A NOTE ON THE INTEGRAL POINTS ON SOME HYPERBOLAS

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Abstract. In this paper, we study the Lie-generalized Fibonacci sequence and the root system of rank 2 symmetric hyperbolic Kac-Moody algebras. We derive several interesting properties of the Lie-Fibonacci sequence and relationship between them. We also give a couple of sufficient conditions for the existence of the integral points on the hyperbola $\mathbf{h}^{a}: x^2 - axy + y^2 = 1$ and $\mathbf{h}_k : x^2 - axy + y^2 = -k \ (k \in \mathbb{Z}_{>0})$. To list all the integral points on that hyperbola, we find the number of elements of $\Omega_k$.

1. Introduction

Let $A$ be a symmetric Cartan matrix $A = \left( \begin{array}{cc} 2 & -a \\ -a & 2 \end{array} \right)$ with $a \geq 3$ and $\mathfrak{g} = \mathfrak{g}(A)$ denote the associated symmetric rank 2 hyperbolic Kac-Moody Lie algebra over the field of complex numbers. Let $\Pi = \{\alpha_0, \alpha_1\}$ denote the set of simple roots with $\Delta$ its root system. A root $\alpha \in \Delta$ is called a real root if there exists $w \in W$ such that $w(\alpha)$ is a simple root, and a root which is not real is called an imaginary root. We denote by $\Delta^re, \Delta^re_+, \Delta^im$, and $\Delta^im_+$ the set of all real, positive real, imaginary and positive imaginary roots, respectively. We also denote by $\Delta^im_+^{+k}$ the set of all positive imaginary roots of the algebra $\mathfrak{g}(A)$ with square length $-2k$. In [2], A.J. Feingold show that the Fibonacci numbers are intimately related to the rank 2 hyperbolic GCM Lie algebras. In [5], S.J. Kang and D.J. Melville show that all the roots of a given length are Weyl conjugate to roots in a small region. These information help in determining the sufficient conditions for the existence of integral points on the hyperbola $\mathbf{h}_k : x^2 - axy + y^2 = -k \ (k \in \mathbb{Z}_{>0})$.

In this paper, we give some results on the Lie-Fibonacci sequence and symmetric hyperbolic Kac-Moody algebra of rank 2.

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In section 2, we derive several interesting properties of the Lie-Fibonacci sequence. And then we give the following results:

1. If $n$ increases, then the ratio of two successive Lie-Fibonacci number approaches \( \frac{a - 2 + \sqrt{a^2 - 4}}{2} \), or \( \frac{1}{a - 2} \left( \frac{a - 2 + \sqrt{a^2 - 4}}{2} \right) \)

(which is the golden ratio if $a = 3$).

2. Two successive Lie-Fibonacci numbers $F_{n}^{(a)}$ and $F_{n+1}^{(a)}$ are relatively prime.

In section 3, we give some definitions and known results on the Kac-Moody algebras and the study of their elementary properties. We derive the relations among the Lie-Fibonacci numbers. We also give some sufficient conditions for the existence of integral points. We find the number of elements of $\Omega_k$ for some $k$. Lastly, we give the following theorem:

**Theorem.** Let \( x^2 - axy + y^2 = -(a-2)\gamma^2 \) for $a \geq 3$ and $\gamma \in \mathbb{Z}_{>0}$ be the hyperbola. If $a + 2 = \gamma^2$, and $a - 2$ is a square free integer, then $|\Omega_{(a-2)\gamma^2}| = 2$.

This procedure finds all the integral points on these hyperbolas far more easily than the traditional number-theoretic algorithm.

2. **Lie-Fibonacci Sequence**

In this section, we introduce the Lie generalized Fibonacci sequence \( \{F_{n}^{(a)}\} \), and generalize the several interesting properties of the Fibonacci sequence \( \{F_{n}\} \).

Define a new sequence \( \{F_{n}^{(a)}\} \) by the recurrence relations

\[
\begin{align*}
F_{0}^{(a)} &= F_{1}^{(a)} = 1, \\
F_{2n+2}^{(a)} &= aF_{2n}^{(a)} - F_{2n-2}^{(a)}, \\
F_{2n+1}^{(a)} &= F_{2n+2}^{(a)} - F_{2n}^{(a)} \quad (n > 0).
\end{align*}
\]

Clearly \( \{F_{n}^{(3)}\} = \{F_{n}\} \), the Fibonacci sequence defined by:

\[
\begin{align*}
F_{0} &= 0, \quad F_{1} = 1, \quad F_{n+2} = F_{n} + F_{n+1}.
\end{align*}
\]

We call this sequence \( \{F_{n}^{(a)}\} \), the Lie-Fibonacci sequence, and $F_{n}^{(a)}$ the Lie-Fibonacci number.

It is well known that there are many interesting identities for the Fibonacci sequence. In this section, we derive several similar identities for the Lie-Fibonacci sequence. Among the several known results concerning Fibonacci numbers, we quote below some interesting ones:
Proposition 2.1 ([7]). Let \( \{ F_n \} \) be the Fibonacci sequence. Then we have the followings.
(a) \( F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n} \).
(b) \( F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \).
(c) \( F_1 + F_2 + \cdots + F_n = F_{n+2} - 1 \).
(d) \( F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{n+1}F_n = (-1)^{n+1}F_{n-1} + 1 \).
(e) \( F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1} \).

To prove several identities for the Lie-Fibonacci sequence, we need the following Proposition.

Lemma 2.2 ([8]). For any positive integer \( n \), we have
(a) \( F_{2n+3}^{(a)} = a F_{2n+1}^{(a)} - F_{2n-1}^{(a)} \).
(b) \( F_{2n}^{(a)} + F_{2n+1}^{(a)} = F_{2n+2}^{(a)} \).
(c) \( F_{2n-1}^{(a)} + (a-2)F_{2n}^{(a)} = F_{2n+1}^{(a)} \).

We deduce from Proposition 2.2 the following theorem.

Theorem 2.3. Let \( \{ F_n^{(a)} \} \) be the Lie-Fibonacci sequence. Then we have the following.
(a) \( F_1^{(a)} + F_4^{(a)} + F_5^{(a)} + \cdots + F_{2n-1}^{(a)} = F_{2n}^{(a)} \).
(b) \( F_2^{(a)} + F_4^{(a)} + \cdots + F_{2n}^{(a)} = \frac{1}{a^2} (F_{2n+1}^{(a)} - 1) \).
(c) \( F_1^{(a)} + F_2^{(a)} + \cdots + F_{2n-1}^{(a)} = \frac{1}{a-2} (F_{2n+1}^{(a)} - 1) \).
(d) \( F_1^{(a)} + (a-2)F_2^{(a)} + F_3^{(a)} + (a-2)F_4^{(a)} + \cdots + F_{2n-1}^{(a)} + (a-2)F_{2n}^{(a)} = F_{2n+2}^{(a)} - 1 \).
(e) \( F_1^{(a)} - F_2^{(a)} + F_3^{(a)} - F_4^{(a)} + \cdots + F_{2n-1}^{(a)} - F_{2n}^{(a)} = \frac{1}{a^2-1} (1 - F_{2n-1}^{(a)}) \).
(f) \( (F_1^{(a)})^2 + (a-2)(F_2^{(a)})^2 + F_3^{(a)} + \cdots + (a-2)l_n (F_n^{(a)})^2 = F_{2n}^{(a)} F_{2n+1}^{(a)} \), where
\[ l_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \]
(g) \( (F_{2n}^{(a)})^2 = F_{2n}^{(a)} F^{(a)}_{2n+2} - F_{2n}^{(a)} F_{2n+1}^{(a)} \).

Proof. Since \( F_{2n+1}^{(a)} = F_{2n+2}^{(a)} - F_{2n}^{(a)} \), we have
\[
F_1^{(a)} + F_3^{(a)} + F_5 + \cdots + F_{2n-1}^{(a)} \\
= F_1^{(a)} + (F_4^{(a)} - F_2^{(a)}) + (F_6^{(a)} - F_4^{(a)}) + \cdots + (F_{2n}^{(a)} - F_{2n-2}^{(a)}) \\
= F_1^{(a)} - F_2^{(a)} + F_{2n}^{(a)} \\
= F_{2n}^{(a)}.
\]
which proves part (a). For part (b), using Proposition 2.2(e), we have:

\[
F(a)^2 + F(a)^4 + F(a)^6 + \cdots + F(a)^{2n} = \frac{1}{a - 2} \left\{ (F(a)^3 - F(a)^1) + (F(a)^5 - F(a)^3) + \cdots + (F(a)^{2n+1} - F(a)^{2n-1}) \right\}
\]

\[
= \frac{1}{a - 2} (F(a)^{2n+1} - 1),
\]

the desired result. Using parts (a) and (b), we have

\[
F(a)^1 + F(a)^2 + \cdots + F(a)^{2n-1} = (F(a)^1 + F(a)^3 + F(a)^5 + \cdots + F(a)^{2n-1}) + (F(a)^2 + F(a)^4 + \cdots + F(a)^{2n-2})
\]

\[
= F(a)^{2n} + F(a)^2 + F(a)^4 + \cdots + F(a)^{2n-2} = \frac{1}{a - 2} (F(a)^{2n+1} - 1),
\]

which proves part (c). In a similar manner, we can derive parts (d),(e) and (f). □

It is well known that two successive Fibonacci numbers \(F_n\) and \(F_{n+1}\) are disjoint. The following Theorem shows that the Lie-Fibonacci numbers have the same property.

**Theorem 2.4.** Let \(\{F_n^{(a)}\}\) be the Lie-Fibonacci sequence. Then two successive Lie-Fibonacci numbers \(F_{2n}^{(a)}\) and \(F_{2n+1}^{(a)}\) are relatively prime.

**Proof.** Clearly, \(F_1^{(a)}\) and \(F_2^{(a)}\) are relatively prime. Let \(d\) be a gcd of \(F_{2n}^{(a)}\) and \(F_{2n+1}^{(a)}\) \((n \geq 1)\). Since \(F_{2n+2}^{(a)} = F_{2n}^{(a)} + F_{2n+1}^{(a)}\), \(d\) divides \(F_{2n+2}^{(a)}\). On the other hand, \(F_{2n+2}^{(a)} = aF_{2n}^{(a)} + F_{2n+1}^{(a)}\). Thus \(d\) also divides \(F_{2n}^{(a)}\) and \(F_{2n+1}^{(a)}\). Continuing this process, we arrive at \(d = 1\). □

Let \(\{F_n\}\) be the Fibonacci sequence. Robert Simson stated that

\[
F_{n-1}F_{n+1} - F_n^2 = (-1)^n,
\]

for every positive integer \(n\), as it is to see, by induction on \(n\).

To generalize the Simson’s identity concerning the Fibonacci sequence, we need the following Proposition.

**Proposition 2.5 ([7]).** (Generalization of Binet formula) Let \(\{F_n^{(a)}\}\) be the Lie-Fibonacci sequence, and let \(\alpha = \frac{a + \sqrt{a^2 - 4}}{2}\) be a zero of \(1 - (a^2 - 2)x^2 + x^4\). Then we have the following:
(a) \( F_{2n}^{(a)} = \frac{1}{\sqrt{a^2 - 4}} \left( \left( \frac{(\alpha - 1)^2}{a - 2} \right)^n - \left( \frac{1}{\alpha} \right)^n \right) \)

\[
= \frac{1}{(a - 2)^n \sqrt{a^2 - 4}} \left( \left( \frac{a - 2 + \sqrt{a^2 - 4}}{2} \right)^{2n} - \left( \frac{a - 2 - \sqrt{a^2 - 4}}{2} \right)^{2n} \right),
\]

(b) \( F_{2n+1}^{(a)} = \frac{1}{\sqrt{a^2 - 4}} \left( \left( \frac{(\alpha - 1)^2}{a - 2} \right)^n - \left( \frac{1}{\alpha} \right)^n \right) \)

\[
= \frac{1}{(a - 2)^n \sqrt{a^2 - 4}} \left( \left( \frac{a - 2 + \sqrt{a^2 - 4}}{2} \right)^{2n+1} - \left( \frac{a - 2 - \sqrt{a^2 - 4}}{2} \right)^{2n+1} \right).
\]

**Theorem 2.6.** Let \( \{F_n^{(a)}\} \) be the Lie-Fibonacci sequence and \( n \in \mathbb{Z}_{>0} \). Then we have:

(a) \( F_{2n-1}^{(a)} F_{2n+1}^{(a)} - (a - 2) (F_{2n}^{(a)})^2 = 1. \)
(b) \( (a - 2) F_{2n}^{(a)} F_{2n+2}^{(a)} - (F_{2n+1}^{(a)})^2 = -1. \)
(c) \( F_n^{(a)} F_{n+1}^{(a)} - F_{n-1}^{(a)} F_{n+2}^{(a)} = (-1)^n. \)

**Proof.** Let \( \beta = \frac{1}{\alpha} \), \( \alpha' = \alpha - 1 \) and \( \beta' = \beta - 1 \). Then we have

\[
F_{2n-1}^{(a)} F_{2n+1}^{(a)} - (a - 2) (F_{2n}^{(a)})^2
\]

\[
= \frac{1}{(a - 2)^{n-1} \sqrt{a^2 - 4}} \left( \left( \frac{a - 2 + \sqrt{a^2 - 4}}{2} \right)^{2n-1} - \left( \frac{a - 2 - \sqrt{a^2 - 4}}{2} \right)^{2n-1} \right)
\]

\[
= \frac{1}{(a - 2)^n \sqrt{a^2 - 4}} \left( \left( \frac{a - 2 + \sqrt{a^2 - 4}}{2} \right)^{2n+1} - \left( \frac{a - 2 - \sqrt{a^2 - 4}}{2} \right)^{2n+1} \right)
\]

\[
- (a - 2) \left( \frac{1}{(a - 2)^n \sqrt{a^2 - 4}} \left( \left( \frac{a - 2 + \sqrt{a^2 - 4}}{2} \right)^{2n} - \left( \frac{a - 2 - \sqrt{a^2 - 4}}{2} \right)^{2n} \right) \right)^2
\]

\[
= \frac{1}{(a - 2)^{2n-1} (a^2 - 4)} \left( \alpha'^4 - (\alpha' \beta')^{2n-1} (\beta^2 + a^2) + \beta'^4 - \alpha'^4 + 2(\alpha' \beta')^{2n} - \beta'^4 \right)
\]

\[
= \frac{-(\alpha' \beta')^{2n-1}}{(a - 2)^{2n-1} (a^2 - 4)} (\alpha'^2 + \beta'^2 - 2\alpha' \beta')
\]

\[
= \frac{-(2 - a)^{2n-1}}{(a - 2)^{2n-1} (a^2 - 4)} (\alpha' - \beta')^2
\]

\[
= 1,
\]
which proves part (a). In a similar manner, we can derive parts (b) and (c).

It is well known that as \( n \) increases the ratio \( \frac{F_{n+1}}{F_n} \) approaches \( \frac{1 + \sqrt{5}}{2} \), the golden ratio. The following Theorem shows that the Lie-Fibonacci sequence \( \{F_n^{(a)}\} \) has similar properties.

**Theorem 2.7.** Let \( \{F_n^{(a)}\} \) be the Lie-Fibonacci sequence, and let \( \alpha = \frac{a + \sqrt{a^2 - 4}}{2} \). Then we have

(a) \( \lim_{n \to \infty} \frac{F_{2n+1}^{(a)}}{F_{2n}^{(a)}} = \frac{a - 2 + \sqrt{a^2 - 4}}{2} = \alpha - 1 \).

(b) \( \lim_{n \to \infty} \frac{F_{2n+2}^{(a)}}{F_{2n+1}^{(a)}} = \left( \frac{1}{a - 2} \right) \left( \frac{a - 2 + \sqrt{a^2 - 4}}{2} \right) = \frac{1}{a - 2} (\alpha - 1) \).

In particular, \( \lim_{n \to \infty} \frac{F_{2n+1}^{(3)}}{F_{2n}^{(3)}} = \frac{1 + \sqrt{5}}{2} \), the golden ratio.

**Proof.** Let \( p_n = \frac{F_{2n+2}^{(a)}}{F_{2n}^{(a)}} \). Then we have

\[
 p_n = \frac{a F_{2n}^{(a)} - F_{2n-2}^{(a)}}{F_{2n}^{(a)}}
 = a - \frac{1}{p_{n-1}}
 = a - \frac{1}{a - \frac{1}{p_{n-2}}} \cdots
\]

Therefore,

\[
 \lim_{n \to \infty} p_n \text{ is a zero of } x = a - \frac{1}{x},
\]

and hence,

\[
 \lim_{n \to \infty} p_n = \frac{a - \sqrt{a^2 - 4}}{2}.
\]

Let

\[
 q_n = \frac{F_{2n+1}^{(a)}}{F_{2n}^{(a)}}.
\]

Then we have

\[
 q_n = \frac{F_{2n+2}^{(a)} - F_{2n}^{(a)}}{F_{2n}^{(a)}}
 = p_n - 1.
\]
Therefore,
\[
\lim_{n \to \infty} q_n = \frac{a - \sqrt{a^2 - 4}}{2} - 1 = \frac{a - 2 + \sqrt{a^2 - 4}}{2},
\]
which proves for part (a). In a similar manner, we can derive part (b). \(\square\)

3. Existence of Integral Points on the Hyperbolas

In this section, we study the root system of the rank 2 hyperbolic Kac-Moody algebras \(g(A)\) with symmetric generalized Cartan matrix \(A = \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}\) with \(a \geq 3\). Let \(W\) be the Weyl group of \(g(A)\), generated by simple reflections \(r_1\) and \(r_2\).

We identify an element
\[
(7) \quad \alpha = x\alpha_1 + y\alpha_2 \in \mathbb{Q}\text{ with an ordered pair } (x, y) \in \mathbb{Z} \times \mathbb{Z}.
\]

We call a root \(\alpha \in \mathbb{Z} \times \mathbb{Z}\) the \textit{positive integral point} if \(x, y \in \mathbb{Z} \geq 0\). Define a symmetric bilinear form \((\cdot | \cdot)\) on \(\mathfrak{h}^*\) by the following equation:

\[
(8) \quad (\alpha_1 | \alpha_1) = (\alpha_2 | \alpha_2) = 2, \quad (\alpha_1 | \alpha_2) = -a.
\]

Then for \(\alpha = x\alpha_1 + y\alpha_2\), we have \((\alpha | \alpha) = 2(x^2 -axy + y^2)\).

It is well known that there is a one-to-one correspondence between the set of real roots of \(g(A)\) and the set of integral points on the hyperbola \(x^2 -axy + y^2 = 1\). Since there is no root \(\alpha\) such that \((\alpha | \alpha) = 0\), the imaginary roots of \(g(A)\) correspondence to the set of integral points on the hyperbolas \(\mathfrak{h}_k : x^2 -axy + y^2 = -k\) for \(k \geq 1\). In other words, for each \(k \geq 1\), there is a one-to-one correspondence between the set of all imaginary roots \(\alpha\) with square length \((\alpha | \alpha) = -2k\) and the set of all integral points on the hyperbola \(\mathfrak{h}_k\).

We introduce the sequences of integers \(\{B_n\}\) for \(n \geq 0\) by the recurrence relations

\[
(9) \quad B_0 = 0, \quad B_1 = 1, \quad \text{and} \quad B_{n+2} = aB_{n+1} - B_n \quad \text{for} \quad n \geq 1.
\]

Clearly, we have

\[
(10) \quad F_{2n}^{(a)} = B_n, \quad \text{and} \quad F_{2n-2}^{(a)} = B_n - B_{n-1}.
\]

The following Proposition is well known.

\textbf{Proposition 3.1 ([3])}. \(\Delta_r = \{(B_n, B_{n+1}) \mid (B_{n+1}, B_n) \mid n \geq 0\}\). \textit{Furthermore},

\[
\Delta_r^c = \{(F_{2j}, F_{2j+2}), \ (F_{2j+2}, F_{2j}) \mid j \in \mathbb{Z}_0\}
\]

for \(a = 3\).
For a positive integer $k$, let $\Delta_{+}^{im,k}$ be the set of all positive imaginary roots $\alpha$ of $g(A)$ with square length $(\alpha | \alpha) = -2k$. That is, $\Delta_{+}^{im,k}$ is the set of all positive integral points on the hyperbola $H_k$. The following Proposition gives a nice description of the set of positive imaginary roots of length $-2k$.

**Proposition 3.2 ([5]).**

$$\Delta_{+}^{im,k} = \{(m, n), (n, m), (mB_j+1 - nB_j, mB_j+2 - nB_{j+1}),$$

$$(mB_{j+2} - nB_{j+1}, mB_{j+1} - nB_j), (nB_{j+1} - mB_j, nB_{j+2} - mB_{j+1}),$$

$$(nB_{j+2} - mB_{j+1}, nB_{j+1} - mB_j) \mid (m, n) \in \Omega_k \}$$

where $\Omega_k = \{(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid \frac{2\sqrt{k}}{a^2 - 4} \leq m \leq \sqrt{\frac{k}{a^2 - 4}}, n = \frac{am - \sqrt{(a^2 - 4)m^2 - 4k}}{2} \}$.

Since $F_{2n}^{(a)} = B_n$, and $F_{2n-2}^{(a)} = B_n - B_{n-1}$, Proposition 3.1 and Proposition 3.2 can be rewritten as follows:

**Proposition 3.3.** Let $\{F_n^{(a)}\}$ be the Lie-Fibonacci sequence. Then

(a) The set of all nonnegative integral points on the hyperbola $x^2 - axy + y^2 = 1$

is $\{(F_{2n}^{(a)}, F_{2n+2}^{(a)}), (F_{2n+2}^{(a)}, F_{2n}^{(a)}) \mid n \in \mathbb{Z}_{\geq 0}\}$.

(b) The set of all nonnegative integral points on the hyperbola $x^2 - axy + y^2 = -(a - 2)$

is $\{(1, 1), (F_{2n-1}^{(a)}, F_{2n+1}^{(a)}), (F_{2n+1}^{(a)}, F_{2n-1}^{(a)}) \mid n \in \mathbb{Z}_{\geq 0}\}$.

**Proof.** (a) is immediate consequence of Proposition 3.1 and definition of the Lie-Fibonacci sequence. For (b), after simple calculation, we have $\Omega_k = \{(1, 1)\}$ and hence

$$\Delta_{+}^{im} = \{(1, 1), (F_{2n-1}^{(a)}, F_{2n+1}^{(a)}), (F_{2n+1}^{(a)}, F_{2n-1}^{(a)}) \mid n \in \mathbb{Z}_{\geq 0}\}.$$

□

To list all the integral points on those hyperbolas, we also find the number of elements of $\Omega_k$.

**Proposition 3.4.** ([7]) Let $x^2 - axy + ay^2 = -k$ be the hyperbola and let $k = t\gamma^2$ be any positive integer where $t$ is a square free integer and $\gamma \in \mathbb{Z}_{>0}$. If $(\gamma, \delta) \in \Omega_k$ for some positive integer $\delta$, then
\[ a - 2 \leq t \leq \frac{a^2 - 4}{4} \text{ for } a \geq 3, \]

where

\[
\Omega_k = \left\{ (m, n) \in \Delta_{\text{im}} \left| \begin{array}{c}
\frac{2\sqrt{k}}{\sqrt{a^2 - 4}} \leq m \leq \sqrt{\frac{k}{a - 2}}, \quad n = \frac{am - \sqrt{(a^2 - 4)m^2 - 4k}}{2}
\end{array} \right. \right\}.
\]

Since \( W \) is infinite, \( \Omega_k \neq \emptyset \) implies that there are infinitely many integral points on the hyperbola \( x^2 - axy + y^2 = -k \). Proposition 3.3 tells us that \( \Omega_k \) have crucial information for the set of integral points on the hyperbola \( x^2 - axy + y^2 = -k \). We have the following Lemma.

**Lemma 3.5.** Let \( x^2 - axy + ay^2 = -k \) be the hyperbola. If \( k < a - 2 \), then there is no integral point on that hyperbola.

**Proof.** Since there is no integer \( m \) with \( \frac{2\sqrt{k}}{\sqrt{a^2 - 4}} \leq m \leq \sqrt{\frac{k}{a - 2}} \), we have \( \Omega_k = \emptyset \), and hence we get the desired result. \( \square \)

The following Proposition is obtained by the definition of \( \Omega_k \).

**Proposition 3.6.** Let \( x^2 - axy + y^2 = -(a-2)\gamma^2 \) be the hyperbola for \( a \geq 3 \) and \( \gamma \in \mathbb{Z}_{>0} \). If \( \gamma < \frac{n\sqrt{a + 2}}{\sqrt{a + 2} - 2} \), then \( 1 \leq |\Omega_{(a-2)\gamma^2}| \leq n \). Furthermore, \( \gamma < \frac{\sqrt{a + 2}}{\sqrt{a + 2} - 2} \), then \( |\Omega_{(a-2)\gamma^2}| = 1 \).

**Proof.** Clearly, we have \((\gamma, \gamma) \in \Omega_{(a-2)\gamma^2}\), and hence,

\[ |\Omega_{(a-2)\gamma^2}| \geq 1. \]

Consider the set

\[
\Omega_{(a-2)\gamma^2} = \left\{ (m, n) \in \Delta_{\text{im}} \left| \begin{array}{c}
\frac{2\gamma}{\sqrt{a + 2}} \leq m \leq \gamma, \quad n = \frac{am - \sqrt{(a^2 - 4)m^2 - 4(a-2)\gamma^2}}{2}
\end{array} \right. \right\}.
\]

Since

\[ \gamma < \frac{2\gamma}{\sqrt{a + 2}} < n \quad \text{implies} \quad \gamma < \frac{n\sqrt{a + 2}}{\sqrt{a + 2} - 2}, \]

at most \( n \) positive integers exist between \( \frac{2\gamma}{\sqrt{a + 2}} \) and \( \gamma \), we get the desired result. \( \square \)

**Example 3.7.** Let \( x^2 - 7xy + y^2 = -5 \) be the hyperbola. Then we have \( \Omega_5 = \{(1, 1)\} \).

Therefore,

\[
\Delta_{\text{im}}^+ = \{(1, 1), (F_{2n+1}^{(7)}, F_{2n}^{(7)}), (F_{2n+1}^{(7)}, F_{2n-1}^{(7)}) \mid n \geq 1 \}
\]

\[
= \{(1, 1), (1, 6), (6, 1)(6, 41), (41, 6), \ldots \ldots \}.\]
Example 3.8. Let \( x^2 - 3xy + y^2 = -\gamma^2 \) be the hyperbola. Since \( a = 3 \), 
\[
\gamma < \frac{\sqrt{5}}{\sqrt{5} - 2}
\] implies \( |\Omega_{(a-2)\gamma^2}| = 1 \), thus we have \( |\Omega_{(a-2)\gamma^2}| = 1 \) for \( 1 \leq \gamma \leq 9 \).

Therefore, 
\[
\Omega_{(a-2)\gamma^2} = \{(\gamma, \gamma)\} \text{ for } 1 \leq \gamma \leq 9,
\]
and hence
\[
\Delta^{re} = \{\sigma(\gamma, \gamma) | \sigma \in W\}.
\]

For the case of \( a = 4 \), similarly we have, \( |\Omega_{(a-2)\gamma^2}| = 1 \) for \( 1 \leq \gamma \leq 5 \).

Lemma 3.9. Let \( x^2 - axy + y^2 = -(a-2)\gamma^2 \) for \( a \geq 3 \) and \( \gamma \in \mathbb{Z}_{>0} \) be the hyperbola. If \( a + 2 = \gamma^2 \), then \( |\Omega_{(a-2)\gamma^2}| \geq 2 \).

Proof. Clearly, \( (\gamma, \gamma) \in \Omega_{(a-2)\gamma^2} \). If we substitute \( \gamma^2 \) for \( a + 2 \), then we have \( \gamma \geq 3 \) and
\[
\Omega_{a^2 - 4} = \left\{ (m, n) \in \Delta^m_{+k} | 2 \leq m \leq \gamma, n = \frac{am - \sqrt{(a-2)(m^2 - 4)\gamma}}{2} \right\}.
\]
Thus we have \( \{(2, a), (\gamma, \gamma)\} \subseteq \Omega_{a^2 - 4} \), and hence we get the desired result. \( \square \)

Theorem 3.10. Let \( x^2 - axy + y^2 = -(a-2)\gamma^2 \) for \( a \geq 3 \) and \( \gamma \in \mathbb{Z}_{>0} \) be the hyperbola. If \( a + 2 = \gamma^2 \), and \( a - 2 \) is a square free integer, then \( |\Omega_{(a-2)\gamma^2}| = 2 \).

Proof. If \( (m, n) \in \Omega_{(a-2)\gamma^2} \) for some \( n \in \mathbb{Z}_{>0} \), then we have \( m^2 - 4 = (a-2)l^2 \) for some \( l \in \mathbb{Z}_{>0} \). Since \( a-2 = \gamma^2 - 4 \), and \( m \leq \gamma \), we have \( \gamma^2 - 4 \geq m^2 - 4 = (\gamma^2 - 4)l^2 \), and hence either \( l = 0 \) or \( l = 1 \). This implies that either \( m = 2 \) or \( m = \gamma \), and hence \( \Omega_{a^2 - 4} = \{(2, a), (\gamma, \gamma)\} \). \( \square \)

Example 3.11. Let \( x^2 - 7xy + y^2 = -5 \cdot 3^2 \) be the hyperbola. Then we have
\[
\Omega_{7 \cdot 3^2} = \{(2, 7), (3, 3)\},
\]
and hence
\[
\Delta^m_{+} = \{3(F_{2n+1}^{(7)}, F_{2n+1}^{(7)}), 3(F_{2n+1}^{(7)}, F_{2n-1}^{(7)}), (2F_{2n+2}^{(7)} - 7F_{2n}^{(7)}, 2F_{2n+4}^{(7)} - 7F_{2n}^{(7)}) \}
\]
\[
(2F_{2n+4}^{(7)} - 7F_{2n+2}^{(7)}, 2F_{2n+2}^{(7)} - 7F_{2n}^{(7)})(7F_{2n+2}^{(7)} - 2F_{2n}^{(7)}, 7F_{2n+4}^{(7)} - 2F_{2n}^{(7)}) \}
\]
\[
(7F_{2n+4}^{(7)} - 2F_{2n+2}^{(7)}, 7F_{2n+2}^{(7)} - 2F_{2n}^{(7)}) | n \geq 1 \}.
\]

Corollary 3.12. There are many integral solutions \( x^2 - axy + y^2 = 4 - a^2 \) for \( a \geq 2 \).

Theorem 3.13. If \( a \neq 2 \mod 4 \), then there is a one-to-one correspondence between the set of integral points on the hyperbolas \( x^2 - axy + y^2 = 1 \) and \( (a+2)x^2 - (a-2)y^2 = 4 \).
A NOTE ON THE INTEGRAL POINTS ON SOME HYPERBOLAS

Proof. \((a+2)x^2 - (a-2)y^2 = 4\) is obtained from \(x^2 - axy + y^2 = 1\) by substituting \((x, y) = \frac{1}{2}(x' + y', -x' + y')\), that is \((x', y') = (x - y, x + y)\).

If \(x, y\) are integers, then clearly \(x'\) and \(y'\) are also integers. On the other hand, we need to show that \((x', y') \in \mathbb{Z} \times \mathbb{Z}\) implies that \((x, y) \in 2\mathbb{Z} \times 2\mathbb{Z}\) or \((x, y) \in (2n+1)\mathbb{Z} \times (2n+1)\mathbb{Z}\). If \(a = 4k\), then \((4k + 2)x'^2 - (4k - 2)y'^2 = 4\). That is \((2k + 1)x'^2 = (2k - 1)y'^2 + 2\). This implies \(x'\) and \(y'\) are both even or both odd.

Similarly, we can show in the other cases: \(a \equiv 1 \pmod{4}\) and \(a \equiv 3 \pmod{4}\). □

Example 3.14. Since the set of all nonnegative integral points on the hyperbola \(x^2 - 5xy + y^2 = 1\) is \\{\(0,1\), \((1,0)\), \((1,5)\), \((5,1)\), \((5,24)\), \((24,5)\), \((24,115)\), \((115,24)\)\}, and \\{\((-1,1)\), \((1,1)\), \((-4,6)\), \((4,6)\), \((-19,29)\), \((19,29)\), \((-91,139)\), \((91,139)\)\} is the set of integral points on the hyperbola \(7x^2 - 3y^2 = 4\).

Corollary 3.15. There are infinitely many integral points on the hyperbola \((a+2)x^2 - (a-2)y^2 = 4\) (\(a \geq 3\), \(a \not\equiv 2 \pmod{4}\)).

References


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