MINTY’S LEMMA FOR STRONG IMPLICIT VECTOR VARIATIONAL INEQUALITY SYSTEMS

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ABSTRACT. In this paper, we consider a new Minty’s Lemma for strong implicit vector variational inequality systems and obtain some existence results for systems of strong implicit vector variational inequalities which generalize some results in [1].

1. INTRODUCTION AND PRELIMINARIES

In [2], Huang and Fang introduced system of order complementarity problems and established some existence theorem by using Ky Fan Lemma and then Kassay, Kolumban and Pales [3] introduced and studied Minty and Stampaccia variational inequality system by using Kakutani-Fan-Glicksberg fixed point theorem. Recently, by those works and by using Kakutani-Fan-Glicksberg fixed point theorem, Fang and Huang [1] provided some existence results for systems of strong implicit vector variational inequalities, for a constant cone, in reflexive Banach spaces.

In this paper, we consider a new Minty’s Lemma for strong implicit vector variational inequality systems and obtain some existence results for a system of strong implicit vector variational inequalities which generalize some results in [1].

Throughout this paper, unless other specified, $X_i$ and $Y_i$ are Banach spaces, $K_i \subset X_i$ are nonempty, bounded, closed and convex sets and $C_i \subset Y_i$ be pointed, closed and convex cones with $int C_i \neq \emptyset$. Let $T_i : K \rightarrow L(X_i, \tilde{Y}_i)$, where $\tilde{Y}_1 = Y_2 \times Y_3$, $\tilde{Y}_2 = Y_3 \times Y_1$, $\tilde{Y}_3 = Y_1 \times Y_2$ and $K = \prod_{i=1}^{3} K_i$, and $h_i : K_i \times K_i \rightarrow X_i (i = 1, 2, 3)$ be mappings. A nonempty subset $C$ of a Hausdorff topological vector space $X$ is said to be a pointed convex cone if

$C + \lambda C \subseteq C$ and $C \cap (-C) = \{0\}$, for all $\lambda \geq 0$.


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where $\mathbf{0}$ denotes the zero vector. If $C_1 \subset Y_1$ and $C_2 \subset Y_2$ are pointed convex cones, then $C_1 \times C_2 \subset Y_1 \times Y_2$ is also a pointed convex cone.

Now we consider the following system of strong implicit vector variational inequalities of Stampacchia type (SSIVVI-S) and Minty type (SSIVVI-M);

(SSIVVI-S) Find $x = (x_1, x_2, x_3) \in K$ such that

$$\langle T_i(x), h_i(x_i, y_i) \rangle \geq \bar{C}_i 0, \text{ for } y_i \in K_i \ (i = 1, 2, 3)$$

and

(SSIVVI-M) Find $x = (x_1, x_2, x_3) \in K$ such that

$$\langle T_i(x), h_i(y_i, x_i) \rangle \leq \bar{C}_i 0, \text{ for } y_i \in K_i \ (i = 1, 2, 3),$$

where $\bar{C}_1 = C_2 \times C_3, \bar{C}_2 = C_3 \times C_1, \bar{C}_3 = C_1 \times C_2$, $\bar{x}_1 = (y_1, x_2, x_3), \bar{x}_2 = (x_1, y_2, x_3), \bar{x}_3 = (x_1, x_2, y_3)$.

**Definition 1.1.** Let $T_i : K \to L(X_i, Y_i)$ and $h_i : K_i \times K_i \to X_i$ be mappings. \{T_1, T_2, T_3\} is said to be co-pseudomonotone with respect to \{h_1, h_2, h_3\} if for any \(x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in K,

$$\langle T_i x, h_i(x_i, y_i) \rangle \geq \bar{C}_i 0 \Rightarrow \langle T_i y, h_i(y_i, x_i) \rangle \leq \bar{C}_i 0.$$;

**Example 1.1.** Let $X_i, Y_i = \mathbb{R}, K_i = [0, 10], C_i = \mathbb{R}_+$,

$$T_i(x) = \begin{pmatrix} \frac{2ix_1^2}{x_2 + x_3} \end{pmatrix}$$

$h_i(x_i, y_i) = iy_i - i(x_i + 1)^2$ for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in K$.

Let $x, y \in K$ such that

$$\langle T_i(x), h_i(x_i, y_i) \rangle = \begin{pmatrix} \frac{2ix_1^2}{x_2 + x_3} \end{pmatrix} (iy_i - i(x_i + 1)^2)$$

$$= \begin{pmatrix} \frac{2ix_1^2(iy_i - i(x_i + 1)^2)}{(x_2 + x_3)(iy_i - i(x_i + 1)^2)} \end{pmatrix} \geq \bar{C}_i 0.$$

The inequality above implies

$$iy_i - i(x_i + 1)^2 \geq 0 \Rightarrow y_i \geq (x_i + 1)^2$$

$$\Rightarrow (y_i + 1)^2 \geq y_i + 1 \geq y_i \geq (x_i + 1)^2 \geq x_i$$

It follows that

$$\langle T_i(y), h_i(y_i, x_i) \rangle = \begin{pmatrix} \frac{2iy_1^2}{y_2 + y_3} \end{pmatrix} (ix_i - i(y_i + 1)^2)$$

$$= \begin{pmatrix} \frac{2iy_1^2(ix_i - i(y_i + 1)^2)}{(y_2 + y_3)(ix_i - i(y_i + 1)^2)} \end{pmatrix} \leq \bar{C}_i 0.$$
Hence \( \{T_1, T_2, T_3\} \) is co-pseudomonotone with respect to \( \{h_1, h_2, h_3\} \).

**Definition 1.2.** Let \( T_i : K \rightarrow L(X_i, \hat{Y}_i) \) and \( h_i : K_i \times K_i \rightarrow X_i \) be mappings.

1. \( \{T_1, T_2, T_3\} \) is said to be properly co-quasimonotone of Stampacchia type with respect to \( \{h_1, h_2, h_3\} \) if for all \( m \in \mathbb{N} \), for all vectors \( u_1^i, \ldots, u_m^i \in K_i \), and scalars \( \lambda^1, \ldots, \lambda^m > 0 \) with \( \sum_{j=1}^{m} \lambda^j = 1 \) and \( u_i := \sum_{j=1}^{m} \lambda^j v_i^j \),
\[
\langle T_i \hat{x}_i, h_i(u_i, v_i^j) \rangle \geq \hat{c}_i 0 \quad \text{holds for all } j,
\]
\[
\hat{x}_1 = (u_1, x_2, x_3), \quad \hat{x}_2 = (x_1, u_2, x_3) \quad \text{and} \quad \hat{x}_3 = (x_1, x_2, u_3).
\]

2. \( \{T_1, T_2, T_3\} \) is said to be properly co-quasimonotone of Minty type with respect to \( \{h_1, h_2, h_3\} \) if for all \( m \in \mathbb{N} \), for all vectors \( v_1^i, \ldots, v_m^i \in \hat{K}_i \) and scalars \( \lambda^1, \ldots, \lambda^m > 0 \) with \( \sum_{j=1}^{m} \lambda^j = 1 \) and \( u_i := \sum_{j=1}^{m} \lambda^j v_i^j \),
\[
\langle T_i \overline{x}_i, h_i(v_i^j, u_i) \rangle \leq \hat{c}_i 0 \quad \text{holds for all } i,
\]
\[
\overline{x}_1 = (v_1^i, x_2, x_3), \quad \overline{x}_2 = (x_1, v_2^i, x_3) \quad \text{and} \quad \overline{x}_3 = (x_1, x_2, v_3^i)
\]

**Definition 1.3** ([1]). Let \( X \) and \( Y \) be Banach spaces, and \( K \) be a nonempty, closed and convex subset of \( X \). A mapping \( h : K \rightarrow Y \) is said to be hemicontinuous if, for any fixed \( x, y \in K \), a mapping \( L : [0, 1] \rightarrow Y \) defined by \( L(t) = h((1 - t)x + ty) \) is continuous at \( t = 0^+ \), i.e., \( \lim_{t \to 0^+} L(t) = L(0) \).

The following lemma is obtained from Theorem 3.3 in [4].

**Lemma 1.1.** Let \( X_i \) be reflexive Banach spaces, and let \( T_i : K \rightarrow L(X_i, \hat{Y}_i) \), where \( \hat{Y}_1 = Y_2 \times Y_3, \hat{Y}_2 = Y_3 \times Y_1 \) and \( \hat{Y}_3 = Y_1 \times Y_2 \), and \( h_i : K_i \times K_i \rightarrow X_i \) be mappings satisfying the following conditions \((i = 1, 2, 3)\):

1. \( \langle T_i(x), h_i(x_i, x_i) \rangle \in -\hat{C}_i \quad (i = 1, 2, 3) \);
2. for any given \( x = (x_1, x_2, x_3) \in K \), \( \{T_1, T_2, T_3\} \) are properly co-quasimonotone of Minty type with respect to \( \{h_1, h_2, h_3\} \);
3. \( h_i \) is continuous.

Then the following variational inequality \((VI)\) has a solution:

\( (VI) \) Find \( x_0 = (x_0^1, x_0^2, x_0^3) \in \prod_{i=1}^{3} K_{M_i} \), where \( K_{M_i} = K_i \cap M_i \neq \phi \) for \( M_i \) are finite-dimensional subspaces of \( X_i \) such that
\[
\langle T_i(x^i), g(z_i, x^0_i) \rangle \leq \hat{c}_i 0, \quad z_i \in K_{M_i} \quad \text{for } i = 1, 2, 3
\]
where \( x^1 = (x_1, x_2, x_3), x^2 = (x_1, x_2, x_3) \) and \( x^3 = (x_1, x_2, x_3) \).
Definition 1.4 ([6]). Let \( X, Y \) be Hausdorff topological spaces and \( T : X \to 2^Y \) be a set-valued mapping. \( T \) is said to be upper semicontinuous (shortly, u.s.c.) at \( x_0 \in X \) if for any neighborhood \( N(T(x_0)) \) of \( T(x_0) \), there exists a neighborhood \( N(x_0) \) of \( x_0 \) such that
\[
\forall x \in N(x_0), T(x) \subset N(T(x_0)).
\]
We say that \( T \) is u.s.c. if \( T \) is u.s.c. at every point \( x \in X \).

Lemma 1.2 ([5]). Let \( X \) and \( Y \) be Hausdorff topological spaces, and \( F : X \to 2^Y \) be a multivalued mapping. If \( Y \) is compact and \( F \) is closed, then \( F \) is u.s.c.

Theorem 1.1 ([6, Kakutani-Fan-Glicksberg fixed point theorem]).
Let \( X \) be a nonempty compact convex subset of a locally convex Hausdorff topological vector space \( E \). Assume that \( T : X \to 2^X \) is an u.s.c. mapping with nonempty closed convex values. Then \( T \) has a fixed point on \( X \).

Lemma 1.3 ([1]). Let \( C \) be a pointed, closed and convex cone of a real Banach space \( E \). Then for any, \( a \in -C \) and \( b \notin C \), we have \( t_1a + t_2b \notin C \) for all \( t_1, t_2 > 0 \).

2. Main Results

First, we consider a new Minty’s Lemma for a system of strong implicit vector variational inequalities.

Theorem 2.1. Let \( T_i : K \to L(X_i, \hat{Y}_i) \), and \( h_i : K_i \times K_i \to X_i \) be mappings satisfying the following conditions \( (i = 1, 2, 3) \); for any given \( x = (x_1, x_2, x_3) \in K \)

1. \( \{T_1, T_2, T_3\} \) is co-pseudomonotone with respect to \( \{h_1, h_2, h_3\} \);
2. \( h_i \) is bilinear such that \( h_i(a, b) + h_i(b, a) = 0 \) for \( a, b \in K_i \);
3. for fixed \( v = (v_1, v_2, v_3) \in K \), \( u \mapsto \langle T_i(u), h_i(u, v_i) \rangle \) is hemicontinuous \( (i = 1, 2, 3) \).

Then for a given point \( x \in K \), the following conclusions are equivalent

1. \( \langle T_i(x), h_i(x, y_i) \rangle \geq \bar{\epsilon}_i \) 0, \( y_i \in K_i \);
2. \( \langle T_i(x), h_i(y_i, x_i) \rangle \leq \bar{\epsilon}_i \) 0, \( y_i \in K_i \ ; (i = 1, 2, 3) \).

Proof. (ii) is easily shown from (i) by the condition (1).

Conversely, for any given \( y = (y_1, y_2, y_3) \in K \) and \( t \in (0, 1) \), let \( y^t = x + t(y - x) \).
It follows from (ii) that
\[
\langle T_i(y^t), h_i(y_i^t, x_i) \rangle \leq \bar{\epsilon}_i \ 0.
\]
Now we show that \( \langle T_i(y^t), h_i(y_i^t, y_i) \rangle \geq \bar{\epsilon}_i \ 0 \) for all \( t \in (0, 1) \). Suppose that there
exists some \( s \in (0, 1) \) such that

\[
\langle T_i(y^s), h_i(y_i^s, y_i) \rangle \not\in \mathcal{C}_i 0.
\]

By Lemma 1.3 and the bilinearity of \( h_i \), we have

\[
\langle T_i(y^s), h_i(y_i^s, y_i^s) \rangle = \langle T_i(y^s), h_i(y_i^s, x_i + s(y_i - x_i)) \rangle
\]

\[
= \langle T_i(y^s), h_i((1 + s - s)y_i^s, (1 - s)x_i + sy_i) \rangle
\]

\[
= s\langle T_i(y^s), h_i(y_i^s, y_i) \rangle + (1 - s)\langle T_i(y^s), h_i(y_i^s, x_i) \rangle
\]

\[\not\in \mathcal{C}_i,
\]

which contradicts condition (2).

Hence \( \langle T_i(y^t), h_i(y_i^t, y_i) \rangle \geq \mathcal{C}_i 0 \), for \( t \in (0, 1) \). From condition (3), for fixed \( v \in K \), a mapping \( L_i : K \to \hat{Y}_i \) defined by

\[L_i(u) = \langle T_i u, h_i(u_i, v_i) \rangle
\]

for \( u = (u_1, u_2, u_3) \in K \), is hemicontinuous, i.e., a mapping from \([0, 1]\) to \( \hat{Y}_i \)

\[t \mapsto \langle T_i(x + t(y - x)), h_i(x_i + t(y_i - x_i), y_i) \rangle
\]

is continuous at \( 0^+ \) for all \( x, y \in K \).

Thus

\[\langle T_i x, h_i(x_i, y_i) \rangle = \lim_{t \to 0^+} \langle T_i(x + t(y - x)), h_i(x_i + t(y_i - x_i), y_i) \rangle
\]

\[= \lim_{t \to 0^+} \langle T_i(y^t), h_i(y_i^t, y_i) \rangle \geq \mathcal{C}_i 0, \quad \forall y \in K.
\]

Now, we consider some existence results for systems of strong implicit vector variational inequalities.

**Theorem 2.2.** Let \( X_i \) be reflexive Banach spaces, \( T_i : K \to L(X_i, \hat{Y}_i) \), and \( h_i : K_i \times K_i \to X_i \) be mappings satisfying the following conditions \((i = 1, 2, 3)\);

1. \( \langle T_i(x), h_i(x_i, x_i) \rangle \leq \mathcal{C}_i 0 \) \((i = 1, 2, 3)\);
2. for any given \( x = (x_1, x_2, x_3) \in K \), \( \{T_1, T_2, T_3\} \) are properly co-quasimono-
tone of Minty type with respect to \( \{h_1, h_2, h_3\} \);
3. for any given \( x = (x_1, x_2, x_3) \in K \) and \( z = (z_1, z_2, z_3) \in \prod_{i=1}^3 X_i \), \( \langle T_i(\tilde{x}_i), z_i \rangle \)
   is continuous from the weak topology of \( X_k \) to the norm topology of \( \hat{Y}_1 \), where
   for \( k = 1, l = 3 \), for \( k = 2, l = 1 \) and for \( k = 3, l = 2 \), and \( \tilde{x}_1 = (x_1, \ldots, x_3) \),
   \( \tilde{x}_2 = (x_1, x_2, \ldots) \), \( \tilde{x}_3 = (\ldots, x_2, x_3) \in K \).
4. \( h \) is linear and continuous such that \( h_i(a, b) + h_i(b, a) = 0 \) for \( a, b \in K_i \).
Then the problem \((SSIVVI - M)\) is solvable.

**Proof.** Let \(A_i = \{M_i : M_i\) is a finite dimensional subspace of \(X_i\) with \(K_{M_i} = K_i \cap M_i \neq \emptyset\}\) for \(i = 1, 2, 3\). Define a multivalued mapping \(G : \prod_{i=1}^{3} K_{M_i} \rightarrow 2^{\prod_{i=1}^{3} K_{M_i}}\) by

\[
G(x) = \left\{ x_0 \in \prod_{i=1}^{3} K_{M_i} : x_0 \text{ solves problem } (VI) \right\}, \quad \forall x \in \prod_{i=1}^{3} K_{M_i}.
\]

By Lemma 1.1, \((VI)\) is solvable, \(G(x)\) is nonempty. Since \(K_i\) is bounded, \(K\) is bounded. Now we claim that \(G(x)\) is closed. Let \(\langle (x_1^n, x_2^n, x_3^n) \rangle\) be a sequence in \(G(x)\) converging to \((x_1^0, x_2^0, x_3^0) \in \prod_{i=1}^{3} K_{M_i}\). Then

\[
\langle T_i(x^i), g_i(z_i, x_i^0) \rangle = \left\langle T_i(x_i), h_i \left( z_i, \lim_{n \to \infty} x_i^n \right) \right\rangle
= \left\langle T_i(x^i), \lim_{n \to \infty} h_i(z_i, x_i^n) \right\rangle
= \lim_{n \to \infty} \langle T_i(x^i), g_i(z_i, x_i^n) \rangle \leq \bar{C}_i 0.
\]

Hence \(G(x)\) is closed. And \(G(x)\) is convex, in fact, for \(x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X\) and for \(t \in (0, 1)\),

\[
t(x_1, x_2, x_3) + (1 - t)(y_1, y_2, y_3)
= (tx_1 + (1 - t)y_1, tx_2 + (1 - t)y_2, tx_3 + (1 - t)y_3) \in G(x),
\]

\[
\left\langle T_i(x^i), h_i(z_i, tx_i + (1 - t)y_i) \right\rangle
= \left\langle T_i(x^i), h_i(z_i, tx_i) \right\rangle + \left\langle T_i(x^i), h_i(z_i, (1 - t)y_i) \right\rangle
= \left\langle T_i(x^i), th_i(z_i, x_i) \right\rangle + \left\langle T_i(x^i), (1 - t)h_i(z_i, y_i) \right\rangle
= t\langle T_i(x^i), h_i(z_i, x_i) \rangle + (1 - t)\langle T_i(x^i), h_i(z_i, y_i) \rangle \leq \bar{C}_i 0.
\]

Hence \(G(x)\) is convex. Now we show that \(G \left( \prod_{i=1}^{3} K_{M_i} \right)\) is closed in \(\prod_{i=1}^{3} K_{M_i} \times \prod_{i=1}^{3} K_{M_i}\).

Let \(\langle (x_1^n, x_2^n, x_3^n), (y_1^n, y_2^n, y_3^n) \rangle\) be a sequence in \(\prod_{i=1}^{3} K_{M_i} \times \prod_{i=1}^{3} K_{M_i}\) such that

\[
\lim_{n \to \infty} \langle (x_1^n, x_2^n, x_3^n), (y_1^n, y_2^n, y_3^n) \rangle = \langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle, (y_1^n, y_2^n, y_3^n)
\in G(x_1^n, x_2^n, x_3^n), \forall n \in \mathbb{N}.
\]

Now we show that \((y_1, y_2, y_3) \in G(x_1, x_2, x_3)\).

\[
\langle T_1(z_1, x_2, x_3), h_1(z_1, y_1) \rangle = \left\langle \lim_{n \to \infty} T_1(x_1^n, x_2^n, x_3^n), h_1 \left( x_1, \lim_{n \to \infty} y_1^n \right) \right\rangle
\]
\[
\langle T_1(z_1, x^n_2, x_3), h_1(z_1, y^n_1) \rangle \\
= \lim_{n \to \infty} T_1(z_1, x^n_2, x_3), h_1(z_1, y^n_1) \leq C_1 0.
\]

\[
\langle T_2(x_1, z_2, x_3), h_2(z_2, y_2) \rangle = \langle \lim_{n \to \infty} T_2(x_1, z_2, x^n_3), h_2(z_2, \lim_{n \to \infty} y^n_2) \rangle \\
= \langle \lim_{n \to \infty} T_2(x_1, z_2, x^n_3), \lim_{n \to \infty} h_2(z_2, y^n_2) \rangle \\
= \lim_{n \to \infty} \langle T_2(x_1, z_2, x^n_3), h_2(z_2, y^n_2) \rangle \leq C_2 0.
\]

\[
\langle T_3(x_1, x_2, z_3), h_3(z_3, y_3) \rangle = \langle \lim_{n \to \infty} T_3(x^n_1, x_2, z_3), h_3(z_3, \lim_{n \to \infty} y^n_3) \rangle \\
= \langle \lim_{n \to \infty} T_3(x^n_1, x_2, z_3), \lim_{n \to \infty} h_3(z_3, y^n_3) \rangle \\
= \lim_{n \to \infty} \langle T_3(x^n_1, x_2, z_3), h_3(z_3, y^n_3) \rangle \leq C_3 0.
\]

Hence \(G(\prod_{i=1}^{3} K_{M_i})\) is closed in \(\prod_{i=1}^{3} K_{M_i} \times \prod_{i=1}^{3} K_{M_i}\). Since \(G\) is closed, \(G\) has a closed graph. Since \(G\) is closed and bounded, \(G\) is u.s.c. The Kakutani-Fan-Glicksberg fixed point theorem implies that there exists \(x_0 \in \prod_{i=1}^{3} K_{M_i}\) such that

\[
\langle T_i(x^i_0), h_i(z_i, x^i_0) \rangle \leq C_i 0, \forall z_i \in K_{M_i} (i = 1, 2, 3),
\]

where \(x^0 = (z_1, x^0_2, x_3), x^2_0 = (x_1, z_2, x^0_3)\) and \(x^3_0 = (x^0_1, x_2, z_3)\).

For any \(M := (M_1, M_2, M_3) \in \prod_{i=1}^{3} A_i\), let \(S_M\) be the solution set of the following vector variational inequality:

Find \(x \in K\) such that

\[
\langle T_i(x^i), h_i(z_i, x^i) \rangle \leq C_i 0, \forall z_i \in K_{M_i} (i = 1, 2, 3).
\]

By the similar argument, \(S_M\) is nonempty and bounded for all \(M \in \prod_{i=1}^{3} K_{M_i}\). Denote by \(\overline{S}_M\) the weak closure of \(S_M\) in \(\prod_{i=1}^{3} X_i\). Since \(X_i (i = 1, 2, 3)\) are reflexive, \(\overline{S}_M\) is weakly compact.

Let \(M^k_i \in A_i\) for \(k = 1, \cdots, n\). For any \(M^k = (M^k_1, M^k_2, M^k_3) \in \prod_{i=1}^{3} A_i\) for \(k = 1, \cdots, n\),

\[
S_{L_M} \subset \bigcap_k S_{M^k},
\]

where \(L_M\) denotes the linear subspace spanned by \(\bigcup_k M^k\).

Hence \(\{\overline{S}_M : M \in \prod_{i=1}^{3} A_i\}\) has the finite intersection property.
We claim that

\[ \langle T_i(x^*_i), h_i(z_i, x^*_i) \rangle \leq \overline{c}_i 0, \quad \forall z_i \in K_i \ (i = 1, 2, 3), \]

where \( x^*_i = (x^*_i, x^*_i, x^*_i, x^*_i) \), \( x^*_i = (x_1, z_2, x_3) \) and \( x^*_3 = (x^*_1, x_2, x_3) \).

In fact, for any given \( x_i \in K_i \), choose \( M_i \in A_i \) such that \( x_i, x^*_i \in M_i \). Since \( x^* \in \overline{S}_M \), there exists a net \( \langle x^\alpha \rangle = \langle (x^\alpha_1, x^\alpha_2, x^\alpha_3) \rangle \in S_M \) converging to \( x^* \) weakly in \( S_M \). Hence \( \langle T_i(x^\alpha_i), h_i(z_i, x^\alpha_i) \rangle \leq \overline{c}_i 0 \). By the condition (3),

\[ \langle T_i(x^*_i), h_i(z_i, x^*_i) \rangle \leq \overline{c}_i 0, \quad \forall z_i \in K_i \ (i = 1, 2, 3). \]

\[ \square \]

**Theorem 2.3.** Let \( X_i \) are reflexive Banach spaces, \( T_i : K \to L(X_i, \hat{Y}_i) \) and \( h_i : K_i \times K_i \to X_i \) be mappings \( (i = 1, 2, 3) \).

1. For fixed \( v = (v_1, v_2, v_3) \in K, \ u \mapsto \langle T_i(u), h_i(u_i, v_i) \rangle \) is hemi-continuous \((i = 1, 2, 3)\);
2. For any given \( x = (x_1, x_2, x_3) \in K \) and \( \{T_1, T_2, T_3\} \) is co-pseudomonotone and properly co-quasimonotone of Stampacchia type with respect to \( \{h_1, h_2, h_3\}\);
3. \( \langle T_i(x), h_i(x_i, x_i) \rangle \geq \overline{c}_i 0 \) for all \( x \in K \);
4. For any given \( x \in K \) and \( z = (z_1, z_2, z_3) \in \prod_{i=1}^3 X_i, \ \langle T_i(x_i), z_i \rangle \) is continuous from the weak topology of \( X_k \) to the norm topology of \( \hat{Y}_i \), where for \( k = 1, l = 3, \) for \( k = 2, l = 1 \) and for \( k = 3, l = 2 \), and \( \bar{x}_1, \bar{x}_2, \bar{x}_3 \in K_i \);
5. \( h \) is bilinear and continuous such that \( h_i(a, b) + h_i(b, a) = 0 \) for \( a, b \in K_i \).

Then (SSIVVI – S) is solvable.

**Proof.** From the existence results for strong implicit vector variational inequality (in [4]), there exists \( x^* = (x^*_1, x^*_2, x^*_3) \in K \) such that

\[ \langle T_i(x^*_i), h_i(x_i, x^*_i) \rangle \leq \overline{c}_i 0, \quad \forall x_i \in K_i \ (i = 1, 2, 3), \]

where \( x_1^* = (x_1, x_2^*, x_3) \), \( x_2^* = (x_1, x_2, x_3^*) \) and \( x_3^* = (x_1^*, x_2, x_3) \).
From conditions (1), (2) and (3) imply that $T_i(x_i^*)$ $(i = 1, 2, 3)$ satisfy all the assumptions of Minty’s lemma. Hence,

$$
\langle T_i(x^*), h_i(x_i^*, x_i) \rangle \geq \xi_i 0, \ \forall x_i \in K_i \ (i = 1, 2, 3).
$$

\[\square\]

By putting $h_i(x, y) = y - g_i(x)$, where $g_i : K_i \to X_i$ $(i = 1, 2, 3)$, in Theorem 2.2 and Theorem 2.3, we obtain the following Corollary 2.1 and Corollary 2.2, respectively, which extend some results in [1].

**Corollary 2.1.** Let $T_i : K \to L(X_i, Y_i)$, $g_i : K_i \to X_i$ $(i = 1, 2, 3)$ be mappings.

1. for any given $x = (x_1, x_2, x_3) \in K$, $\{T_1, T_2, T_3\}$ is properly co-quasimonotone of Minty type with respect to $\{g_1, g_2, g_3\}$;

2. for any given $x \in K$ and $z = (z_1, z_2, z_3) \in \prod_{i=1}^{3} X_i$, $\langle T_i(x_i), z_i \rangle$ is continuous from the weak topology of $X_k$ to the norm topology of $Y_i$, where for $k = 1$, $l = 3$, for $k = 2$, $l = 1$ and for $k = 3$, $l = 2$, and $x_1$, $x_2$, $x_3 \in K$.

Then there exists $x^* = (x_1^*, x_2^*, x_3^*) \in K$ such that

$$
\langle T_i(x_i^*), x_i^* - g_i(x) \rangle \leq \xi_i 0, \ \forall x_i \in K_i \ (i = 1, 2, 3).
$$

**Corollary 2.2.** Let $T_i : K \to L(X_i, Y_i)$, $g_i : K_i \to X_i$ $(i = 1, 2, 3)$ be mappings.

1. for fixed $u = (u_1, u_2, u_3) \in K$, $u \mapsto \langle T_i(u), h_i(u_i, u_i) \rangle$ is hemicontinuous $(i = 1, 2, 3)$;

2. for any given $x \in K$, $\{T_1, T_2, T_3\}$ is co-pseudomonotone with respect to $\{g_1, g_2, g_3\}$;

3. for any given $x \in K$, $\{T_1, T_2, T_3\}$ is properly co-quasimonotone of Stampacchia type with respect to $\{g_1, g_2, g_3\}$;

4. for any given $x \in K$ and $z \in \prod_{i=1}^{3} X_i$, $\langle T_i(x_i), z_i \rangle$ is continuous from the weak topology of $X_k$ to the norm topology of $Y_i$, where for $k = 1$, $l = 3$, for $k = 2$, $l = 1$ and for $k = 3$, $l = 2$, and $x_1$, $x_2$, $x_3 \in K$.

Then there exists $x^* \in K$ such that

$$
\langle T_i(x_i^*), x_i - g_i(x_i^*) \rangle \geq \xi_i 0, \ \forall x_i \in K_i \ (i = 1, 2, 3).
$$
REFERENCES


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