APPROXIMATING COMMON FIXED POINTS OF A SEQUENCE OF ASYMPTOTICALLY QUASI-\textit{f}-\textit{g}-NONEXPANSIVE MAPPINGS IN CONVEX NORMED VECTOR SPACES

BYUNG-SOO LEE

ABSTRACT. In this paper, we introduce asymptotically quasi-\textit{f}-\textit{g}-nonexpansive mappings in convex normed vector spaces and consider approximating common fixed points of a sequence of asymptotically quasi-\textit{f}-\textit{g}-nonexpansive mappings in convex normed vector spaces.

1. INTRODUCTION AND PRELIMINARIES

Now we introduce asymptotically quasi-\textit{f}-\textit{g}-nonexpansive mappings and asymptotically \textit{f}-\textit{g}-nonexpansive mappings with convex normed vector spaces.

Definition 1.1. Let $(X, \| \cdot \|)$ be a normed vector space, $T : (X, \| \cdot \|) \to (X, \| \cdot \|)$ be a self-mapping and $f, g : (X, \| \cdot \|) \to (0, \infty)$ be functions.

(i) $T$ is said to be asymptotically \textit{f}-\textit{g}-nonexpansive if there exist two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in $X$ such that

$$\lim_{n \to \infty} f(x_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} g(y_n) = 0$$

satisfying

$$\|T^n x - T^n y\| \leq f(x_n) \cdot \|x - y\| + g(y_n) \quad \text{for} \quad x, y \in X$$

(ii) $T$ is said to be asymptotically quasi-\textit{f}-\textit{g}-nonexpansive if there exist two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in $X$ such that

$$\lim_{n \to \infty} f(x_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} g(y_n) = 0$$
satisfying

$$\|T^n x - p\| \leq f(x_n) \cdot \|x - p\| + g(y_n) \quad \text{for} \quad p \in F(T) \quad \text{and} \quad x \in X,$$

where $F(T)$ is the set of fixed points of $T$.

**Example 1.1.** Let $(X, \| \cdot \|)$ be the 2-dimensional Euclidean normed vector space $(\mathbb{R}^2, \| \cdot \|)$, $T : (\mathbb{R}^2, \| \cdot \|) \to (\mathbb{R}^2, \| \cdot \|)$ be a self-mapping defined by $T((x_1, x_2)) = (\frac{1}{2}x_1, \frac{1}{2}x_2)$ for $(x_1, x_2) \in \mathbb{R}^2$ and $f, g : (\mathbb{R}^2, \| \cdot \|) \to (0, \infty)$ be two functions defined by $f((x_1, x_2)) = \frac{1}{x_1^2 + x_2^2}$, $g((x_1, x_2)) = x_1^2 + x_2^2$ for $x = (x_1, x_2) \in \mathbb{R}^2$, respectively. Take two sequences $\langle x_n \rangle = ((x_{1n}, x_{2n}))$ and $\langle y_n \rangle = ((y_{1n}, y_{2n}))$ in $\mathbb{R}^2$ such that $x_{1n} = \frac{1}{\sqrt{3}}$, $x_{2n} = \frac{\sqrt{2}}{\sqrt{3}}$ and $y_{1n} = \frac{1}{n}$, $y_{2n} = \frac{1}{2n}$ for $n \in \mathbb{N}$, respectively. Then $F(T) = \{(0,0)\}$. For $x = (x_1, x_2) \in \mathbb{R}^2$ and $p = (0,0) \in F(T)$, we have

$$\|T^n x - p\| = \left\| \left( \frac{1}{2^n}x_1, \frac{1}{3^n}x_2 \right) \right\| = \sqrt{\frac{1}{2^{2n}}x_1^2 + \frac{1}{3^{2n}}x_2^2},$$

$$f(x_n) \cdot \|x - p\| + g(y_n) = \frac{1}{\left( \frac{1}{\sqrt{3}} \right)^2 + \left( \frac{\sqrt{2}}{\sqrt{3}} \right)^2} \cdot \sqrt{x_1^2 + x_2^2} + \left( \frac{1}{n^2} + \frac{1}{(2n)^2} \right),$$

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \frac{1}{\left( \frac{1}{\sqrt{3}} \right)^2 + \left( \frac{\sqrt{2}}{\sqrt{3}} \right)^2} = 1,$$

$$\lim_{n \to \infty} g(y_n) = \lim_{n \to \infty} \left( \frac{1}{n^2} + \frac{1}{4n^2} \right) = 0.$$

Thus, we have

$$\|T^n x - p\| \leq f(x_n) \cdot \|x - p\| + g(y_n) \quad \text{for} \quad p \in F(T) \quad \text{and} \quad x \in X,$$

which shows that the mapping $T$ is asymptotically quasi-$f$-$g$-nonexpansive.

**Definition 1.2 ([1, 3]).** Let $(X, \| \cdot \|)$ be a normed vector space. A mapping $W : X^3 \times I^3 \to X$ is called a *convex structure* on $X$, if it satisfies the following condition:

For any $(x, y, z) \in X^3$ and $(a, b, c) \in I^3$ with $a + b + c = 1$,

$$\|W(x, y, z; a, b, c) - u\| \leq a \cdot \|x - u\| + b \cdot \|y - u\| + c \cdot \|z - u\|$$

for all $u \in X$, where $I = [0,1]$. 
A normed vector space $(X, \| \cdot \|)$ with a convex structure $W$ is called a \textit{convex normed vector space} and is denoted as $(X, \| \cdot \|, W)$. A nonempty subset $C$ of a convex normed vector space $(X, \| \cdot \|, W)$ is said to be a \textit{convex subset} of $(X, \| \cdot \|)$, if $W(x, y, z; a, b, c) \in C$ for $(x, y, z) \in C^3$ and $(a, b, c) \in I^3$ with $a + b + c = 1$.

2. Main Results

A convex normed vector space becomes a convex metric space if we define a metric $d$ by $d(x, y) = \|x - y\|$ for $x, y \in X$. When we speak about metric properties in a normed vector space, we referring to this metric. It should be pointed out that each normed vector space is a special example of convex metric space, but there exist some convex metric spaces which can not be embedded into any normed spaces [5].

Now, we introduce a new implicit iteration process;

\begin{equation}
    x_{n+1} = W(x_n, T_n^a x_n, T_n^b x_n; \alpha_n, \beta_n, \gamma_n),
\end{equation}

where $T_i : C \to C$ is an asymptotically quasi-$f_i$-$g_i$-nonexpansive mapping of a nonempty convex subset $C$ of $(X, \| \cdot \|, W)$ for functions $f_i, g_i : (X, \| \cdot \|) \to (0, \infty)$ ($i \in \mathbb{N}$) and sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in $I$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ ($n \in \mathbb{N}$).

Now we consider the approximating common fixed points of a sequence of quasi-$f$-$g$-nonexpansive mappings in convex normed vector spaces.

\textbf{Lemma 2.1} ([4]). Let $\langle a_n \rangle$, $\langle b_n \rangle$ and $\langle \delta_n \rangle$ be sequences of nonnegative real numbers satisfying the following inequality $a_{n+1} \leq (1 + \delta_n)a_n + b_n$, $n \geq 1$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n \to \infty} a_n$ exists.

\textbf{Theorem 2.1}. Let $C$ be a nonempty closed convex subset of a real complete convex normed vector space $(X, \| \cdot \|, W)$. Let $\langle T_i \rangle : i \in \mathbb{N}$ be a sequence of asymptotically quasi-$f_i$-$g_i$-nonexpansive mappings of $C$ with sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in $X$ such that $\lim_{n \to \infty} f_i(x_n) = 1$, $\lim_{n \to \infty} g_i(y_n) = 0$ and $g_i(y_1) = 0$ for $i \in \mathbb{N}$. Suppose that $F = \bigcap_{i=1}^{\infty} F(T_i)$ is nonempty closed. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $I$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for $n \geq 1$. Starting from an arbitrarily given $x_0 \in K$, we define the sequence $\langle x_n \rangle_{n \geq 1}$ by (2.1). Then the following are equivalent,

(i) $\langle x_n \rangle$ converges strongly to a common fixed point of the mappings $\langle T_i \rangle : i \in \mathbb{N}$,
(ii) $\lim d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p \in F} \|x_n - p\|$. 


Proof. Obviously, (ii) implies (i). Now we show that (i) implies (ii). For \( p \in F \),

\[
\|x_{n+1} - p\| = \|W(x_n, T^n x_n, T^n x_{n+1}; \alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}) - p\| \\
\leq \alpha_{n+1} \cdot \|x_n - p\| + \beta_{n+1} \cdot \|T^n x_n - p\| + \gamma_{n+1} \cdot \|T^n x_{n+1} - p\| \\
(2.2)
\]

Thus by Lemma 2.1, \( \lim_{n \to \infty} \|T^n x_n - p\| = 0 \).

The inequality (2.3) shows that

\[
\sum_{n=0}^{\infty} s_n < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} t_n < \infty,
\]

where \( s_n = \frac{(1 - \alpha_{n+1}) \cdot \varepsilon}{1 - \gamma_{n+1} \cdot (1 + \varepsilon)} \) and \( t_n = \frac{2 \cdot \varepsilon}{1 - \gamma_{n+1} \cdot (1 + \varepsilon)} \) for \( n \geq K \).

From (2.2) we obtain, for \( n \geq K \),

\[
\|x_{n+1} - p\| \leq \alpha_{n+1} \cdot \|x_n - p\| + \beta_{n+1} \cdot (1 + \varepsilon) \cdot \|x_n - p\| + \gamma_{n+1} \cdot (1 + \varepsilon) \cdot \|x_{n+1} - p\| + 2 \cdot \varepsilon.
\]

Hence

\[
(1 - \gamma_{n+1} \cdot (1 + \varepsilon)) \cdot \|x_{n+1} - p\| \leq \alpha_{n+1} \cdot \|x_n - p\| + \beta_{n+1} \cdot (1 + \varepsilon) \cdot \|x_n - p\| + 2 \cdot \varepsilon
\]

\[
= (\alpha_{n+1} + \beta_{n+1} \cdot (1 + \varepsilon)) \cdot \|x_{n+1} - p\| + 2 \cdot \varepsilon \quad \text{for} \quad n \geq K.
\]

The inequality (2.3) shows that

\[
\|x_{n+1} - p\| \leq \left( \frac{\alpha_{n+1} + \beta_{n+1} \cdot (1 + \varepsilon)}{1 - \gamma_{n+1} \cdot (1 + \varepsilon)} \right) \|x_n - p\| + \frac{2 \cdot \varepsilon}{1 - \gamma_{n+1} \cdot (1 + \varepsilon)}
\]

\[
= (1 + s_n) \|x_n - p\| + t_n.
\]

Since \( p \in F \) is arbitrary,

\[
\|d(x_{n+1}, F)\| \leq (1 + s_n) \|d(x_n, F)\| + t_n.
\]

Thus by Lemma 2.1, \( \lim_{n \to \infty} \|d(x_{n+1}, F)\| \) exists. Since \( \lim_{n \to \infty} \|d(x_n, F)\| = 0 \) and \( \sum_{n=0}^{\infty} t_n < \infty \), for arbitrary positive number \( \varepsilon \), there exists a natural number \( N_0 \in \mathbb{N} \) such that

\[
\|d(x_n, F)\| \leq \frac{\varepsilon}{4L} \quad \text{for} \quad n \geq N_0
\]

and

\[
\sum_{n=N_0}^{\infty} t_n \leq \frac{\varepsilon}{2L}, \quad \text{where} \quad L = \sum_{j=1}^{m} s_{n+m-j}.
\]
In particular, there exists a point \( p_1 \in F \) and \( N_1 > N_0 \) such that
\[
\| x_{N_1} - p_1 \| \leq \frac{\varepsilon}{4L}.
\]

On the other hand, from the fact that
\[
\| x_n - p \| \leq (1 - s_n)\| x_n - p \| + t_n
\]
and the inequality \( 1 + x \leq e^x \) for \( x \geq 0 \), we obtain that
\[
\| x_{n+m} - p \| \leq (1 + s_{n+m-1}) \cdot \| x_{n+m-1} - p \| + t_{n+m-1}
\leq e^{s_{n+m-1}}[(1 + s_{n+m-2}) \cdot \| x_{n+m-2} - p \| + t_{n+m-2}] + t_{n+m-1}
\leq e^{s_{n+m-1}} \sum_{j=1}^{2} s_{n+m-j} \cdot [(1 + s_{n+m-j}) \cdot \| x_{n+m-j} - p \| + t_{n+m-j}]
+ e^{s_{n+m-1}} \cdot t_{n+m-j} + t_{n+m-1}
\leq e^{s_{n+m-1}} \sum_{j=1}^{3} s_{n+m-j} \cdot [(1 + s_{n+m-j}) \cdot \| x_{n+m-j} - p \| + t_{n+m-j}]
+ e^{s_{n+m-1}} \cdot t_{n+m-j} + t_{n+m-1}
\leq \vdots
\leq e^{s_{n+m-1}} \cdot \| x_m - p \| + e^{s_{n+m-1}} \cdot t_n + \cdots + e^{s_{n+m-1}} \cdot t_{n+m-1}
+ e^{s_{n+m-1}} \cdot t_{n+m-3} + e^{s_{n+m-1}} \cdot t_{n+m-2} + t_{n+m-1}
\leq L \cdot \| x_n - p \| + L \cdot \sum_{j=1}^{n+m-1} t_j.
\]

Thus for \( n > N_1 \),
\[
\| x_{n+m} - p_1 \| = \| x_{N_1 + (n+m-N_1)} - p_1 \|
\leq L \cdot \| x_{N_1} - p_1 \| + L \cdot \sum_{j=N_1}^{n+m-1} t_j
\]

(2.4)
and

\[ \|x_n - p_1\| = \|x_{N_1} + (n - N_1) - p_1\| \]
\[ \leq L \cdot \|x_{N_1} - p_1\| + L \cdot \sum_{j=N_1}^{n-1} t_j \]

Hence from (2.4) and (2.5), we obtain that

\[ \|x_{n+m} - x_n\| \leq \|x_{n+m} - p_1\| + \|p_1 - x_n\| \]
\[ \leq L \cdot \|x_{N_1} - p_1\| + L \cdot \sum_{j=N_1}^{n+m-1} t_j + L \cdot \|x_{N_1} - p_1\| + L \cdot \sum_{j=N_1}^{n-1} t_j \]
\[ \leq 2L \cdot \|x_{N_1} - p_1\| + L \left( \sum_{j=N_1}^{n+m-1} t_j + \sum_{j=N_1}^{n-1} t_j \right) \]
\[ \leq 2L \cdot \frac{\varepsilon}{4L} + 2 \cdot L \cdot \sum_{j=N_1}^{n+m-1} t_j \]
\[ \leq 2L \cdot \frac{\varepsilon}{4L} + 2 \cdot L \cdot \frac{\varepsilon}{2L} \]
\[ = \varepsilon \]

Hence \( \{x_n\} \) is a Cauchy sequence in a closed convex subset \( C \) of a real complete convex normed vector space \( (X, \| \cdot \|, W) \). Therefore the sequence \( \{x_n\} \) converges to a point of \( C \). Let \( \lim_{n \to \infty} x_n = p^* \). Now we will show that \( p^* \in F \). Let \( \{p_n\} \) be a sequence in \( F \) such that \( p_n \to p' \). Since

\[ \|p' - T_i p'\| \leq \|p' - p_n\| + \|T_i p' - p_n\| \]
\[ \leq \|p' - p_n\| + |f_i(x_1)| \cdot \|p' - p_n\| + |g_i(y_1)| \to 0 \quad \text{as} \quad n \to \infty, \]

\( \|p' - T_i p'\| = 0 \) for \( i \in \mathbb{N} \). Thus \( p' \in F \), which means that \( F \) is closed. Since

\[ d(p^*, F) = \lim_{n \to \infty} d(p_n, F) = 0, \]

we have \( p^* \in F \), which completes the proof.

\[ \square \]

Remark 2.1. (i) We obtain the same results for asymptotically \( f-g \)-nonexpansive mappings as a corollary.

(ii) We obtain the same results for asymptotically (quasi) nonexpansive mappings [2, 4, 6].
REFERENCES


Department of Mathematics, Kyungsung University, Busan 608-736, Korea
Email address: bslee@ks.ac.kr