AN EXTENSION WHICH IS A WEAKLY LINDELÖFF SPACE

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Abstract. In this paper, we construct an extension \((kX, kX)\) of a space \(X\) such that \(kX\) is a weakly Lindelöff space and for any continuous map \(f : X \rightarrow Y\), there is a continuous map \(g : kX \rightarrow kY\) such that \(g|_X = f\). Moreover, we show that \(\nu X\) is Lindelöff if and only if \(kX = \nu X\) and that for any \(P'\)-space \(X\) which is weakly Lindelöff, \(kX = \nu X\).

1. Introduction

All spaces in this paper are assumed to be Tychonoff spaces and \(\beta X(\nu X, \text{resp.})\) denotes the Stone-Čech compactification (the Hewitt realcompactification, resp.) of a space \(X\).

One of the many characterizations of \((\beta X, \beta X)\) is following:

1. \(\beta X\) is a compact space, and
2. for any continuous map \(f : X \rightarrow Y\), there is a continuous map \(f^\beta : \beta X \rightarrow \beta Y\) such that \(f^\beta|_X = f\) ([5]).

There have been many ramifications from the Stone-Čech compactifications of spaces. In fact, realcompactifications of spaces and zero-dimensional compactifications of zero-dimensional spaces have been studied by various authors ([3], [5]).

The purpose to write this paper is to construct an extension of a space which has similar properties to the above extensions. We first construct an extension \((kX, kX)\) of a space \(X\) such that \(\nu X \subseteq kX \subseteq \beta X\) and \(kX\) is a weakly Lindelöff space. We show that for any continuous map \(f : X \rightarrow Y\), there is a continuous map \(g : kX \rightarrow kY\) such that \(g|_X = f\). Blasco ([1], [2]) showed that for a paracompact (or separable) space \(X\), \(\nu X\) is a Lindelöff space if and only if every separating nest generated intersection ring on \(X\) is complete. We show that \(\nu X\) is Lindelöff if and

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only if $kX = vX$. Using these, we then show that $kX = X$ if and only if $X$ is Lindelöf. Finally, we will show that for any $P'$-space $X$ which is weakly Lindelöf, $kX = vX$.

For the terminology, we refer to [3] and [5].

2. AN EXTENSION WHICH IS A WEAKLY LINDELÖFF SPACE

For any space $X$, let $Z(X)$ be the set of all zero-sets in $X$. A $Z(X)$-filter is called a $z$-filter on $X$.

**Definition 2.1.** Let $X$ be a space and $F$ a $z$-filter on $X$. Then $F$ is called

1. real if it has the countable intersection property, and
2. free (fixed, resp.) if $\cap \{F \mid F \in F\} = \emptyset$ (or $\cap \{F \mid F \in F\} \neq \emptyset$, resp.).

A space $X$ is called a realcompact space if every real $z$-ultrafilter on $X$ is fixed. It is known that for any real $z$-ultrafilter $F$ on a space $X$, $\cap \{cl_x^X(F) \mid F \in F\} \neq \emptyset$ ([3]).

Let $X$ be a space and $kX = vX \cup \{p \in \beta X - vX \mid$ there is a real $z$-filter $F$ on $X$ such that $\cap \{cl_x^X(F) \mid F \in F\} = \emptyset$ and $p \in \cap \{cl_{\beta X}(F) \mid F \in F\}\}$. Let $X$ be a set and $F \subseteq P(X)$. For any $A \subseteq X$, let $F_A$ denote the set $\{F \cap A \mid F \in F\}$. **Proposition 2.2.** Let $X$ be a space. Then we have the following:

1. $vX \subseteq kX \subseteq \beta X$,
2. $k(vX) = kX$, and
3. $kX$ is realcompact if for any non-empty zero-set $Z$ in $kX$, $Z \cap X \neq \emptyset$.

**Proof.** (1) It is trivial.

(2) Let $p \in kX - vX$. Then there is a real $z$-filter $F$ on $X$ such that $\cap \{cl_x^X(F) \mid F \in F\} = \emptyset$ and $p \in \cap \{cl_{\beta X}(F) \mid F \in F\}$. Let $F_p = \{cl_x^X(F) \mid F \in F\}$. Note that for any zero-set $Z$ in $X$, $cl_x^X(Z)$ is a zero-set in $vX$ and for any sequence $(Z_n)$ in $Z(X)$, $cl_x^X(\cap \{Z_n \mid n \in N\}) = \cap \{cl_x^X(Z_n) \mid n \in N\}$ ([3]). Hence $F_p$ is a real $z$-filter $F$ on $vX$. Note that $\cap \{cl_x^X(H) \mid H \in F_p\} = \{cl_x^X(F) \mid F \in F\} = \emptyset$ and $p \in \cap \{cl_{\beta X}(H) \mid H \in F_p\} = \cap \{cl_{\beta X}(F) \mid F \in F\}$. Since $v(vX) = vX$ and $\beta (vX) = \beta X, p \in k(vX)$. Hence $kX \subseteq k(vX)$.

Let $q \in k(vX)$ and $q \notin vX$. Since $v(vX) = vX$, there is a real $z$-filter $G$ on $vX$ such that $\cap \{G \mid G \in G\} = \emptyset$ and $q \in \cap \{cl_{\beta X}(G) \mid G \in G\}$. Then $G$ is a real $z$-filter on $X$ and $\cap \{cl_x^X(H) \mid H \in G_X\} = \cap \{G \mid G \in G\} = \emptyset$. Since $q \in \cap \{cl_{\beta X}(H) \mid H \in G_X\} = \cap \{cl_{\beta X}(G) \mid G \in G\}, q \in kX$. Hence $k(vX) \subseteq kX$.

(3) Take any real $z$-ultrafilter $F$ on $kX$. By the assumption, for any $F \in F$,
Let $F \cap X \neq \emptyset$ and so $\mathcal{F}_X$ is a z-filter on $X$. Let $Z$ be a zero-set in $X$ such that for any $F \in \mathcal{F}$, $Z \cap F \neq \emptyset$. Since $X \subseteq kX \subseteq \beta X$, there is a zero-set $B$ in $kX$ such that $Z = B \cap X$. Then for any $F \in \mathcal{F}$, $F \cap B \neq \emptyset$. Since $\mathcal{F}$ is a z-ultrafilter on $kX$, $B \in \mathcal{F}$ and $B \cap X = Z \in \mathcal{F}_X$. Hence $\mathcal{F}_X$ is a z-ultrafilter on $X$. Since $\mathcal{F}_X$ is real, $\cap \{cl_vX(F \cap X) \mid F \in \mathcal{F}\} = \{q\}$ for some $q \in vX$. Note that $\cap \{cl_vX(F \cap X) \mid F \in \mathcal{F}\} = \cap \{cl_vX(F \cap vX) \mid F \in \mathcal{F}\}$ and for any $F \in \mathcal{F}$, $cl_vX(F \cap vX) \subseteq F$. Hence $q \in \cap \{F \mid F \in \mathcal{F}\}$ and so $\cap \{F \mid F \in \mathcal{F}\} \neq \emptyset$. Thus $kX$ is a realcompact space.

Let $S$ be a subspace of a space $X$. Then $S$ is called $C(C^*, \text{ resp.})$-embedded in $X$ if for any real-valued (bounded, resp.) continuous function $f$ on $S$, there is a real-valued (bounded, resp.) continuous function $g$ on $X$ such that $g|_S = f$.

Note that $X$ is a dense $C$-embedded subspace of $Y$ if and only if $X \subseteq Y \subseteq vX$, equivalently, $vX = vY$ and that a dense subspace $X$ of a space $Y$ is $C^*$-embedded in $Y$ if and only if $\beta X = \beta Y$ ([3]). Using these, we have the following:

**Proposition 2.3.** Let $X$ be a dense $C$-embedded subspace of $Y$. Then $kX = kY$.

**Proof.** Since $X$ is a dense $C$-embedded subspace of $Y$, $vX = vY([3])$. Let $p \in kX - vX$. Then there is a real $z$-filter $\mathcal{F}$ on $X$ such that $\cap \{cl_vX(F) \mid F \in \mathcal{F}\} = \emptyset$ and $p \in \cap \{cl_{\beta X}(F) \mid F \in \mathcal{F}\}$. Let $G = \{G \in Z(Y) \mid G \cap X \in \mathcal{F}\}$. Then $G_X = \mathcal{F}$ and since $vX = vY$, $G$ is a real $z$-filter on $Y$.

Let $G \in G$ and $x \in vX - cl_vX(G \cap X)$. Then there is a zero-set neighborhood $Z$ of $x$ in $vX$ such that $G \cap Z \cap X = \emptyset$. Since $X \subseteq Y \subseteq vX$, there is a zero-set $H$ in $vX$ such that $G = H \cap Y$. Since $H \cap Z \cap X = \emptyset$ and $H \cap Z$ is a zero-set in $vX$, $H \cap Z = \emptyset([5])$. Hence $G \cap Z = \emptyset$ and $x \notin cl_vX(G)$. Thus $cl_vX(G) \subseteq cl_vX(G \cap X)$. Clearly, $cl_vX(G \cap X) \subseteq cl_vX(G)$ and so $cl_vX(G \cap X) = cl_vX(G)$.

Since $\cap \{cl_vX(G \cap X) \mid G \in G\} = \emptyset$, $\cap \{cl_{\beta Y}(G) \mid G \in G\} = \emptyset$. Since $X$ is $C^*$-embedded in $Y$, $\beta X = \beta Y$ and $p \in \cap \{cl_{\beta Y}(G) \mid G \in G\}$. Hence $p \in kY$ and so $kX \subseteq kY$.

Similarly, we have $kY \subseteq kX$.

For any space $X$, let $k_X : X \rightarrow kX$ denote the inclusion map. Then $(kX, k_X)$ is an extension of $X$.

Note that for any continuous map $f : X \rightarrow Y$, there is a unique continuous map $f^v : vX \rightarrow vY$ such that $f^v |_X = f$.

**Proposition 2.4.** Let $f : X \rightarrow Y$ be a continuous map. Then there is a unique continuous map $g : kX \rightarrow kY$ such that $g \circ k_X = k_Y \circ f$. 

AN EXTENSION WHICH IS A WEAKLY LINDELÖFF SPACE
Proof. Note that there is a continuous map \( h : \beta X \to \beta Y \) such that \( h \circ \beta_X = \beta_Y \circ f \) and \( h(vX) \subseteq vY \). Let \( p \in kX - vX \). Then there is a real \( z \)-filter \( F \) on \( X \) such that \( \cap \{ \text{cl}_v(X)\} F \subseteq F \) and \( p \in \cap \{ \text{cl}_\beta_X(F)\} F \subseteq F \}. \) Since \( F \) is a real \( z \)-filter on \( X \), \( \mathcal{G} \) is a real \( z \)-filter on \( Y \). Let \( G \subseteq \mathcal{G} \). Then \( h^{-1}(G) \subseteq \mathcal{F} \). Since \( p \in \text{cl}_\beta_X(h^{-1}(G)) \subseteq \text{cl}_\beta_Y(h^{-1}(G)) \subseteq \text{cl}_\beta_Y(G) \). Hence \( p \in \cap \{ \text{cl}_\beta_X(G)\} G \subseteq \mathcal{G} \) and so \( h(p) \in kY \).

Let \( g : kX \to kY \) be the restriction and corestriction of \( h \) with respect to \( kX \) and \( kY \), respectively. Then \( g : kX \to kY \) is a continuous map and \( g \circ kX = kY \circ f \). Since \( kX : X \to kX \) is a dense embedding, such an \( g \) is unique. \( \square \)

It is well-known that a space \( X \) is Lindel"off if and only if for any real \( z \)-filter \( F \) in \( X \), \( \cap \{ F \mid F \in \mathcal{F} \} \neq \emptyset \).

**Proposition 2.5.** Let \( X \) be a space. Then the following are equivalent:

1. \( vX = kX \),
2. \( vX \) is a Lindel"off space,
3. for any free real \( z \)-filter \( F \) on \( X \), \( \cap \{ \text{cl}_v(X)\} F \subseteq F \} \neq \emptyset \), and
4. for any free real \( z \)-filter \( F \) on \( X \), there is a free real \( z \)-ultrafilter \( \mathcal{A} \) on \( X \) such that \( \mathcal{F} \subseteq \mathcal{A} \).

Proof. (1) \( \Rightarrow \) (2) Take any real \( z \)-filter \( \mathcal{G} \) on \( vX \). Then \( \mathcal{G}_X \) is a real \( z \)-filter on \( X \). Suppose that \( \cap \{ G \cap X \mid G \in \mathcal{G} \} = \emptyset \). Then \( \cap \{ \text{cl}_\beta_X(G \cap X) \mid G \in \mathcal{G} \} \neq \emptyset \). Pick \( p \in \cap \{ \text{cl}_\beta_X(G \cap X) \mid G \in \mathcal{G} \} \). Then \( p \in kX \) and since \( kX = vX \), \( p \in vX \). Hence \( p \in \cap \{ \text{cl}_\beta_X(G \cap X) \mid G \in \mathcal{G} \} \) \( vX = \cap \{ G \mid G \in \mathcal{G} \} \) and so \( \cap \{ G \mid G \in \mathcal{G} \} \neq \emptyset \). Thus \( vX \) is a Lindel"off space.

(2) \( \Rightarrow \) (3) It is trivial.

(3) \( \Rightarrow \) (4) Let \( \mathcal{F} \) be a free real \( z \)-filter on \( X \). By the assumption, \( \cap \{ \text{cl}_v(X)\} F \subseteq F \} \neq \emptyset \). Pick \( p \in \cap \{ \text{cl}_v(X)\} F \subseteq F \} \}. \) Let \( \mathcal{A}_p = \{ A \in Z(X) \mid p \in \text{cl}_v(X)\} \}). \) Then \( \mathcal{A}_p \) is a free real \( z \)-ultrafilter on \( X \) and \( \mathcal{F} \subseteq \mathcal{A}_p \).

(4) \( \Rightarrow \) (1) Let \( p \in kX - vX \). Then there is a real \( z \)-filter \( F \) on \( X \) such that \( \cap \{ \text{cl}_v(X)\} F \subseteq F \} \neq \emptyset \) and \( p \in \cap \{ \text{cl}_\beta_X(G \cap X) \mid G \in \mathcal{G} \} \). Since \( \mathcal{F} \) is free, by (4), there is a free real \( z \)-ultrafilter \( \mathcal{A} \) on \( X \) such that \( \mathcal{F} \subseteq \mathcal{A} \). Since \( \cap \{ \text{cl}_v(X)\} A \subseteq A \} \neq \emptyset \), \( \cap \{ \text{cl}_v(X)\} F \subseteq F \} \neq \emptyset \) and this is a contradiction. \( \square \)

By Proposition 2.2. and Proposition 2.5., we have the following:

**Corollary 2.6.** Let \( X \) be a space. Then \( kX = X \) if and only if \( X \) is Lindel"off.

Recall that a space \( X \) is called a pseudo-compact space if every real-valued
continuous function on $X$ is bounded, equivalently, $vX = \beta X$.

**Corollary 2.7.** If $X$ is a pseudo-compact space, then $kX = \beta X$.

Let $X$ be a space. The collection $\mathcal{R}(X)$ of all regular closed sets in $X$, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows: For any $A \in \mathcal{R}(X)$ and any $\mathcal{F} \subseteq \mathcal{R}(X)$,

\[ \bigvee \mathcal{F} = \text{cl}_X((\bigcup \{F \mid F \in \mathcal{F}\}), \quad \bigwedge \mathcal{F} = \text{cl}_X(\text{int}_X(\bigcap \{F \mid F \in \mathcal{F}\})), \quad A' = \text{cl}_X(X - A). \]

A sublattice of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$, $X$ and is closed under finite joins and finite meets ([7]).

An $\mathcal{R}(X)$-filter $\mathcal{A}$ is said to have the countable meet property if for any sequence $(A_n)$ in $\mathcal{R}(X)$, $\bigwedge \{A_n \mid n \in \mathbb{N}\} \neq \emptyset$.

Let $Z(X)^\# = \{\text{cl}_X(\text{int}_X(A)) \mid A \in Z(X)\}$. Then $Z(X)^\#$ is a sublattice of $\mathcal{R}(X)$.

A space $X$ is called a weakly Lindelöf space if for any open cover $\mathcal{U}$ of $X$, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\cup \{V \mid V \in \mathcal{V}\}$ is dense in $X$.

A space $X$ is a weakly Lindelöf space if and only if for any $Z(X)^\#$-filter $\mathcal{A}$ with the countable meet property, $\cap \{A \mid A \in \mathcal{A}\} \neq \emptyset$ ([4]).

**Theorem 2.8.** Let $X$ be a space. Then $kX$ is a weakly Lindelöf space.

**Proof.** Take any $Z(X)^\#$-filter $\mathcal{U}$ on $kX$ with the countable meet property. Let $\mathcal{F} = \{Z \in Z(kX) \mid \text{cl}_X(\text{int}_X(Z)) \in \mathcal{U}\}$. Clearly, $\emptyset \notin \mathcal{F} \neq \emptyset$. For any $A, B \in \mathcal{F}$, $\text{cl}_X(\text{int}_X(A \cap B)) = \text{cl}_X(\text{int}_X(A)) \cap \text{cl}_X(\text{int}_X(B)) \in \mathcal{U}$ and hence $A \cap B \in \mathcal{F}$. Thus $\mathcal{F}$ is a $z$-filter on $kX$. By the definition of $\mathcal{F}$, for any $F \in \mathcal{F}$, $F \cap X \neq \emptyset$. Hence $\mathcal{F}_X$ is also a $z$-filter on $X$. Let $(A_n)$ be a sequence in $\mathcal{F}_X$. For any $n \in \mathbb{N}$, there is a $B_n \in \mathcal{F}$ such that $A_n = B_n \cap X$. Since $\mathcal{U}$ has the countable meet property, $\text{cl}_X(\text{int}_X(\cap \{B_n \mid n \in \mathbb{N}\})) \neq \emptyset$ and since $X$ is dense in $kX$, $\text{cl}_X(\text{int}_X(\cap \{B_n \mid n \in \mathbb{N}\})) \cap X \neq \emptyset$. Since $\text{cl}_X(\text{int}_X(\cap \{B_n \mid n \in \mathbb{N}\})) \cap X = \text{cl}_X(\text{int}_X(\cap \{B_n \cap X \mid n \in \mathbb{N}\})) = \text{cl}_X(\text{int}_X(\cap \{B_n \mid n \in \mathbb{N}\})) = \text{cl}_X(\text{int}_X(\cap \{A_n \mid n \in \mathbb{N}\}))$, $\cap \{A_n \mid n \in \mathbb{N}\} \neq \emptyset$ and so $\mathcal{F}_X$ has the countable intersection property. Note that $\cap \{\text{cl}_{vX}(F \cap X) \mid F \in \mathcal{F}\} \neq \emptyset$. Pick $x \in \cap \{\text{cl}_{vX}(F \cap X) \mid F \in \mathcal{F}\}$. Let $U \in \mathcal{U}$. Suppose that $x \notin U$. Since $\mathcal{U}$ is a closed set in $kX$, there is a zero-set $Z$ in $kX$ such that $x \notin Z$ and $U \subseteq Z$. Then $Z \cap X \in \mathcal{F}_X$ and since $\text{cl}_{vX}(Z \cap X) = Z \cap vX$, $\text{cl}_{vX}(Z \cap X) = Z \cap vX$. AN EXTENSION WHICH IS A WEAKLY LINDELÖFF SPACE 277
since \( cl_{\nu X}(Z \cap X) = Z \cap \nu X, \ x \in Z \). This is a contradiction and so \( x \in U \). Hence \( x \in \cap \{U \mid U \in \mathcal{U} \} \).

Assume that \( \cap \{cl_{\nu X}(F \cap X) \mid F \in \mathcal{F} \} = \emptyset \). Let \( p \in \cap \{cl_{\beta X}(F \cap X) \mid F \in \mathcal{F} \} \). Then \( p \in kX \). Let \( U \in \mathcal{U} \). Suppose that \( p \notin U \). Then there is a zero-set \( B \) in \( \beta X \) such that \( p \notin B \) and \( U \subseteq B \). Since \( B \cap X \in \mathcal{F}_X \), \( p \in cl_{\beta X}(B \cap X) \subseteq B \). This is a contradiction and so \( p \in U \). Hence \( p \in \cap \{U \mid U \in \mathcal{U} \} \).

Thus \( \cap \{U \mid U \in \mathcal{U} \} \neq \emptyset \) and \( kX \) is a weakly Lindelöf space.

A space \( X \) is called a \( P' \)-space if for any non-empty zero-set \( Z \) in \( X \), \( int_X(Z) \neq \emptyset \), equivalently, every zero-set in \( X \) is a regular closed set in \( X \). Clearly, a space \( X \) is a \( P' \)-space if and only if \( \nu X \) is a \( P' \)-space. If \( X \) is a realcompact and locally compact space, then \( \beta X - X \) is a \( P' \)-space ([6]).

**Proposition 2.9.** Let \( X \) be a \( P' \)-space. Then \( X \) is a weakly Lindelöf space if and only if \( X \) is a Lindelöf space.

**Proof.** Suppose that \( X \) is a weakly Lindelöf space. Let \( \mathcal{F} \) be a real \( z \)-filter on \( X \). Since \( X \) is a \( P' \)-space, \( Z(X) = Z(X)^\# \) and since \( Z(X) \) is closed under countable intersections, for any sequence \( (A_n) \) in \( Z(X) \),

\[
\bigwedge \{A_n \mid n \in N\} = cl_X(int_X(\cap \{A_n \mid n \in N\})) = \cap \{A_n \mid n \in N\}.
\]

Hence \( \mathcal{F} \) is a \( Z(X)^\# \)-filter with the countable meet property. Since \( X \) is a weakly Lindelöf space, \( \cap \{F \mid F \in \mathcal{F} \} \neq \emptyset \) and hence \( X \) is a Lindelöf space.

The converse is trivial.

A space with a dense weakly Lindelöf space is also a weakly Lindelöf space. Using this, Proposition 2.9. and Proposition 2.5., we have the following:

**Corollary 2.10.** For any \( P' \)-space \( X \) which is weakly Lindelöf, \( \nu X \) is a Lindelöf space and \( \nu X = kX \).

**References**


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