SURFACES WITH POINTWISE 1-TYPE GAUSS MAP OF THE SECOND KIND

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ABSTRACT. In this article, we study generalized slant cylindrical surfaces (GSCS’s) with pointwise 1-type Gauss map of the first and second kinds. Our main results state that the right circular cones are the only rational kind GSCS’s with pointwise 1-type Gauss map of the second kind.

1. INTRODUCTION AND PRELIMINARIES

During the late 1970’s, B.-Y. Chen introduced the notion of finite type submanifolds in Euclidean or pseudo-Euclidean space, and then the notion has become a useful tool for investigating and characterizing a lot of important submanifolds ([3, 4]). The notion of finite type submanifolds in Euclidean or pseudo-Euclidean space was extended to Gauss maps of submanifolds ([1, 2, 6]).

Suppose that a submanifold $M$ of Euclidean or pseudo-Euclidean space has 1-type Gauss map $G$. Then the Gauss map $G$ satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$, where $\Delta$ is the Laplace operator corresponding to the induced metric on $M$ (cf [1, 2, 10]). But, on the important surfaces such as helicoids, catenoids and right circular cones, the Laplacian of the Gauss map take a somewhat different form; namely,

$$\Delta G = f(G + C),$$

where $f$ is a non-constant function and $C$ is a constant vector. For this reason, a submanifold is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1.1) for some smooth function $f$ on $M$ and vector $C$. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1.1) is the zero
vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind ([5, 8]).

For the induced metric on the submanifold $M$, we consider the matrix $g = (g_{ij})$ consisting of the components of the induced metric on $M$ and we denote by $g^{-1} = (g^{ij})$ (resp., $G$) the inverse matrix (resp., the determinant) of the matrix $(g_{ij})$. The Laplacian $\Delta$ on $M$ is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{G}} \sum_{i,j} \frac{\partial}{\partial x^i} \left( \sqrt{G} g^{ij} \frac{\partial}{\partial x^j} \right).$$

Now, we show that the right circular cone has pointwise 1-type Gauss map of the second kind ([5]).

**Example 1.1.** Let’s consider the right circular cone $C_a$ which is parameterized by

$$x(u, v) = (v \cos u, v \sin u, av), \quad a \geq 0.$$ 

Then the Gauss map $G$ and its Laplacian $\Delta G$ are given by

$$G = \frac{1}{\sqrt{1 + a^2}} (a \cos u, a \sin u, -1)$$

and

$$\Delta G = \frac{1}{v^2} \left( G + \left( 0, 0, \frac{1}{\sqrt{1 + a^2}} \right) \right),$$

respectively. This implies that the right circular cone has pointwise 1-type Gauss map of the second kind.

In [5], B.-Y. Chen, M. Choi and Y. H. Kim studied surfaces of revolution with pointwise 1-type Gauss map. In [7], U. Dursun studied flat surfaces in Euclidean 3-space with pointwise 1-type Gauss map.

In [9], the author and Y. H. Kim introduced the class of generalized slant cylindrical surfaces (GSCS’s). This class includes surfaces of revolution and cylindrical surfaces as special cases. In [8], the author studied GSCS’s with pointwise 1-type Gauss map. As a result, he showed that GSCS’s with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature; and the right circular cones are the only polynomial kind GSCS’s with pointwise 1-type Gauss map of the second kind.

In this article, we study the GSCS’s with pointwise 1-type Gauss map of the second kind. As a result, we show that the right circular cones are the only rational kind GSCS’s with pointwise 1-type Gauss map of the second kind.
From now on, all objects are assumed to be connected and smooth, unless mentioned otherwise.

2. GSCS’s with Pointwise 1-type Gauss Map of the First Kind

Consider a fixed unit speed plane curve $X(s) = (x(s), y(s), 0)$. We let $T(s) = X'(s)$ and $N(s) = (-y'(s), x'(s), 0)$ denote the unit tangent and principal normal vector, respectively. The curvature $\kappa(s)$ of $X(s)$ is defined by $T'(s) = \kappa(s)N(s)$ and we have $T(s) \times N(s) = V$, where $V$ denotes the unit vector $(0, 0, 1)$. For a constant $\theta$, we let $Y_\theta(s) = \cos \theta N(s) + \sin \theta V$. Then the ruled surface $M$ defined by

\begin{equation}
F(s, t) = X(s) + tY_\theta(s)
\end{equation}

is regular at $(s, t)$ where $1 - \cos \theta \kappa(s)t$ does not vanish. This ruled surface $M$ is called a slant cylindrical surface (SCS) over $X(s)$ ([9]).

More generally, instead of a line, we consider another unit speed plane curve $W(t) = (z(t), w(t))$. If we let $Y_s(t) = z(t)N(s) + w(t)V$, then the parametrized surface defined by

\begin{equation}
H(s, t) = X(s) + Y_s(t)
\end{equation}

is regular at $(s, t)$ where $1 - \kappa(s)z(t)$ does not vanish. This parametrized surface $M$ is called a generalized slant cylindrical surface (GSCS) over $X(s)$ ([9]).

In case $W(t)$ is a straight line, the GSCS $H(s, t)$ is nothing but an SCS. If $X(s)$ is a straight line, then the GSCS $H(s, t)$ is nothing but a cylindrical surface. Furthermore, we have the following ([8, 9]).

**Proposition 2.1.** If $X(s)$ is a circle, then a GSCS $M$ over $X(s)$ is a surface of revolution.

Thus, we see that cylindrical surfaces and surfaces of revolution are special families of GSCS’s.

**Proposition 2.2.** Let $M$ denote a GSCS given by (2.2). Then we have the following.

1. If the mean curvature $H$ is constant, then $M$ is a surface of revolution.
2. If the Gaussian curvature $K$ is constant, then $M$ is either a surface of revolution or an SCS.

Now, we consider a GSCS $M$ parametrized by (2.2), where $W(t) = (z(t), w(t))$ is a unit speed plane curve, $Y_s(t) = z(t)N(s) + w(t)V$, and $V = (0, 0, 1)$.

Then, we get the following propositions ([8]).
Proposition 2.3. Let $M$ be a GSCS given by (2.2). Suppose that $M$ has pointwise 1-type Gauss map $G$ of the first kind. Then $M$ is a surface of revolution.

Proposition 2.4. Let $M$ be a GSCS given by (2.2). Then the following are equivalent.

1. $M$ has pointwise 1-type Gauss map $G$ of the first kind.
2. $M$ has constant mean curvature.
3. $M$ is a surface of revolution with constant mean curvature.

Remark 2.5. Surfaces of revolution with constant mean curvature are also known as surfaces of Delaunay (cf. [11, p.115]).

3. GSCS’s with Pointwise 1-type Gauss Map of the Second Kind

Consider a GSCS $M$ parametrized by (2.2). If $M$ is not cylindrical, then $W(t)$ can be parametrized by $W(t) = (t, g(t))$ for some function $g = g(t)$. Hence $M$ is given by

$$H(s, t) = X(s) + tN(s) + g(t)V.$$ 

If $g(t)$ is a polynomial (resp., rational) in $t$, then $M$ is said to be of polynomial (resp., rational) kind ([5]).

$$H_s = Q(s,t)T(s), H_t = N(s) + g'(t)V,$$
$$G(s, t) = \frac{1}{P(t)} \{-g'(t)N(s) + V\}, P(t) = \sqrt{1 + g'(t)^2}.$$ 

The Laplacian $\Delta$ on $M$ is given by

$$\Delta f = -P^{-4}Q^{-3}\{\kappa'(s)TP^4f_s + P^4Qf_{ss}$$
$$- (P^2Q^2\kappa(s) + Q^3g'g'')f_t + P^2Q^3f_t\}.$$ 

Hence, it follows from (3.2) and (3.3) that

$$\Delta G = -\kappa'(s)g'P^{-1}Q^{-3}T(s)$$
$$- P^{-7}Q^{-2}\{Q^{-2}g'P^6 + \kappa(s)g''P^2Q$$
$$+ g'(g'')^2Q^2 - g''P^2Q^2 + 3g'(g'')^2Q^2\}N(s)$$
$$- P^{-7}Q^{-1}\{(3g')^2g''^2 - (g'')^2 - g'g'' - (g')^3g'''Q + \kappa(s)g'g''P^2\}V.$$ 

Suppose that the Gauss map $G$ satisfies (1.1) with nonzero constant vector $C$. Hereafter, we may assume that $f \neq 0$, because otherwise, $M$ is a plane. Letting $C = C_1(s)T(s) + C_2(s)N(s) + C_3V$, we have the following:
\[(3.5) \quad PQ^3 C_1(s)f(s,t) + \kappa'(s)g'(t) = 0, \]
\[(3.6) \quad P^6 Q^2 f(s,t)\{-g'(t) + PC_2(s)\} + \kappa(s)^2 g'P^6 \]
\[ + \kappa(s)g''P^2 Q + 4g'(g'')^2 Q^2 - g'''P^2 Q^2 = 0, \]
and
\[(3.7) \quad P^6 Qf(s,t)\{1 + C_3P\} + \{3(g')^2(g'')^2 \]
\[- (g'')^2 - g'g''' - (g')^3 g''\} Q + \kappa(s)g''P^2 = 0. \]

Suppose that \(M\) is a GSCS of rational kind, that is, \(g(t)\) is a rational function in \(t\). Then both of \(g(t)\) and \(g'(t)\) are rational functions in \(t\). Denote by \(g'(t) = r(t)/q(t)\), where \(r(t)\) and \(q(t)\) are relatively prime polynomials.

**Lemma 3.1.** Suppose that \(C_1(s)\) vanishes identically. Then \(M\) is a right circular cone.

**Proof.** If \(C_1(s)\) vanishes identically, then \((3.5)\) shows that \(\kappa'(s)g'(t)\) vanishes identically. If \(g'(t) = 0\), then \(M\) is a plane. Otherwise, \(\kappa(s)\) is a constant. First, suppose that \(\kappa(s)\) is a nonzero constant. Then Proposition 2.1 shows that \(M\) is a surface of rotation. Thus, it follows from [5] that \(M\) is a right circular cone.

Now, suppose that \(\kappa(s)\) vanishes identically. Then \(X(s)\) is a straight line with constant vector fields \(T\) and \(N\). Hence \(M\) is a cylindrical surface over a plane curve \(W(t) = (t, g(t))\) with constants \(C_2\) and \(C_3\). Furthermore, \((3.6)\) and \((3.7)\) reduce to, respectively,
\[(3.8) \quad P^6 f(s,t)\{-g'(t) + PC_2\} + 4g'(g'')^2 - g'''P^2 = 0, \]
and
\[(3.9) \quad P^6 f(s,t)\{1 + C_3P\} + \{3(g')^2(g'')^2 - (g'')^2 - g'g''' - (g')^3 g''\} = 0. \]

By eliminating \(f(s,t)\), we get
\[(3.10) \quad \sqrt{1 + (g')^2}\{C_2 A - C_3 D - C_2 B\} = \{g' A - g' B + D\}, \]
where
\[(3.11) \quad A = 3(g')^2(g'')^2 - (g')^3 g''' , \quad B = (g'')^2 + g' g'' , \quad D = 4g'(g'')^2 - (g'')^2 g''' - g'' . \]

From \((3.10)\), we also obtain
\[(3.12) \quad \{1 + (g')^2\}\{C_2 A - C_3 D - C_2 B\}^2 = \{g' A - g' B + D\}^2. \]
Case 1. First, suppose that \( g'(t) = r(t)/q(t) \) satisfies \( \deg r(t) > \deg q(t) \). Then we put \( g'(t) = r(t)/q(t) = s(t) + u(t)/q(t) \), where \( s(t), q(t) \) and \( u(t) \) are polynomials given by

\[
q(t) = t^m + \cdots + q_m, \quad s(t) = s_0 t^l + \cdots + s_l, \quad l \geq 1,
\]

\[
u(t) = u_0 t^n + \cdots + u_n, \quad n < m.
\]

By comparing the leading coefficients of both sides of \( q(t)^{14} \) times of (3.12), we get \( C_2^2 = 1 \), and hence again we get \( C_3 = 0 \). This shows that the leading coefficient of \( q(t)^{14} \{ A^2 - 2qA \} \) becomes zero, which is a contradiction.

Case 2. Second, suppose that \( g'(t) = u(t)/q(t) \) satisfies \( \deg u(t) < \deg q(t) \), where \( q(t) \) and \( u(t) \) are relatively prime polynomials given in (3.13).

By comparing the leading coefficients of both sides of \( q(t)^{14} \) times of (3.12), we get \( C_2^2 = 1 \), and hence again we get \( C_2 = 0 \). This shows that the leading coefficient of \( q(t)^{14} g'D(q'D + 2B) \) becomes zero, which is a contradiction.

Case 3. Finally, suppose that \( g'(t) = r(t)/q(t) \) satisfies \( \deg r(t) = \deg q(t) \), where \( q(t) \) and \( r(t) \) are relatively prime polynomials given in (3.12). Hence we have \( g'(t) = r(t)/q(t) = a + u(t)/q(t) \) for some nonzero constant \( a \) and a polynomial \( u(t) \) with \( \deg u < \deg q \).

In this case, first suppose that \( C_2^2 A - C_3^2 D - C_2 B = 0 \) in (3.10), then we have

\[
g'A - g'B + D = 0.
\]

Hence we get \( A - B = D = 0 \). Thus, from \( D = 0 \) we get

\[
g'' = \frac{4g'(g'')^2}{1 + (g')^2},
\]

and hence we obtain

\[
A - B = -(g'')^2 (1 + (g')^2) = 0.
\]

This shows that \( g'(t) \) is a linear function, and hence \( M \) is nothing but a plane.

Now, suppose that \( g(t) \) is not a linear function. Then, the above discussion shows that \( P = \sqrt{1 + g'(t)^2} \) is a rational function in \( t \). Hence, there exists a polynomial \( p(t) \) satisfying

\[
(1 + a^2)q^2 + 2aq + u^2 = p^2.
\]

Thus, we see that \( q(t), u(t) \) and \( p(t) \) satisfy

\[
(p - \sqrt{1 + a^2}q)(p + \sqrt{1 + a^2}q) = u(2aq + u).
\]

Since the leading coefficient of \( p(t) \) is \( \pm \sqrt{1 + a^2} \), without loss of generality, we may assume that the leading term of \( p(t) \) is given by \( \sqrt{1 + a^2} t^m \). Then, by considering
the leading terms of polynomials in (3.17), we get
\[ p + \sqrt{1 + a^2 q} = \frac{a}{\sqrt{1 + a^2}}(2aq + u), \]
and hence
\[ p - \sqrt{1 + a^2 q} = \frac{\sqrt{1 + a^2}}{a}. \]
From (3.19), we get
\[ p = \sqrt{1 + a^2 q} + \frac{\sqrt{1 + a^2}}{a} u. \]
By substituting \( p \) in (3.20) into (3.18), we obtain
\[ 2aq + u = 0, \]
which is a contradiction. \( \square \)

**Lemma 3.2.** Suppose that \( M \) is a GSCS of rational kind with pointwise 1-type Gauss map. Then \( C_1(s) \) vanishes identically.

**Proof.** Suppose that \( C_1(s) \neq 0 \) on an interval \( I \). Then we have \( \kappa'(s)g'(t) \neq 0 \) on \( I \). It follows from (3.5) and (3.7) that
\[ C_3\kappa'(s)g'P^6 + \kappa'(s)g'P^5 \]
\[ = C_1(s)Q^3\{3(g')^2(g'')^2 - (g')^3g''\} + C_1(s)\kappa(s)g''P^2Q^2 \]
\[ - C_1(s)Q^3\{(g'')^2 - g'g''\}. \]
From now on, we proceed on the interval \( I \). Since \( g'(t) \) and \( P^2 \) are rational functions, (3.22) shows that \( P = \sqrt{1 + g'(t)^2} \) is also a rational function in \( t \). Hence, there exists a polynomial \( p(t) \) satisfying \( q^2(t) + r^2(t) = p^2(t) \), where \( q(t), r(t) \) and \( p(t) \) are relatively prime. We put
\[ R(t) = C_3\kappa'(s)g'P^6, \quad R_1(t) = -\kappa'(s)g'P^5, \]
\[ R_2(t) = C_1(s)Q^3\{3(g')^2(g'')^2 - (g')^3g''\}, \quad R_3(t) = C_1(s)\kappa(s)g''P^2Q^2, \quad R_4(t) = -C_1(s)Q^3\{(g'')^2 - g'g''\}. \]
Then, for each \( i = 1, 2, 3, 4 \), \( R_i \) is a rational function, which satisfies
\[ R(t) = R_1(t) + R_2(t) + R_3(t) + R_4(t). \]
Since, for each \( i = 1, 2, 3, 4 \), \( q^6R_i(t) \) is a polynomial, it follows from (3.23) and (3.24) that
\[ q^6R(t) = \frac{C_3\kappa'(s)r(t)p(t)^6}{q(t)}. \]
is a polynomial in \( t \). Since \( p(t), q(t), r(t) \) are relatively prime, it follows from \( \kappa'(s) \neq 0 \)
that \( C_3 = 0 \). Thus (3.22) becomes
\[
\begin{align*}
\{ & \kappa'(s) \over C_1(s) \}^2 \over (g')^2 \{ 1 + (g')^2 \}^5 \\
= & Q^3 \{ 3 (g')^2 (g'')^2 - (g')^3 g''' - (g'')^2 - g' g''' \} + \kappa(s) g' g''' Q^2 \{ 1 + (g')^2 \}^2.
\end{align*}
\]

We also get from (3.5) and (3.6) that
\[
S_1(t) + S_2(t) = 0,
\]
where we denote
\[
\begin{align*}
S_1(t) = & \kappa'(s) g'(t)^2 P^5 - \kappa'(s) C_2(s) g'(t) P^6 + C_1(s) \kappa(s)^2 g'(t) P^6 Q, \\
S_2(t) = & \kappa(s) C_1(s) g''(t) P^2 Q^2 + 4 C_1(s) g'(g'')^2 Q^3 - C_1(s) g'''(t) P^2 Q^3. \\
\end{align*}
\]
Since \( q(t)^5 S_2(t) \) is a polynomial in \( t \), (3.27) shows that
\[
q(t)^5 S_1(t) = {q(t)^7 S_1(t) \over q(t)^2} = {r p^5 (Ar + Bp) \over q(t)^2}
\]
is a polynomial in \( t \), where we denote
\[
A = \kappa'(s), \quad B = C_1(s) \kappa(s)^2 Q - \kappa'(s) C_2(s).
\]
Since \( p(t), q(t), r(t) \) are relatively prime, we see that
\[
Ar + Bp = u(t) q(t)^2,
\]
where \( u(t) \) is a polynomial in \( t \).

Case 1. Suppose that \( \deg q(t) \geq \deg r(t) \). Then we have \( \deg p(t) = \deg q(t) \). Since \( \kappa(s) \neq 0 \), we have \( \deg B(t) = 1 \). Hence (3.31) shows that \( \deg q(t) = 1 \), and hence \( g'(t) = r(t)/q(t) \) is a linear fractional function in \( t \). But, in this case, \( q^2(t) + r^2(t) \)
can not be a square of a linear function. This is a contradiction.

Case 2. Suppose that \( \deg q(t) < \deg r(t) \). Then we put \( g'(t) = r(t)/q(t) = s(t) + u(t)/q(t) \), where \( s(t), q(t) \) and \( u(t) \) are polynomials given in (3.13).

Note that \( Q \) is a polynomial in \( t \) given by \( Q = 1 - \kappa(s) t \). Then, it is straightforward to show that the highest degree of left side of \( q(t)^{12} \) times of (3.26) is \( 12(m + l) \)
with leading coefficient \( \{ \kappa'(s)/C_1(s) \}^2 s_0^{12} \), and the highest degree of right side of 
\( q(t)^{12} \) times of (3.26) is \( 12m + 8l + 2 \). This shows that \( \kappa'(s) = 0 \) on \( I \), which is a contradiction. □

Summarizing above, we obtain

**Theorem 3.3.** Suppose that a GSCS \( M \) of rational kind has pointwise 1-type Gauss map \( G \) of the second kind. Then \( M \) is a right circular cone.
Hence, combining the results in [5] and [10], we get

**Corollary 3.4.** Suppose that a GSCS $M$ has pointwise 1-type Gauss map $G$ of the second kind. Then the following are equivalent.

1. $M$ is of rational kind.
2. $M$ is of polynomial kind.
3. $M$ is a right circular cone.

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**References**