INEQUALITIES OF OPERATOR POWERS

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ABSTRACT. Duggal-Jeon-Kubrusly(2] introduced Hilbert space operator $T$ satisfying property $|T|^2 \leq |T^2|$, where $|T| = (T^*T)^{1/2}$. In this paper we extend this property to general version, namely property $B(n)$. In addition, we construct examples which distinguish the classes of operators with property $B(n)$ for each $n \in \mathbb{N}$.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T$ in $\mathcal{L}(\mathcal{H})$ is said to be $p$-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$, $p \in (0, \infty)$. If $p = 1$, then $T$ is hyponormal. The Löwner-Heinz inequality(3]) implies that every $p$-hyponormal operator is a $q$-hyponormal operator for $0 < q \leq p$. In particlar, $T$ is said to be $\infty$-hyponormal if $T$ is $p$-hyponormal for every $p > 0$ (7). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $A(p)$-operator if $(T^* |T|^2 T)^{1/(p+1)} \geq |T|^2$ $(0 < p < \infty)$ where $|T| = (T^*T)^{1/2}$. It is well known that every $p$-hyponormal is $A(p)$-operator(3]).

In [2], Duggal-Jeon-Kubrusly studied operators $T$ on $\mathcal{H}$ satisfying property

$$|T|^2 \leq |T^2|.$$ (1)

In this paper we extend this property to a general version, namely property $B(n)$ whose definition will be introduced in Section 3. The operator satisfying (1) will be equivalent to property $B(2)$. It follows from (1) that an operator $T$ in $\mathcal{L}(\mathcal{H})$ has property $B(2)$ if and only if $T$ is $A(1)$-operator. Because only a few examples for property (1) have been known, it is worthwhile to find such examples.

In this paper, we construct examples which distinguish property $B(n)$ of an operator in $\mathcal{L}(\mathcal{H})$ for each $n \geq 2$. Also, we see the relationships between property $B(n)$ for each $n \geq 2$ and hyponormality of an operator $T$ on $\mathcal{H}$ from some simple examples. In addition, we show mutually disjoint ranges of property $B(n)$ for $n \geq 2$ of operator in the 2-dimensional space.

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2. Property $B(n)$

For $n \geq 2$, an operator $T \in \mathcal{L}(H)$ has the property $B(n)$ if $|T^n| \geq |T|^n$. If $T$ is a $p$-hyponormal operator for $p > 0$, then $T^{*n}T^n \geq (T^*T)^n$ for all positive integer $n < p([8])$. Hence we have the following proposition.

**Proposition 2.1.** If $T$ is $\infty$-hyponormal, then $T$ has property $B(n)$ for all $n \geq 2$.

**Proof.** Since $T$ is $\infty$-hyponormal, $T$ is $(n+1)$-hyponormal for all $n \in \mathbb{N}$. By the above known result, we have that $T^{*n}T^n \geq (T^*T)^n$, which implies that $|T^n|^2 \geq |T|^{2n}$. By Löwner-Heinz inequality([3]), $|T^n| \geq |T|^n$. Thus, $T$ has property $B(n)$. □

**Theorem 2.2.** Let $W_\alpha$ be a weighted shift with weight sequence $\alpha = \{\alpha_k\}_{k=0}^\infty$. Then $W_\alpha$ has property $B(n)$ if and only if

$$|\alpha_{k+1}| \cdot |\alpha_{k+2}| \cdots |\alpha_{k+n-1}| \geq |\alpha_k|^{n-1}$$

for all $k = 0, 1, \cdots$.

**Proof.** If $W_\alpha$ has property $B(n)$, then we have $|W_\alpha^n| \geq |W_\alpha|^n$. Hence by simple computation, we have

$$|W_\alpha^n|^2 = (W_\alpha^*)^nW_\alpha^n = \text{Diag}\{|\alpha_0\alpha_1 \cdots \alpha_{n-1}|^2, |\alpha_1\alpha_2 \cdots \alpha_n|^2, |\alpha_2\alpha_3 \cdots \alpha_{n+1}|^2, \ldots\}$$

and

$$|W_\alpha|^{2n} = (W_\alpha^*W_\alpha)^n = \text{Diag}\{|\alpha_0|^{2n}, |\alpha_1|^{2n}, |\alpha_2|^{2n}, \ldots\}.$$

Thus $W_\alpha$ has property $B(n)$, which is equivalent to $|\alpha_k\alpha_{k+1} \cdots \alpha_{k+n-1}| \geq |\alpha_k|^n$ $(k \geq 0)$, i.e.,

$$|\alpha_{k+1}| \cdot |\alpha_{k+2}| \cdots |\alpha_{k+n-1}| \geq |\alpha_k|^{n-1} \quad (k \geq 0).$$

This completes the proof. □

**Corollary 2.3.** Let $W_\alpha$ be a weighted shift with weight sequence $\alpha$. Then we have the following statements.

(i) $W_\alpha$ has property $B(2)$ if and only if $W_\alpha$ is hyponormal.

(ii) If $W_\alpha$ is hyponormal, then $W_\alpha$ has property $B(n)$ for all $n \geq 2$.

**Proof.** (i) By the Theorem 2.2 and the fact in [1], we may have that $W_\alpha$ has property $B(2) \iff \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \iff W_\alpha$ is hyponormal.

(ii) Using the above fact (i), we can easily obtain that

$$|\alpha_{k+1}| \cdot |\alpha_{k+2}| \cdots |\alpha_{k+n-1}| \geq |\alpha_k|^{n-1} \quad (k \geq 0).$$

By Theorem 2.2, $W_\alpha$ has property $B(n)$ for each $n \geq 2$. □
Theorem 2.4. Let $W_\alpha$ be a weighted shift with weight sequence
\[ \alpha : x \equiv \alpha_0, \ y \equiv \alpha_1, \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \cdots \leq \alpha_n \leq \cdots \]
for $x$, $y \geq 0$ and $\alpha_2 > 0$. For all $n \geq 2$, if we set
\[ B_n := \{(x, y) : W_\alpha \text{ has property } B(n)\}, \]
then we have
(i) $B_n = \{(x, y) : 0 \leq x \leq (\alpha_2 \alpha_3 \cdots \alpha_{n-1} y)^{\frac{1}{n-1}}, 0 \leq y \leq (\alpha_2 \alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}}\}$.
(ii) $B_m \subseteq B_n$ for $2 \leq m < n$.
(iii) $\bigcap_{n=2}^{\infty} B_n = \{(x, y) : 0 \leq x \leq y \leq \alpha_2\}$.

Proof. (i) By Theorem 2.2 and the condition of $0 < \alpha_k \leq \alpha_{k+1}$ for all $k \geq 2$, we have that $W_\alpha$ has property $B(n)$, which is equivalent to $\alpha_1 \alpha_2 \cdots \alpha_{n-1} \geq \alpha_0^{n-1}$ and $\alpha_2 \alpha_3 \cdots \alpha_n \geq \alpha_1^{n-1}$, and that is
\[ \alpha_2 \cdots \alpha_{n-1} y \geq x^{n-1} \text{ and } 0 \leq y \leq (\alpha_2 \alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}} \]
for each $n \geq 2$.
(ii) Put $f(n, x) := \frac{x^{n-1}}{\alpha_2 \alpha_3 \cdots \alpha_{n-1}}$ for all $n \geq 2$ and $x > 0$. Then
\[ \frac{\partial f(n, x)}{\partial x} = (n-1)x^{n-2} \alpha_2 \alpha_3 \cdots \alpha_{n-1} > 0 \text{ and } \frac{\partial^2 f(n, x)}{\partial x^2} = \frac{(n-1)(n-2)x^{n-3}}{\alpha_2 \alpha_3 \cdots \alpha_{n-1}} \geq 0 \]
for all $n \geq 2$ and $x > 0$. So the function $f(n, x)$ is strictly increasing function about $x > 0$ and for all $n \geq 2$.

Suppose $2 \leq m < n$. For $0 < x < (\alpha_m \alpha_{m+1} \cdots \alpha_{n-1})^{\frac{1}{n-m}}$, we have
\[ f(n, x) - f(m, x) = \frac{x^{n-1}}{\alpha_2 \cdots \alpha_{m-1}} \left( \frac{x^{n-m}}{\alpha_m \cdots \alpha_{n-1}} - 1 \right) < 0. \]
\[ \text{i.e. } f(m, x) > f(n, x) \text{ for } 2 \leq m < n \text{ and } x \in (0, (\alpha_m \alpha_{m+1} \cdots \alpha_{n-1})^{\frac{1}{n-m}}). \]

Let we set $a_n := (\alpha_2 \alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}}$ for each $n \geq 2$. Then, using the assumption $0 < \alpha_k \leq \alpha_{k+1}$ for all $k \geq 2$, we obtain
\[ a_{n+1} - a_n = (\alpha_2 \alpha_3 \cdots \alpha_{n+1})^{\frac{1}{n}} - (\alpha_2 \alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}} = (\alpha_2 \cdots \alpha_n)^{\frac{1}{n}} \left[ (\alpha_n^{\frac{1}{n-1}} - (\alpha_2 \cdots \alpha_n)^{\frac{1}{n-1}}) \right] \geq (\alpha_2 \cdots \alpha_n)^{\frac{1}{n}} \left[ (\alpha_n^{\frac{1}{n}} - (\alpha_n^{\frac{1}{n}})^{\frac{1}{n-1}}) \right] \geq 0. \]

Therefore the sequence $\{(\alpha_2 \alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}} : n = 2, 3, \ldots \}$ is an increasing sequence. Since $B_n = \{(x, y) : 0 \leq f(n, x) \leq y, 0 \leq y \leq a_n\}$ for each $n \geq 2$, we completes the proof of (ii).
(iii) From the facts (i) and (ii), the assertion (iii) is obvious. \[ \square \]
Remark 2.5. For the weighted shift $W_\alpha$ in Theorem 2.4, we note the following facts:

$W_\alpha$ is $\infty$-hyponormal $\iff$ $W_\alpha$ is hyponormal
$\iff 0 \leq x \leq y$ and $0 \leq y \leq \alpha_2$
$\iff W_\alpha$ has the property $B(n)$ for all $n \geq 2$.

In general, but the converse of Proposition 2.1 is not true (see Example 3.3).

3. Examples

The following example explains that for a weighted shift $W_\alpha$ with weight sequence $\alpha$, there is no relation with the property $B(n)$ and $B(m)$ for $m, n > 2$ with $m \neq n$.

Example 3.1. Consider a positive bounded sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$,

$\alpha_0 = \frac{2}{3}, \alpha_1 = \frac{40}{81}, \alpha_2 = \frac{9}{16}, \alpha_3 = \frac{16000}{59049}, \alpha_4 = \frac{4782969}{1600000}, \alpha_{n+1} = \alpha_n + \frac{1}{n^2} \ (n \geq 4)$.

Let $W_\alpha$ be the weighted shift with the above weight sequence $\alpha$. Then $W_\alpha$ has property $B(3)$ but not property $B(4)$. In fact, from simple calculations, we have

$\alpha_2 k = \alpha_k + 1 \alpha_{k+1} \alpha_{k+2} \ (k = 0, 1, 2)$

and $\alpha_2 k \leq \alpha_k + 1 \alpha_{k+1} \alpha_{k+2}$ for all $k \geq 3$. So $W_\alpha$ satisfies property $B(3)$. But $\alpha_3 = \frac{8}{27} > \alpha_1 \alpha_2 \alpha_3 = \frac{64000}{531441}$. Therefore $W_\alpha$ does not satisfy property $B(4)$.

For the distinction of property $B(n)$, we introduce the following example which classify them clearly for each $n \geq 2$.

Example 3.2. Let $W_\alpha$ be the Bergman shift with weight sequence

$\alpha : \sqrt{x}, \sqrt{y}, \sqrt{x \frac{3}{4}}, \sqrt{y \frac{4}{5}}, \sqrt{x \frac{5}{6}}, \ldots, \sqrt{n+1 \frac{k+1}{k+2}}, \ldots \ (k \geq 2)$.

Then by Theorem 2.2 we may obtain the following assertion:

$W_\alpha$ has property $B(n) \iff 0 \leq x \leq \left(\frac{3y^{n+1}}{n+1}\right)^{-\frac{1}{n-1}}$ and $0 \leq y \leq \left(\frac{3}{n+2}\right)^{-\frac{1}{n-1}}$

for each $n \geq 2$. Hence

$B_n = \left\{ (x, y) \mid 0 \leq x \leq \left(\frac{3y^{n+1}}{n+1}\right)^{-\frac{1}{n-1}}, 0 \leq y \leq \left(\frac{3}{n+2}\right)^{-\frac{1}{n-1}} \right\}$.

Now, we claim that $B_m \subset B_n$ for $2 \leq m < n$. First, we write $f(n, x) := \frac{n+1}{n+1} x^n - 1$ for all $n \geq 2$ and $x > 0$. By the derivative of $f(n, x)$ about $x$, we can see that the function $f(n, x)$ is strictly increasing function about $x$ for all $n \geq 2$. Suppose $2 \leq m < n$. For $0 < x < (\frac{m+1}{n+1})^{-\frac{1}{n-m}}$, we have

$f(n, x) - f(m, x) = \frac{1}{3} x^{m-1} \left(x^{n-m} - (m+1)\right) < 0$,

which is that, for $2 \leq m < n, f(m, x) > f(n, x)$ on $(0, (\frac{m+1}{n+1})^{-\frac{1}{n-m}})$.
Also, by simple calculations, we have that the sequence \( \left\{ \left( \frac{3}{n+2} \right)^{n-1} : n = 2, 3, \ldots \right\} \) is an increasing sequence and converges to 1. Therefore we have \( \mathcal{B}_m \subsetneq \mathcal{B}_n \). In fact, we can easily show the disjoint ranges of properties \( B(n) \) for each \( n \geq 2 \) of \( W_\alpha \) by usual way.

For each integer \( n \geq 2 \), we consider the following block matrix of operators in \([6]\) and \([5]\).

**Example 3.3.** Let \( C = (c_{ij}) \) be an \( m \times m \) matrix with \( c_{ij} = 1/m \) \( (1 \leq i, j \leq m) \) and let \( D \equiv D(x_1, x_2, \ldots, x_m) := \text{Diag}\{x_1, x_2, \ldots, x_m\} \) with \( x_i \geq 0 \), \( i = 1, \ldots, m \). We define an operator \( T(x_1, x_2, \ldots, x_m) \) on \( \mathcal{H} \equiv \mathbb{C}^m \otimes \ell_2(\mathbb{Z}) \) by

\[
T := T(x_1, x_2, \ldots, x_m) = \begin{pmatrix}
\ddots & & \\
& O & C \\
& O & \left[ \begin{array}{cc} O & O \\
O & D \\
\end{array} \right] \\
& D & O \\
& \ddots & \ddots \\
\end{pmatrix},
\]

where \([ \cdot ]\) denotes the center of the two sided infinite matrix. We note that \( C^p = C \) for every \( p > 0 \). By simple calculations, we have that

\[
(CD^kC)^{\frac{1}{2}} = \sqrt{\frac{x_1^k + x_2^k + \ldots + x_m^k}{m}} \cdot C
\]

and

\[
(CD^kC)^{\frac{1}{2}} - C \geq 0 \iff x_1^k + x_2^k + \ldots + x_m^k \geq m,
\]

for \( x_i \geq 0 \) \( (i = 1, 2, \ldots, m) \) and \( k \geq 1 \). Hence \( T \) has property \( B(n) \iff (T^{n*}T^m)^{\frac{1}{2}} \geq (T^*T)^{\frac{1}{2}} \), which is equivalent to

\[
(CD^2C)^{\frac{1}{2}} \geq C, \ (CD^4C)^{\frac{1}{2}} \geq C, \ldots, (CD^{2(n-1)}C)^{\frac{1}{2}} \geq C.
\]

For an integer \( n \geq 2 \), we denote

\[
\mathcal{E}_n = \{ (x_1, x_2, \ldots, x_m) : T \text{ has property } B(n) \text{ for } x_i \geq 0 \}.
\]

If \( x_i \) satisfy \( x_i \geq m \frac{1}{2^{(n-1)}} \) for some \( i \), then \( x_1^{2(l-1)} + x_2^{2(l-1)} + \ldots + x_m^{2(l-1)} \geq m \). So we have that \( x_1^{2(n-1)} + x_2^{2(n-1)} + \ldots + x_m^{2(n-1)} \geq m \) for \( 2 \leq l < n \). Suppose \( 0 < x_i < m \frac{1}{2^{(n-1)}} \) for all \( i = 1, 2, \ldots, m-1 \) for all \( n \geq 2 \). Then we obtain that the function

\[
\phi_m(n, x_1, x_2, \ldots, x_{m-1}) := (m - x_1^{2(n-1)} - x_2^{2(n-1)} - \ldots - x_m^{2(n-1)}) \frac{1}{2^{(n-1)}}
\]
is strictly decreasing with respect to all $n \geq 2$ on $(0, m^{-\frac{1}{m-1}}) \times \cdots \times (0, m^{-\frac{1}{m-1}})$ (see [6] and [5] for the detail methods). Therefore we have

$$E_n = \{ (x_1, \ldots, x_m) : (CD)^{2(j-1)}C \frac{1}{2} \geq C, 2 \leq j \leq n, x_i \geq 0, 1 \leq i \leq m \}$$

$$= \bigcap_{2 \leq j \leq n} \{ (x_1, \ldots, x_m) : x_1^{2(j-1)} + x_2^{2(j-1)} + \cdots + x_m^{2(j-1)} \geq m, x_i \geq 0, 1 \leq i \leq m \}$$

$$= E_2.$$

Moreover, we have that

$$E_2 = \{ (x_1, \ldots, x_m) : x_1^2 + x_2^2 + \cdots + x_m^2 \geq m, x_i \geq 0, 1 \leq i \leq m \}$$

$$= \{ (x_1, \ldots, x_m) : T \text{ is } A(1)-\text{operator} \}$$

and $T$ is $\infty$-hyponormal (see [5]). Therefore we have this implication: $T$ is $\infty$-hyponormal $\Rightarrow$ $T$ is hyponormal $\Rightarrow$ $T$ has property $B(2)$, and the converse is not true.

**References**