FALLING SUBALGEBRAS AND IDEALS IN BH-ALGEBRAS

Eun Mi Kim\textsuperscript{a} and Sun Shin Ahn\textsuperscript{b,∗}

Abstract. Based on the theory of a falling shadow which was first formulated by Wang([14]), a theoretical approach of the ideal structure in BH-algebras is established. The notions of a falling subalgebra, a falling ideal, a falling strong ideal, a falling n-fold strong ideal and a falling translation ideal of a BH-algebra are introduced. Some fundamental properties are investigated. Relations among a falling subalgebra, a falling ideal and a falling strong ideal, a falling n-fold strong ideal are stated. A relation between a fuzzy subalgebra/ideal and a falling subalgebra/ideal is provided.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3,4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have some connections with other areas: D. Mundici [8] proved MV-algebras are categorically equivalent to bounded commutative algebra, and J. Meng [9] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim [5] introduced the notion of a BH-algebra, which is a generalization of BCK/BCI-algebras. They defined the notions of ideal, maximal ideal and translation ideal and investigated some properties. E. H. Roh and S. Y. Kim [11] estimated the number of BH\textsuperscript{∗}-subalgebras of order i in a transitive BH\textsuperscript{∗}-algebras by using Hao’s method. S. S. Ahn and J. H. Lee ([2]) defined the notion of strong ideals in BH-algebra and studied some properties of it. They considered the notion of a rough set in BH-algebras. S. S. Ahn and E. M. Kim ([1]) introduced the notion of n-fold strong ideal in BH-algebra and gave some related properties of it.

In this paper we introduced the notions of a falling subalgebra, a falling ideal, a falling strong ideal, a falling n-fold strong ideal and a falling translation ideal of a
BH-algebra. We investigate some fundamental properties. Also we give relations among a falling subalgebra, a falling ideal and a falling strong ideal, a falling n-fold strong ideal. We study a relation between a fuzzy subalgebra/ideal and a falling subalgebra/ideal.

2. Preliminaries

By a BH-algebra ([5]), we mean an algebra $(X; *, 0)$ of type (2,0) satisfying the following conditions:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$.

For brevity, we also call $X$ a BH-algebra. In $X$ we can define an order relation “$\leq$” by $x \leq y$ if and only if $x * y = 0$. A non-empty subset $S$ of a BH-algebra $X$ is called a subalgebra of $X$ if, for any $x, y \in S$, $x * y \in S$, i.e., $S$ is closed under binary operation.

**Definition 2.1** ([5]). A non-empty subset $A$ of a BH-algebra $X$ is called an ideal of $X$ if it satisfies:

(I1) $0 \in A$,

(II) $x * y \in A$ and $y \in A$ imply $x \in A$, $\forall x, y \in X$.

An ideal $A$ of a BH-algebra $X$ is said to be a translation ideal of $X$ if it satisfies:

(III) $x * y \in A$ and $y * x \in A$ imply $(x * z) * (y * z) \in A$ and $(z * x) * (z * y) \in A$, $\forall x, y, z \in X$.

Obviously, $\{0\}$ and $X$ are ideals of $X$. For any elements $x$ and $y$ of a BH-algebra $X$, $x * y^n$ denotes $(\cdots ((x * y) * y) * \cdots) * y$ in which $y$ occurs $n$ times.

**Definition 2.2.** A non-empty subset $A$ of a BH-algebra $X$ is called a strong ideal ([2]) of $X$ if it satisfies (II) and

(I4) $(x * y) * z \in A$ and $y \in A$ imply $x * z \in A$ for all $x, y, z \in X$.

A non-empty subset $A$ of a BH-algebra $X$ is called an $n$-fold strong ideal ([1]) of $X$ if it satisfies (II) and

(I5) for every $x, y, z \in X$ there exists a natural number $n$ such that $x * z^n \in A$ whenever $(x * y) * z^n \in A$ and $y \in A$.

**Definition 2.3** ([11]). A BH-algebra $X$ is called a BH*-algebra if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$. 
Definition 2.4. A BH-algebra \((X; *, 0)\) is said to be transitive if \(x * y = 0\) and \(y * z = 0\) imply \(x * z = 0\) for all \(x, y, z \in X\).

We now review some fuzzy logic concepts. A fuzzy set in a set \(X\) is a function \(\mu : X \to [0, 1]\). For a fuzzy set \(\mu\) in \(X\) and \(t \in [0, 1]\), define \(U(\mu; t)\) to be the set \(\{ x \in X | \mu(x) \geq t \}\), which is called a level subset of \(\mu\).

Definition 2.5. A fuzzy set \(\mu\) in a BH-algebra \(X\) is called a fuzzy BH-ideal (here call it a fuzzy ideal) ([6]) of \(X\) if

\[(\text{FI1}) \quad \mu(0) \geq \mu(x), \forall x \in X,\]
\[(\text{FI2}) \quad \mu(x) \geq \min\{\mu(x * y), \mu(y)\}, \forall x, y \in X.\]

A fuzzy set \(\mu\) in a BH-algebra \(X\) is called a fuzzy translation BH-ideal ([6]) of \(X\) if it satisfies (FI1), (FI2) and

\[(\text{FI3}) \quad \min\{\mu((x+z) * (y+z)), \mu((z+x) * (z+y))\} \geq \min\{\mu(x*y), \mu(y*x)\}, \forall x, y, z \in X.\]

A fuzzy set \(\mu\) in a BH-algebra \(X\) is called a fuzzy strong ideal ([7]) of \(X\) if it satisfies (FI1) and

\[(\text{FI4}) \quad \mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y)\}, \forall x, y, z \in X.\]

A fuzzy set \(\mu\) in a BH-algebra \(X\) is called a fuzzy \(n\)-fold strong ideal ([7]) of \(X\) if it satisfies (FI1) and

\[(\text{FI5}) \quad \mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y)\}, \forall x, y, z \in X.\]

We now display the basic theory on falling shadows. We refer the reader to the papers [12, 13, 14] for further information regarding the theory of falling shadows.

Given a universe of discourse \(U\), let \(\mathcal{P}(U)\) denote the power set of \(U\). For each \(u \in U\), let

\[(2.1) \quad \hat{u} := \{ E | u \in E \text{ and } E \subseteq U \},\]

and for each \(E \in \mathcal{P}(U)\), let

\[(2.2) \quad \hat{E} := \{ \hat{u} | u \in E \}.\]

An ordered pair \((\mathcal{P}(U), \mathcal{B})\) is said to be a hyper-measurable structure on \(U\) if \(\mathcal{B}\) is a \(\sigma\)-field in \(\mathcal{P}(U)\) and \(\hat{U} \subseteq \mathcal{B}\). Given a probability space \((\Omega, \mathcal{A}, P)\) and a hyper-measurable structure \((\mathcal{P}(U), \mathcal{B})\) on \(U\), a random set on \(U\) is defined to be a mapping \(\xi : \Omega \to \mathcal{P}(U)\) which is \(\mathcal{A}\)-\(\mathcal{B}\) measurable, that is,

\[(2.3) \quad (\forall C \in \mathcal{B}) (\xi^{-1}(C) = \{ \omega | \omega \in \Omega \text{ and } \xi(\omega) \in C \} \in \mathcal{A}).\]
Suppose that $\xi$ is a random set on $U$. Let
\[
\tilde{H}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for each } u \in U.
\]
Then $\tilde{H}$ is a kind of fuzzy set in $U$. We call $\tilde{H}$ a falling shadow of the random set $\xi$, and $\xi$ is called a cloud of $\tilde{H}$.

For example, $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0,1]$ and $m$ is the usual Lebesgue measure. Let $\tilde{H}$ be a fuzzy set in $U$ and $\tilde{H}_t := \{ u \in U \mid \tilde{H}(u) \geq t \}$ be a $t$-cut of $\tilde{H}$. Then
\[
\xi : [0,1] \to \mathcal{P}(U), \ t \mapsto \tilde{H}_t
\]
is a random set and $\xi$ is a cloud of $\tilde{H}$. We shall call $\xi$ defined above as the cut-cloud of $\tilde{H}$.

3. FALLING SUBALGEBRAS/IDEALS IN $BH$-ALGEBRAS

In what follows let $X$ denote a $BH$-algebra unless otherwise specified.

**Definition 3.1.** Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let
\[
\xi : \Omega \to \mathcal{P}(X),
\]
be a random set. If $\xi(\omega)$ is a subalgebra (resp., ideal, strong ideal, $n$-fold strong ideal and translation ideal) of a $BH$-algebra $X$ for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\tilde{H}$ of the random set $\xi$, i.e.,
\[
\tilde{H}(x) = P(\omega \mid x \in \xi(\omega))
\]
is called a falling subalgebra (resp., falling ideal, falling strong ideal, falling $n$-fold ideal and falling translation ideal) of $X$.

**Example 3.2.** (1) Let $X := \{0,1,2,3\}$ be a $BH$-algebra([5]) with the following table:

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For a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:
\[
\xi : \Omega \to \mathcal{P}(X), \ t \mapsto \begin{cases} 
\emptyset & \text{if } t \in [0,0.3), \\
\{0,1,2\} & \text{if } t \in [0.3,0.8), \\
X & \text{if } t \in [0.8,1].
\end{cases}
\]
Then $\xi(t)$ is an ideal of $X$ for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling ideal of $X$. If we take $t \in [0.3,0.8)$, then $\xi(t) = \{0,1,2\}$ is neither a subalgebra nor a translation ideal of $X$ since $0 \ast 2 = 3 \notin \{0,1,2\}$ and $1 \ast 2 = 2, 2 \ast 1 = 2 \in \{0,1,2\}$, $(1 \ast 1) \ast (2 \ast 1) = 0 \ast 2 = 3 \notin \{0,1,2\}$. Hence $\tilde{H}$ is neither a falling subalgebra nor a falling translation ideal of $X$.

(2) Let $X := \{0,1,2\}$ be a $BH$-algebra([5]) with the following table:

$$
\begin{array}{c|ccc}
* & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
2 & 2 & 1 & 0 \\
\end{array}
$$

For a probability space $(\Omega, A, P) = ([0,1], A, m)$, define a random set $\xi : [0,1] \rightarrow \mathcal{P}(X)$ as follows:

$$
\xi : \Omega \rightarrow \mathcal{P}(X), \ t \mapsto \begin{cases} 
\emptyset & \text{if } t \in [0,0.4), \\
\{0,1\} & \text{if } t \in [0.4,0.7), \\
X & \text{if } t \in [0.7,1]. 
\end{cases}
$$

Then $\xi(t)$ is a subalgebra of $X$ for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling subalgebra of $X$. If we take $t \in [0.4,0.7)$, then $\xi(t) = \{0,1\}$ is not an ideal of $X$ since $2 \ast 1 = 1, 1 \in \{0,1\}$ and $2 \notin \{0,1\}$. Hence $\tilde{H}$ is not a falling ideal of $X$.

(3) Let $X := \{0,1,2,3\}$ be a $BH$-algebra([5]) with the following table:

$$
\begin{array}{c|cccc}
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 3 \\
3 & 3 & 3 & 3 & 0 \\
\end{array}
$$

For a probability space $(\Omega, A, P) = ([0,1], A, m)$, define a random set $\xi : [0,1] \rightarrow \mathcal{P}(X)$ as follows:

$$
\xi : \Omega \rightarrow \mathcal{P}(X), \ t \mapsto \begin{cases} 
\emptyset & \text{if } t \in [0,0.2), \\
\{0,1\} & \text{if } t \in [0.2,0.7), \\
X & \text{if } t \in [0.7,1]. 
\end{cases}
$$

Then $\xi(t)$ is both a subalgebra and a translation ideal of $X$ for all $t \in [0,1]$. Hence $\tilde{H}$ is both a falling subalgebra and a falling translation ideal of $X$.

**Lemma 3.3 ([6,7]).** A fuzzy set $\mu$ in a $BH$-algebra $X$ is a fuzzy subalgebra(resp., fuzzy ideal, fuzzy strong ideal, fuzzy $n$-fold strong ideal, and fuzzy translation ideal) of $X$ if and only if for every $t \in [0,1]$, $\mu_t$ is either empty or a subalgebra(resp., ideal, strong ideal, $n$-fold strong ideal, and translation ideal) of $X$. 

Theorem 3.4. Let $X$ be a BH-algebra. Then every fuzzy ideal(resp., fuzzy subalgebra, fuzzy strong ideal, fuzzy $n$-fold strong ideal, and fuzzy translation ideal) of $X$ is a falling ideal(resp., falling subalgebra, falling strong ideal, falling $n$-fold strong ideal, and falling translation ideal) of $X$.

Proof. Let $\tilde{H}$ be any fuzzy ideal(resp., fuzzy subalgebra, fuzzy strong ideal, fuzzy $n$-fold strong ideal, and fuzzy translation ideal) of $X$. By Lemma 3.3, $\tilde{H}_t$ is an ideal(resp., subalgebra, strong ideal, $n$-fold strong ideal, and translation ideal) of $X$ for all $t \in [0, 1]$. Let $\xi(t) : [0, 1] \to \mathcal{P}(X)$ be a random set and $\xi(t) = \tilde{H}_t$. Then $\tilde{H}$ is a falling ideal(resp., falling subalgebra, falling strong ideal, falling $n$-fold strong ideal, and falling translation ideal) of $X$. \hfill $\square$

The converse of Theorem 3.4 is not true in general as seen in the following example.

Example 3.5. Let $X \coloneqq \{0, 1, 2, 3, 4\}$ be a $BH$-algebra([2]) with the following table:

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</table>

For a probability space $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, define a random set $\xi : [0, 1] \to \mathcal{P}(X)$ as follows:

$$\xi : \Omega \to \mathcal{P}(X), \ t \mapsto \begin{cases} 
\{0, 1\} & \text{if } t \in [0, 0.2), \\
\{0, 2\} & \text{if } t \in [0.2, 0.5), \\
\{0, 3, 4\} & \text{if } t \in [0.5, 0.8) \\
X & \text{if } t \in [0.8, 1].
\end{cases}$$

Then $\xi(t)$ is a subalgebra of $X$ for all $t \in [0, 1]$ and

$$\tilde{H}(x) = \begin{cases} 
1 & \text{if } x = 0, \\
0.4 & \text{if } x = 1, \\
0.5 & \text{if } x = 2, \\
0.5 & \text{if } x = 3, \\
0.5 & \text{if } x = 4.
\end{cases}$$

Hence $\tilde{H}$ is a falling subalgebra of $X$, but not a fuzzy subalgebra of $X$ since $\tilde{H}(3 \ast 2) = \tilde{H}(1) = 0.4 \not\leq 0.5 = \min\{\tilde{H}(3), \tilde{H}(2)\}$. 
For a probability space \((\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)\), define a random set \(\eta : [0, 1] \to \mathcal{P}(X)\) as follows:

\[
\eta : \Omega \to \mathcal{P}(X), \quad t \mapsto \begin{cases} 
\{0\} & \text{if } t \in [0, 0.2), \\
\emptyset & \text{if } t \in [0.2, 0.3), \\
\{0, 1\} & \text{if } t \in [0.3, 0.5), \\
\{0, 2\} & \text{if } t \in [0.5, 0.8), \\
X & \text{if } t \in [0.8, 1].
\end{cases}
\]

Then \(\eta(t)\) is an ideal and a subalgebra of \(X\) for all \(t \in [0, 1]\) and

\[
\tilde{H}(x) = \begin{cases} 
0.9 & \text{if } x = 0, \\
0.4 & \text{if } x = 1, \\
0.5 & \text{if } x = 2, \\
0.2 & \text{if } x = 3, \\
0.2 & \text{if } x = 4.
\end{cases}
\]

Hence \(\tilde{H}\) is a falling ideal and a falling subalgebra of \(X\), but not a fuzzy ideal of \(X\) since \(\tilde{H}(3) = 0.2 \not\leq 0.4 = \min\{\tilde{H}(3 \ast 2), \tilde{H}(2)\}\).

**Proposition 3.6.** In a BH*-algebra \(X\), every falling ideal of \(X\) is a falling subalgebra of \(X\).

**Proof.** Let \(\tilde{H}\) be a falling ideal of a BH*-algebra \(X\). Then \(\xi(\omega)\) is an ideal of \(X\) for any \(\omega \in \Omega\) with \(\xi(\omega) \neq \emptyset\). Let \(x, y \in X\) be such that \(x, y \in \xi(\omega)\). Since \((x \ast y) \ast x = 0\) for any \(x, y \in X\), we have \((x \ast y) \ast x = 0 \in \xi(\omega)\). It follows from (12) that \(x \ast y \in \xi(\omega)\). Hence \(\xi(\omega)\) is a subalgebra of \(X\). Thus \(\tilde{H}\) is a falling subalgebra of \(X\). \(\square\)

In a BH-algebra \(X\), Proposition 3.6 is not true in general (see Example 3.2(1)).

**Theorem 3.7.** In a BH-algebra, every falling n-fold strong ideal is a falling ideal.

**Proof.** Let \(\tilde{H}\) be a falling n-fold strong ideal of a BH-algebra \(X\). Then \(\xi(\omega)\) is an n-fold strong ideal of \(X\) for any \(\omega \in \Omega\) with \(\xi(\omega) \neq \emptyset\). Let \(x, y, z \in X\) be such that \((x \ast y) \ast z^n \in \xi(\omega)\) and \(y \in \xi(\omega)\) for any positive integer \(n\). Putting \(z := 0\) and \(n := 1\) in the above statement, we have \(x \ast y = (x \ast y) \ast 0^1\) and \(y \in \xi(\omega)\). It follows from (15) that \(x = x \ast 0^1 \in \xi(\omega)\), i.e., \(\xi(\omega)\) is an ideal of \(X\). Therefore \(\tilde{H}\) is a falling ideal of \(X\). \(\square\)

**Corollary 3.8.** In a BH-algebra, every falling strong ideal is a falling ideal.

**Proof.** Put \(n := 1\) in Theorem 3.7. \(\square\)
The converse of Corollary 3.8 is not true in general as seen in the following example.

**Example 3.9.** Let \( X := \{0, a, b, c, d\} \) be a BH-algebra([2]) with the following table:

\[
\begin{array}{c|ccccc}
* & 0 & a & b & c & d \\
\hline
0 & 0 & 0 & 0 & 0 & d \\
a & a & 0 & a & 0 & 0 \\
b & b & b & 0 & 0 & 0 \\
c & c & c & a & 0 & 0 \\
d & d & c & d & c & 0 \\
\end{array}
\]

For a probability space \((\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)\), define a random set \( \xi : [0,1] \to \mathcal{P}(X) \) as follows:

\[
\xi : \Omega \to \mathcal{P}(X), \ t \mapsto \begin{cases} 
\emptyset & \text{if } t \in [0, 0.4), \\
\{0, a\} & \text{if } t \in [0.4, 1]. 
\end{cases}
\]

Then \( \xi(t) \) is a subalgebra and an ideal of \( X \) for all \( t \in [0,1] \). Hence \( \tilde{H} \) is a falling subalgebra and a falling ideal of \( X \). If we take \( t \in [0, 0.4) \), then \( \xi(t) = \{0, a\} \) is not a strong ideal of \( X \) since \((d * a) * b = a \in \{0, a\}, a \in \{0, a\} \) and \( d * b = d \notin \{0, a\} \).

**Therefore \( \tilde{H} \) is not a falling strong ideal of \( X \).**

**Corollary 3.10.** In a BH*-algebra, every falling \( n \)-fold strong ideal is a falling subalgebra.

**Proof.** It follow from Proposition 3.6 and Theorem 3.7. \( \square \)

The converse of Corollary 3.10 is not true in general as seen in the following example.

**Example 3.11.** Let \( X := \{0, a, b, c\} \) be a BH*-algebra([1]) with the following table:

\[
\begin{array}{c|ccccc}
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 \\
b & b & b & 0 & 0 \\
c & c & b & b & 0 \\
\end{array}
\]

For a probability space \((\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)\), define a random set \( \xi : [0,1] \to \mathcal{P}(X) \) as follows:

\[
\xi : \Omega \to \mathcal{P}(X), \ t \mapsto \begin{cases} 
\emptyset & \text{if } t \in [0, 0.3), \\
\{0, a, b\} & \text{if } t \in [0.3, 0.8), \\
X & \text{if } t \in [0.8, 1]. 
\end{cases}
\]
Then $\xi(t)$ is an $n$-fold strong ideal of $X$ for all $t \in [0, 1]$ and for every positive integer $n$. Hence $\tilde{H}$ is a falling $n$-fold strong ideal of $X$ for every positive integer $n$.

Define a random set $\xi : [0, 1] \to \mathcal{P}(x)$ as follows:

$$
\xi : \Omega \to \mathcal{P}(X), \ t \mapsto \begin{cases} 
\{0, c\} & \text{if } t \in [0, 0.3), \\
\{0, b\} & \text{if } t \in [0.3, 0.8), \\
X & \text{if } t \in [0.8, 1].
\end{cases}
$$

Then $\xi(t)$ is a subalgebra of $X$ for all $t \in [0, 1]$. Hence $\tilde{H}$ is a falling subalgebra of $X$. If we take $t \in [0.3, 0.8)$, then $\xi(t) = \{0, b\}$ is not an $n$-fold strong ideal of $X$ since $(c * b) * 0^n = b * 0^n = b \in \{0, b\}$ and $c * 0^n = c \notin \{0, b\}$. Thus $\tilde{H}$ is not a falling $n$-strong ideal of $X$ for every positive integer $n$.

**Theorem 3.12.** Let $X$ be a BH-algebra. Assume that the falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathcal{P}(X)$ is a falling subalgebra of $X$. Then $\tilde{H}$ is a falling $n$-fold strong ideal of $X$ if and only if for each $\omega \in \Omega$, the following is valid:

$$(3.1) \quad (\forall x \in \xi(\omega))(\forall y, z \in X)(y * z^n \notin \xi(\omega) \Rightarrow (y * x) * z^n \notin \xi(\omega)).$$

**Proof.** Suppose that $\tilde{H}$ is a falling $n$-fold strong ideal of a BH-algebra $X$. Then $\xi(\omega)$ is an $n$-fold strong ideal of $X$ for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Let $x, y, z \in X$ with $x \in \xi(\omega)$ and $y * z^n \notin \xi(\omega)$. If $(y * x) * z^n \in \xi(\omega)$, then $y * z^n \in \xi(\omega)$ since $\xi(\omega)$ is an $n$-fold strong ideal of $X$. This is a contradiction. Thus $(y * x) * z^n \notin \xi(\omega)$ for all positive integer $n$.

Conversely, let $\tilde{H}$ be a falling subalgebra of $X$ satisfying (3.1). Then $\xi(\omega)$ is a subalgebra of $X$ for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence $0 \in \xi(\omega)$. Let $x, y, z \in X$ be such that $(y * x) * z^n \in \xi(\omega)$ and $x \in \xi(\omega)$. If $y * z^n \notin \xi(\omega)$, then $(y * x) * z^n \notin \xi(\omega)$ by (3.1). This is a contradiction and so $\tilde{H}$ is a falling $n$-fold strong ideal of $X$. □

**Corollary 3.13.** Let $X$ be a BH-algebra. Assume that the falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathcal{P}(X)$ is a falling subalgebra of $X$. Then $\tilde{H}$ is a falling strong ideal of $X$ if and only if for each $\omega \in \Omega$, the following is valid:

$$(\forall x \in \xi(\omega))(\forall y, z \in X)(y * z \notin \xi(\omega) \Rightarrow (y * x) * z \notin \xi(\omega)).$$

**Proof.** Put $n := 1$ in Theorem 3.12. □

**Corollary 3.14.** Let $X$ be a BH-algebra. Assume that the falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathcal{P}(X)$ is a falling subalgebra of $X$. Then $\tilde{H}$ is a falling ideal of $X$ if and only if for each $\omega \in \Omega$, the following is valid:
(\forall x \in \xi(\omega))(\forall y \in X)(y \notin \xi(\omega) \Rightarrow y \ast x \notin \xi(\omega)).

**Proof.** Put \( z := 0 \) in Corollary 3.13. \( \square \)

Let \( (\Omega, A, P) \) be a probability space and \( \tilde{H} \) a falling shadow of a random set \( \xi : \Omega \rightarrow \mathcal{P}(X) \). For any \( x \in X \), let

(3.2) \[ \Omega(x; \xi) := \{ \omega \in \Omega \mid x \in \xi(\omega) \}. \]

Then \( \Omega(x; \xi) \in A \).

**Lemma 3.15.** If \( \tilde{H} \) is a falling subalgebra of a BH-algebra \( X \), then

(3.3) \[ (\forall x \in X)(\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x \ast y; \xi)). \]

**Proof.** If \( \Omega(x; \xi) = \emptyset \), then it is clear. Assume that \( \Omega(x; \xi) \neq \emptyset \) and let \( \omega \in \Omega \) be such that \( \omega \in \Omega(x; \xi) \). Then \( x \in \xi(\omega) \), and so \( 0 = x \ast x \in \xi(\omega) \) since \( \xi(\omega) \) is a subalgebra of \( X \). Hence \( \omega \in \Omega(0; \xi) \), and therefore \( \Omega(x; \xi) \subseteq \Omega(0; \xi) \) for all \( x \in X \). \( \square \)

Combing Proposition 3.6 and Lemma 3.15, we have the following corollary.

**Corollary 3.16.** If \( \tilde{H} \) is a falling ideal of a BH*-algebra \( X \), then (3.3) is valid.

**Theorem 3.17.** If \( \tilde{H} \) is a falling subalgebra of a BH-algebra \( X \), then

(\forall x, y \in X)(\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x \ast y; \xi)).

**Proof.** Let \( \omega \in \Omega(x; \xi) \cap \Omega(y; \xi) \) for any \( x, y \in X \). Then \( x \in \xi(\omega) \) and \( y \in \xi(\omega) \). Since \( \xi(\omega) \) is a subalgebra of \( X \), \( x \ast y \in \xi(\omega) \). Hence \( \omega \in \Omega(x \ast y, \xi) \). Thus \( \Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x \ast y; \xi) \). \( \square \)

**Theorem 3.18.** If \( \tilde{H} \) is a falling ideal of a BH-algebra \( X \), then

(i) (\forall x, y \in X)(x \leq y \Rightarrow \Omega(y; \xi) \subseteq \Omega(x; \xi)).

(ii) (\forall x, y \in X)(\Omega(x \ast y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)).

**Proof.** (i) Let \( x, y \in X \) with \( x \leq y \) and \( \omega \in \Omega(y; \xi) \). Then \( y \in \xi(\omega) \) and \( 0 = x \ast y \in \xi(\omega) \). Since \( \xi(\omega) \) is an ideal of \( X \), \( x \in \xi(\omega) \), i.e., \( \omega \in \Omega(x; \xi) \). Hence (i) holds.

(ii) Let \( \omega \in \Omega(x \ast y; \xi) \cap \Omega(y; \xi) \) for any \( x, y \in X \). Then \( x \ast y \in \xi(\omega) \) and \( y \in \xi(\omega) \). Since \( \xi(\omega) \) is an ideal of \( X \), \( x \in \xi(\omega) \). Hence \( \omega \in \Omega(x; \xi) \). Thus (ii) holds. \( \square \)

**Theorem 3.19.** If \( \tilde{H} \) is a falling \( n \)-fold strong ideal of a BH-algebra \( X \), then

(i) (\forall x, y, z \in X)(x \ast y \leq z^n \Rightarrow \Omega(y; \xi) \subseteq \Omega(x \ast z^n; \xi)),

(ii) (\forall x, y, z \in X)(\Omega((x \ast y) \ast z^n; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x \ast z^n; \xi))

for any positive integer \( n \).
Proof. (i) Let \( x, y, z \in X \) with \( \omega \in \Omega(y; \xi) \) and \( x * y \leq z^n \) for any integer \( n \). Then \( y \in \xi(\omega) \) and \( (x * y) * z^n = 0 \in \xi(\omega) \). Since \( \xi(\omega) \) is an \( n \)-fold strong ideal of \( X \), we have \( x * z^n \in \xi(\omega) \). Hence \( \omega \in \Omega(x * z^n; \xi) \). Thus (i) holds.

(ii) Let \( x, y, z \in X \) be such that \( \omega \in \Omega((x * y) * z^n; \xi) \cap \Omega(y; \xi) \). Then \( (x * y) * z^n \in \xi(\omega) \) and \( y \in \xi(\omega) \). Since \( \xi(\omega) \) is an \( n \)-fold strong ideal of \( X \), we have \( x * z^n \in \xi(\omega) \), i.e., \( \omega \in \Omega(x * z^n; \xi) \). Thus (ii) holds. \( \square \)

**Corollary 3.20.** If \( \tilde{H} \) is a falling strong ideal of a BH-algebra \( X \), then

(i) \( (\forall x, y, z \in X) (x * y \leq z \Rightarrow \Omega(y; \xi) \subseteq \Omega(x * z; \xi)) \).

(ii) \( (\forall x, y, z \in X) (\Omega((x * y) * z; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x * z; \xi)) \).

Proof. Since the 1-fold strong ideal is precisely a strong ideal, these two conditions hold by Theorem 3.19. \( \square \)

**Theorem 3.21.** If \( \tilde{H} \) is a falling translation ideal of a BH-algebra \( X \), then

(i) \( (\forall x, y, z \in X) (x \leq y \Rightarrow \Omega(y * x; \xi) \subseteq \Omega((x * z) * (y * z); \xi) \cap \Omega((z * x) * (z * y); \xi)) \).

(ii) \( (\forall x, y, z \in X) (\Omega(x * y; \xi) \cap \Omega(y * x; \xi) \subseteq \Omega((x * z) * (y * z); \xi) \cap \Omega((z * x) * (z * y); \xi)) \).

Proof. (i) Let \( x, y, z \in X \) be such that \( \omega \in \Omega(y * x; \xi) \) and \( x \leq y \). Then \( y * x \in \xi(\omega) \) and \( 0 = x * y \in \xi(\omega) \). Since \( \xi(\omega) \) is a translation ideal of \( X \), we have \( (x * z) * (y * z) \in \xi(\omega) \) and \( (z * x) * (z * y) \in \xi(\omega) \). Hence \( \omega \in \Omega((x * z) * (y * z); \xi) \cap \Omega((z * x) * (z * y); \xi) \). Hence (i) holds.

(ii) Let \( x, y, z \in X \) be such that \( \omega \in \Omega(x * y; \xi) \cap \Omega(y * x; \xi) \). Then \( x * y \in \xi(\omega) \) and \( y * x \in \xi(\omega) \). Since \( \xi(\omega) \) is a translation ideal of \( X \), we have \( (x * z) * (y * z) \in \xi(\omega) \) and \( (z * x) * (z * y) \in \xi(\omega) \). Hence \( \omega \in \Omega((x * z) * (y * z); \xi) \cap \Omega((z * x) * (z * y); \xi) \). Thus (ii) holds. \( \square \)

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**References**


a Department of Mathematics Education, Dongguk University, Seoul 100-715, Korea
Email address: duchil@hanmail.net

b Department of Mathematics Education, Dongguk University, Seoul 100-715, Korea
Email address: sunshine@dongguk.edu