DOMINATED SPLITTING WITH STABLY EXPANSIVE

MANSEOB LEE

Abstract. In this paper, we show that if a transitive set Λ is $C^1$-stably expansive, then Λ admits a dominated splitting.

1. Introduction

In this paper, we study dominated splitting - a weak form of hyperbolicity. More precisely, using results of [2] and [3], we show that if a closed set have the same property then it admits dominated splitting.

Let $M$ be a closed $C^\infty$ manifold, and let Diff($M$) be the space of diffeomorphisms of $M$ endowed with the $C^1$-topology. Denote by $d$ the distance on $M$ induced from a Riemannian metric $\| \cdot \|$ on the tangent bundle $TM$. Let $f \in \text{Diff}(M)$, and let $\Lambda \subset M$ be a closed $f$-invariant set. We say that $f|_{\Lambda}$ is expansive if there is a constant $e > 0$ such that for any pair of distinct points $x, y \in \Lambda$, $d(f^n(x), f^n(y)) > e$ for some $n \in \mathbb{Z}$. Let $f \in \text{Diff}(M)$, and let $\Lambda \subset M$ be a closed $f$-invariant set. We say that $\Lambda$ is locally maximal if there is a compact neighborhood $U$ of $\Lambda$ such that $\bigcap_{n \in \mathbb{Z}} f^n(U) = \Lambda(U)$.

We say that $\Lambda$ admits a dominated splitting if the tangent bundle $T_{\Lambda}M$ has a continuous $Df$-invariant splitting $E \oplus F$ and there exists constant $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_xf^n|_{E(x)}\| \cdot \|D_xf^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. 

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Definition 1.1. We say that an $f$-invariant set $\Lambda$ is $C^1$-stably expansive if there exists a $C^1$-neighborhood $U(f)$ of $f$ and a compact neighborhood $U$ of $\Lambda$ such that:

- $\Lambda(U) = \bigcap_{n \in \mathbb{Z}} f^n(U)$,
- for any $g \in U(f)$, $g|_{\Lambda_g(U)}$ is expansive, where $\Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U)$ is called the continuation of $\Lambda$.

Let $f \in \text{Diff}(M)$, and let $\Lambda \subset M$ be a closed $f$-invariant set. Then we say that $\Lambda$ is called a transitive set if there exists a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. Mañé [6] studied the case in which for $f \in \text{Diff}(M)$ there is a $C^1$-neighborhood $U(f)$ of $f$ such that for any $g \in U(f)$, $g$ is expansive. He proved in the case $f$ is quasi-Anosov, that is, for all $v \in TM$, $v \neq 0$,

\[\{Df^n(v)\| : n \in \mathbb{Z}\}\] is bounded. Thus we can restate the above facts are follows.

**Theorem A.** $M$ is $C^1$-stably expansive if and only if $f$ satisfies quasi-Anosov.

In this paper, we get a problem which if a transitive set $\Lambda$ is $C^1$-stably expansive then is $\Lambda$ is hyperbolic? Unfortunately, it is not true. Indeed, for any hyperbolic periodic points $p, q \in \Lambda$, we don’t know that $W^s(p) \neq \phi$ and $W^u(q) \neq \phi$. Therefore, our aim is to characterize closed sets by making use of the $C^1$-stably expansive property. We are now in position to state main theorem.

**Theorem B.** Let $\Lambda$ be a transitive set. If $\Lambda$ is $C^1$-stably expansive, then $\Lambda$ admits a dominated splitting.

2. Introduction Some Results.

We use Mañé’s result which is on a uniformly family of periodic sequences of linear maps of $\mathbb{R}^n (n = \dim M)$. Let $GL(n)$ be the group of linear isomorphisms of $\mathbb{R}^n$. If a sequence $\xi : \mathbb{Z} \to GL(n)$ is periodic if there is $k > 0$ such that $\xi_{j+k} = \xi_j$ for $k \in \mathbb{Z}$. We call a finite subset $A = \{\xi_i : 0 \leq i \leq k - 1\} \subset GL(n)$ is a periodic family with period $k$. For a periodic family $A = \{\xi_i : 0 \leq i \leq n - 1\}$, we denote

$C_A = \xi_{n-1} \circ \xi_{n-2} \circ \cdots \circ \xi_0$.

**Definition 2.1.** We say that the periodic family $A = \{\xi_i : 0 \leq i \leq n - 1\}$ admits a $l$-dominated splitting, if there is a splitting $\mathbb{R}^n = E \oplus F$ which satisfies:

(a) $E$ and $F$ are $C_A$ invariant, i.e., $C_A(E) = E$ and $C_A(F) = F$,

(b) For any $k = 0, 1, 2, \ldots$,
Given any periodic family \( \mathcal{A} = \{ \xi_i : 0 \leq i \leq n-1 \} \) which satisfies the following property: given any periodic family \( \mathcal{A} = \{ \xi_i : 0 \leq i \leq n-1 \} \) which satisfies the period \( n \geq n_1 \) and \( \max\{\|\xi_i\|, \|\xi_i^{-1}\|\} \leq K \), for all \( i = 0, 1, \ldots, n-1 \), one can find a periodic family \( \mathcal{B} = \{ \zeta_i : 0 \leq i \leq n-1 \} \) such that \( \max\{\|\zeta_i - \xi_i\|, \|\zeta_i^{-1} - \xi_i^{-1}\|\} \leq \varepsilon \), for any \( i = 0, 1, \ldots, n-1 \), and \( \det(C_A) = \det(C_B) \) and the eigenvalues of \( C_B \) are all real, multiplicity one and different moduli.

**Theorem 2.3.** Given any \( \varepsilon > 0 \) and \( K > 0 \), there is positive integers \( n_2 \geq 0 \) and \( l \geq 0 \) which satisfies the following property: given any periodic family \( \mathcal{A} = \{ \xi_i : 0 \leq i \leq n-1 \} \) which satisfies the period \( n \geq n_2 \) and \( \max\{\|\xi_i\|, \|\xi_i^{-1}\|\} \leq K \), for all \( i = 0, 1, \ldots, n-1 \), if \( \mathcal{A} \) does not admits any \( l \)-dominated splitting, then one can find a periodic family \( \mathcal{B} = \{ \zeta_0, \zeta_1, \ldots, \zeta_n-1 \} \) such that \( \max\{\|\zeta_i - \xi_i\|, \|\zeta_i^{-1} - \xi_i^{-1}\|\} \leq \varepsilon \), for any \( i = 0, 1, \ldots, n-1 \), and \( \det(C_A) = \det(C_B) \) and the eigenvalues of \( C_B \) are all real, and have same modulus.

To prove Theorem B, we need another lemma about uniformly contracting family. Let \( \mathcal{A} = \{ \xi_i : 0 \leq i \leq k-1 \} \subset GL(n) \) be a periodic family. We say the sequence \( \mathcal{A} \) is uniformly contracting family if there is a constant \( \delta > 0 \) such that for any \( \delta \)-perturbation of \( \mathcal{A} \) are sink, i.e., for any \( \mathcal{B} = \{ \zeta_i : 0 \leq i \leq k-1 \} \) with \( \|\zeta_i - \xi_i\| < \delta \), all eigenvalue of \( C_B \) have moduli less than 1. Similarly, we can define the uniformly expanding periodic family. The following theorem is well known.

**Theorem 2.4 ([7]).** For any \( \delta > 0 \) and \( K > 0 \), there are constants \( C > 0 \), \( 0 < \lambda < 1 \) and positive integer \( m \) such that if \( \mathcal{A} = \{ A_0, A_1, \ldots, A_{n-1} \} \) a uniformly contracting periodic family which satisfies

\[
\max_{i=0,1,\ldots,n-1} \{ \| A_i \|, \| A_i^{-1} \| \} < K
\]

for \( n > m \), then
\[ \prod_{j=0}^{k-1} \prod_{i=0}^{m-1} A_{i+j} \leq C\lambda^k, \]

where \( k = \lfloor n/m \rfloor \).

3. PROOF OF THEOREM B

Let \( M \) be as before, and let \( f \in \text{Diff}(M) \). In this section, we will use the notation of pre-sink (resp. pre-source). A periodic point \( p \) is called a pre-sink (resp. pre-source) if \( Df^n(p)(p) \) has an multiplicity one eigenvalue equal to +1 or −1 and the other eigenvalues has norm less than 1 (resp. bigger than 1).

**Remark 3.1** ([1, Theorem 2.2.23 and 2.2.26]). Let \( f \in \text{Diff}(M) \).
- Let \( \mathcal{I} \) be a small arc. Then \( f : \mathcal{I} \to \mathcal{I} \) is not expansive.
- Let \( \mathcal{C} \) be a small circle. Then \( f : \mathcal{C} \to \mathcal{C} \) is not expansive.

Recall that if \( \Lambda \) is \( C^1 \)-stably expansive then there are a \( C^1 \)-neighborhood \( U(f) \) and a compact neighborhood \( U \) of \( \Lambda \) such that for any \( g \in U(f) \), \( \Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U) \) is expansive for \( g \).

**Lemma 3.2.** Let \( \Lambda \) be a closed set of \( f \in \text{Diff}(M) \), and let \( U(f) \) and \( U \) be as above. If \( \Lambda \) is \( C^1 \)-stably expansive, then for any \( g \in U(f) \), \( g \) has neither pre-sink nor pre-source with the orbit staying in \( U \).

**Proof.** Suppose that \( f \) is \( C^1 \)-stably expansive on \( \Lambda \). Then there are a \( C^1 \)-neighborhood \( U(f) \) of \( f \) and a compact neighborhood \( U \) of \( \Lambda \) such that for any \( g \in U(f) \), \( g \) is expansive on \( \Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U) \). Assume that there is \( g \in U(f) \) such that \( g \) has a pre-sink \( p \) with \( O(p) \subset U \). For simplicity, we may assume \( p \) is fixed point of \( g \) (other case is similar).

By making use of the Franks’ Lemma, we linearize \( g \) at \( p \) with respect to the exponential coordinates \( \exp_p \), i.e, choose \( \epsilon_1 > 0 \) and \( \alpha > 0 \) with \( B_{\alpha}(p) \subset U \) and there exists \( g_1 \) \( C^1-\epsilon_1 \) nearby \( g \) such that

\[
g_1(x) = \begin{cases} 
\exp_p \circ D_p g(p) \circ \exp_p^{-1}(x) & \text{if } x \in B_{\alpha}(p), \\
g(x) & \text{if } x \notin B_{4\alpha}(p).
\end{cases}
\]

Then \( g_1(p) = g(p) = p \).

Since \( p \) is pre-sink of \( g \), \( D_p g \) has a multiplicity one eigenvalue such that \( |\lambda| = 1 \) and other eigenvalues of \( D_p g \) are with modulus less than 1. Denote by \( E_{\lambda}^g \) the eigenspace corresponding to \( \lambda \), and \( E_p^s \) the eigenspace corresponding to the eigenvalues with modulus less than 1. Thus \( T_p M = E_{\lambda}^g \oplus E_p^s \). Then we get two cases: \( \lambda \) is real or complex.
Case 1: $\lambda$ is real. Then $\dim E^c_p = 1$. For simplicity, we suppose that $\lambda = 1$. There is a small arc $I_p \subset B_\alpha(p) \cap \exp_p(E^c_p(\alpha))$ center at $p$ such that $g_1|I_p = id$, where $id$ is identity map. Here $E^c_p(\alpha)$ is the $\alpha$-ball in $E^c_p$ center at the origin $O_p$. Clearly, $I_p \subset \Lambda_{g_1}(U)$.

Note that for a set $A \subset M$, if $M$ is expansive then $A$ have to expansive. By the definition of the $C^1$-stably expansivity, $g_1|\Lambda_{g_1}(U)$ is expansive. Moreover, by Remark 3.1, $g_1|I_p$ is not expansive. This is a contradiction. Therefore, if $\Lambda$ is $C^1$-stably expansive of $f$ then it does not have pre-sink.

Case 2: $\lambda$ is complex. Then $\dim E^c_p = 2$. Since the eigenvalue $\lambda$ is complex, there is a small circle $C_p \subset B_\alpha(p) \cap \exp_p(E^c_p(\alpha))$ center at $p$ such that $g_1|C_p$ is conjugate to an irrational rotation map. Here $E^c_p(\alpha)$ is the $\alpha$-ball in $E^c_p$ center at the origin $O_p$. Clearly, $C_p \subset \Lambda_{g_1}(U)$. Thus by the notion of $C^1$-stably expansivity, $g_1|C_p$ has to be expansive. Again by Remark 3.1, the rotation map $g_1 : C_p \to C_p$ is not expansive. This is a contradiction.

Therefore, if $\Lambda$ is $C^1$-stably expansive of $f$ then it does not have pre-sink. Similarly, $f$ does not have pre-source. ∎

The following lemma is well known result. In fact, we make using the $C^1$-closing lemma and property of transitive set. Hereafter, we consider transitive sets is non-trivial, that is, the set is not one orbit.

**Lemma 3.3** ([8]). Let $\Lambda$ be a transitive set. There exist a sequence $\{g_n\}_{n \in \mathbb{N}}$ of diffeomorphism and a periodic orbit $P_n$ of $g_n$ with period $\pi(P_n) \to \infty$ as $n \to \infty$ such that $g_n \to f$ in the $C^1$-topology and $\lim H P_n = \Lambda$, where $\lim H$ is the Hausdorff limit and $\pi(P_n)$ is the period of $P_n$.

From Lemma 3.3, we can choose $p_n \in P_n$ such that we get a periodic family $\mathcal{A}_n = \{D_{p_n} f, D_{f(p_n)} f, \ldots, D_{f^n(p_n) - 1(p_n)} f\}$.

**Lemma 3.4** ([4]). Let $\Lambda$, $P_n$ be as in Lemma 3.3, and $\mathcal{A}_n$ be given as above. Then for any $\epsilon > 0$ there exists an integer $n_0(\epsilon) > 0$ such that for any $n > n_0(\epsilon)$, $\mathcal{A}_n$ is neither $\epsilon$-uniformly contracting nor $\epsilon$-uniformly expanding.

Let $\mathcal{U}_0(f)$ be given by Lemma 3.2, and let $g \in \mathcal{U}_0(f)$. We consider the periodic family of linear maps $\mathcal{A} = \{D_p g : \text{for any } p \in P(g) \cap \Lambda_g(U)\}$. Let $\mathcal{B} = \{\xi_p : \text{for any } p \in P(g) \cap \Lambda_g(U)\}$ be a family of periodic sequence of linear maps closed to $\mathcal{A}$, and for any $p \in P(g) \cap \Lambda_g(U)$, consider the linear map

$$\mathcal{C}_B = \xi_{g^{\pi(p) - 1(p)}} \circ \cdots \circ \xi_{p}.$$
and denote by $\lambda_s(CB)$, $\lambda_u(CB)$ its eigenvalues. Here $\xi_{g^i(p)}$ is a linear map nearby $D_{g^i(p)}g$ for $0 \leq i \leq \pi(p) - 1$ and $|\lambda_s(CB)| \leq |\lambda_u(CB)|$.

**Lemma 3.5** ([4]). Let $\Lambda, P_n$ be as in Lemma 3.3. Then for any $\epsilon > 0$ there are $n(\epsilon), l(\epsilon) > 0$ such that for any $n > n(\epsilon)$ if $P_n$ does not admits a $l(\epsilon)$ dominated splitting, then choose $g$ $C^1$-nearby $f$ and preserving the orbit of $P_n$ such that $P_n$ is pre-sink or pre-source respecting $g$.

From Lemma 3.2 and Lemma 3.5, we can get the following Proposition 3.6.

**Proposition 3.6.** Let $\Lambda$ be a transitive set. Then if $\Lambda$ is $C^1$-stably expansive, then we can choose $N, l > 0$ such that for any $n > N$, $P_n$ admits a $l$-dominated splitting.

**Proof.** Let $\Lambda$ be a transitive set. Suppose that $\Lambda$ is $C^1$-stably expansive. Then by Franks’ Lemma, and by the notion of the $C^1$-stably expansivity, there are a $C^1$-neighborhood $U(f)$ of $f$ and a compact neighborhood $U$ of $\Lambda$ such that for any $g \in U_0(f) \subset U(f)$, $g|_{\Lambda(g(U))}$ is expansive. By Lemma 3.2, $g$ has neither pre-sink nor pre-source. And, by Lemma 3.5, $P_n$ is neither pre-sink nor pre-source respecting $g$. Therefore, by Lemma 3.5, $P_n$ admits a $l$-dominated splitting. □

By Proposition 3.6 and the following proposition, we directly obtain Theorem B.

**Proposition 3.7** ([2]). Let $g_n$ convergent to $f$ and if $\Lambda_{g_n}$ be a closed $g_n$-invariant set of $g_n$ and $\lim \Lambda_{g_n} = \Lambda$. Then if $\Lambda_{g_n}$ admits a $l$-dominated splitting respecting $g_n$, then $\Lambda$ admits a $l$-dominated splitting respecting $f$.

End of proof of Theorem B. Let $\Lambda$ be a transitive set of $f \in \text{Diff}(M)$. Then by Lemma 3.3, there exists a sequence $\{g_n\}_{n \in \mathbb{Z}}$ of diffeomorphism and a periodic orbit $P_n$ of $g_n$ such that $g_n \rightarrow f$ in the $C^1$-topology and $P_n \rightarrow \Lambda$ in the Hausdorff limit. By Proposition 3.6, $P_n$ admits a $l$-dominated splitting. Thus by Proposition 3.7, $\Lambda$ admits a $l$-dominated splitting. □

**References**


Department of Mathematics, Mokwon University, Daejeon, 302-729, Korea

Email address: lmsds@mokwon.ac.kr