t), Y_{i}(t)) \sim H, (X_{i}(t), Y_{i}(t)) \sim H', H >_{APND} H'$ and two ordered elements belong to $\beta^{*}$, then the corresponding elements in $\beta_{1}, f_{2}$ maintain the same order, and so $P_{H_{1}}(f_{1}(X_{i}(t)), f_{2}(Y_{i}(t))) >_{APND} P_{H_{2}}(f_{1}(X_{i}(t)), f_{2}(Y_{i}(t)))$. Thus we can obtain that $P_{H_{1}}(f_{1}(X_{i}(t)), f_{2}(Y_{i}(t)), i = 1, 2, \ldots, n) >_{APND} P_{H_{2}}(f_{1}(X_{i}(t)), f_{2}(Y_{i}(t)), i = 1, 2, \ldots, n)$.

In the following theorem we show that APND ordering is preserved under limits in distributions.

**Theorem 3.5.** Suppose that $H_{n} >_{APND} H'_{n}$ such that $H_{n} \rightarrow H, H'_{n} \rightarrow H'$ weakly as $n \rightarrow \infty$ for every $n$. Then $H >_{APND} H'$

**Proof.** For fixed $t_{i}, t_{j} \geq 0, a_{i} \in E_{i}, a_{j} \in E_{j}, i \neq j, i, j \geq 1$, define the following hitting times $T_{i}(a_{i}) = \inf \{ t : X_{i}(t) \leq a_{i} \}$ and $S_{i}(a_{i}) = \inf \{ t : Y_{i}(t) \leq a_{i} \}$, then $P_{H}(T_{i}(a_{i}) > t_{i}, T_{j}(a_{j}) > t_{j})$
Suppose that \( (X_{ni}(t), X_{nj}(t))^H_n \) and \( (Y_{ni}(t), Y_{nj}(t))^H_n \) are sequence of bivariate vector process, for \( n \geq 1, i \neq j, i,j \geq 1 \), and let \( H_n > \text{APND} H'_n \). Let \( H_n \) and \( H'_n \) converge weakly to another vector process \( (X_i(t), X_j(t))^H \) and \( (Y_i(t), Y_j(t))^H \) respectively (with respect to any Skorohod(1956) topology as \( n \to \infty \)), and if \( (X_{ni}(t), X_{nj}(t))^H_n, Y_{ni}(t), Y_{nj}(t))^H_n \), \( (X_i(t), X_j(t))^H \) and \( (Y_i(t), Y_j(t))^H \) have sample paths that are right-continuous on \( R_+ \) with finite left limits at all \( t \), then, by Theorem 3.5, we have the limiting distribution \( H > \text{APND} H' \).

Example 4.2. Suppose that stochastic vector process \( (X_1(t), X_2(t)) \) be APND with increasing sample paths and coefficients \( q_1 \), and let \( Z(t) \) be independent and have increasing sample paths of \( (X_1(t), X_2(t)) \). If we define \( X(t) = X_1(t) + \lambda_1 Z(t) \) and \( Y(t) = X_2(t) + \lambda_2 Z(t) \), where \( \lambda_1 \geq 0, \lambda_2 \leq 0 \), then \( X_1(t) + \lambda_1 Z(t) \) is SI in \( Z(t) \) and \( X_2(t) + \lambda_2 Z(t) \) is SD in \( Z(t) \). Therefore, since \( (X(t), Y(t)) \) given \( Z(t) \) is APND, by Theorem 3.2, we can obtain that stochastic vector process \( (X(t), Y(t)) \) is APND.

Application 4.3. Consider a system with two components which is subjected to shocks. Let \( N(t) \) be the number of shocks received by time \( t \) and let \( \sum_{i=1}^{N(t)} X_i, \sum_{i=0}^{N(t)} Y_i, \sum_{i=0}^{N_1(t)} U_i \) and \( \sum_{i=0}^{N_2(t)} V_i \) be total damages to components 1, 2, 3 and 4 by time \( t \), respectively. If \( X_i, Y_i, U_i \) and \( V_i \) are damages to components 1, 2, 3 and 4 by shock \( i \), respectively, then we can obtain Theorem 3.1.

Application 4.4. Consider a following bivariate vector process comes from the fact that a Brownian motion has continuous paths. Let \( \{X_n, Y_n\}_{n \geq 1} \) and \( \{(V_n, W_n)_{n \geq 1}\} \) be a bivariate process such that \( (X_n, Y_n) > \text{APND} (V_n, W_n) \) for \( i = 1, 2, \ldots \). Suppose that \( \{(X_i, Y_i)_{i \geq 1}\} \) and \( \{U_i, V_i\}_{i \geq 1} \) are independent bivariate processes respectively. Then, from the result given by Pitt (1982) about multivariate normal distribution, we can know that \( \{(X_n, Y_n)_{n \geq 1}\} \) and \( \{(V_n, W_n)_{n \geq 1}\} \) are APND respectively and we can obtain that \( \{(X_n, Y_n)_{n \geq 1}\} > \text{APND} \{(V_n, W_n)_{n \geq 1}\} \).

Acknowledgment. We thank the references for careful reading of our manuscript and for helpful comments.
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