Let $A$ be an algebra and $D$ a derivation of $A$. Then $D$ is called algebraic nil if for any $x \in A$ there is a positive integer $n = n(x)$ such that $D^{n(x)}(P(x)) = 0$, for all $P \in \mathbb{C}[X]$ (by convention $D^{n(x)}(\alpha) = 0$, for all $\alpha \in \mathbb{C}$). In this paper, we show that any algebraic nil derivation (possibly unbounded) on a commutative complex algebra $A$ maps into $N(A)$, where $N(A)$ denotes the set of all nilpotent elements of $A$. As an application, we deduce that any nilpotent derivation on a commutative complex algebra $A$ maps into $N(A)$.

Finally, we deduce two noncommutative versions of algebraic nil derivations inclusion range.

1. Introduction

Let $A$ be a complex algebra. A linear map $D$ from $A$ to $A$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in A$. A derivation of $D$ on $A$ is called nil if for any $x \in A$ there is a positive integer $n = n(x)$ such that $D^{n(x)} = 0$ (see [6]). Here, if the number $n$ can be taken independently of $x$, $D$ is called nilpotent. A derivation $D$ of $A$ is called algebraic nil if for any $x \in A$ there is a positive integer $n = n(x)$ such that $D^{n(x)}(P(x)) = 0$, for all $P \in \mathbb{C}[X]$ (by convention $D^{n(x)}(\alpha) = 0$, for all $\alpha \in \mathbb{C}$).

We will denote by $Q(A)$ the set of all quasinilpotent elements in a Banach algebra $A$. In 1955, Singer and Wermer [12] proved that a continuous derivation on a commutative Banach algebra maps into the (Jacobson) radical, and they conjectured that this result holds even if the derivation is discontinuous. In 1988, Thomas [13] solved the long standing problem by showing that the conjecture is true.

In 1991, Kim and Jun [10] proved that if $D$ is a derivation on a noncommutative Banach algebra $A$ satisfying the condition $[[A, A], A] = 0$ then $D(A) \subset Q(A)$. In
1992, Vukman [15] proved that if $D$ is a linear Jordan derivation on a noncommutative Banach algebra $A$ such that the map $F(x) = [[Dx, x], x]$ is commuting on $A$ then $D = 0$. In 1992, Mathieu and Runde [11] proved that if $D$ is a centralizing derivation on a Banach algebra $A$; then $D(A) \subset rad(A)$. In 1994, Bresar [5] showed that if $D$ is a bounded derivation of a Banach algebra such that $[D(x), x] \in Q(A)$ for every $x \in A$; then $D(A) \subset rad(A)$ where $rad(A)$ denotes the Jacobson radical of $A$.

To the best of our knowledge, there is no inclusion versions for derivations on arbitrary algebra, except the paper of Colville, Davis, and Keimel [9] in which they began studying positive derivations on $f$-rings (i.e., $D(a) \geq 0$, for all $a \geq 0$) and the papers of Boulabiar [4], A. Toumi et al [14] and Ben Amor [2], in which the authors studied exclusively positive and order bounded derivations on Archimedean almost $f$-algebras.

It is well-known that the notion of nil derivations is a generalization of the notion of nilpotent derivations. The latter, because of its close relation with automorphisms and the existence of a Jordan decomposition into semisimple and nilpotent parts for a large family of derivations (it is a generalization of that of algebraic derivations), has received considerable attention (see [6,7,8]). In this paper we shall be concerned principally with the range of algebraic nil derivations $D$ on commutative algebra, on noncommutative archimedean $d$-algebra and on noncommutative algebra $A$ satisfying the following condition: $[[A, A], A] = 0$.

2. The Main Results

To prove our first theorem, we shall need the following algebraic result.

**Proposition 2.1.** Let $A$ be a commutative complex algebra, $n$ be a positive integer, $D$ be a derivation on $A$ and $x \in A$ such that

$$D^n(x), D^n(x^2), D^n(x^n) \in N(A),$$

where $N(A)$ denotes the set of all nilpotent elements of $A$. Then $D(x) \in N(A)$.

**Proof.** Let $x \in A$ with $D^n(x), D^n(x^2), D^n(x^n) \in N(A)$. It follows that

$$D^n(x^2) = \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(x) \in N(A).$$
Since $D^n (x) \in N (A)$, we have

\[ \sum_{k=1}^{n-1} \binom{n}{k} D^k (x) D^{n-k} (x) \in N (A). \]

Moreover, letting $n_1, n_2 \in \mathbb{N}$ such that $n_1 + n_2 = n$, this one has

\[ D^n (x^n) = \sum_{k=0}^{n} \binom{n}{k} D^k (x^{n_1}) D^{n-k} (x^{n_2}) \in N (A). \]

By using the Leibnitz rule for $D^k (x^{n_1})$ and $D^k (x^{n_2})$ in Equality (3) and by using the relation (2), we deduce that

\[ (D (x))^n \in N (A) \]

and then $D (x) \in N (A)$. \hfill \Box

From the above result, we deduce the following:

**Proposition 2.2.** Let $A$ be a commutative complex algebra, $n$ be a positive integer, $D$ be a derivation on $A$ and $x \in A$ such that

\[ D^n (x) = D^n (x^2) = D^n (x^n) = 0. \]

Then $(D (x))^n = 0$.

The below theorem is an immediate consequence of Proposition 2.2.

**Theorem 2.3.** Let $A$ be a commutative complex algebra and let $D$ be an algebraic nil derivation on $A$. Then $D (A)$ is contained in $N (A)$.

Since any nilpotent derivation is algebraic nil, we have the following:

**Corollary 2.4.** Let $A$ be a commutative complex algebra and let $D$ be a nilpotent derivation on $A$. Then $D (A)$ is contained in $N (A)$.

In what follows, we shall deal with the range of algebraic nil derivation on non-commutative algebras. In order to hit this mark, we will need the following lemma.

**Lemma 2.5** ([10, Lemma 3.1]). Let $A$ be a complex algebra satisfying the condition $[[A, A], A] = 0$. Let $A \oplus A$ be the vector space direct sum. Define a multiplication in $A \oplus A$ by setting

\[ (a_1, b_1) (a_2, b_2) = (a_1a_2 + a_2a_1, b_1b_2 + b_2b_1) \]

for all $(a_1, b_1), (a_2, b_2)$ in $A \oplus A$. Then $A \oplus A$ is a commutative algebra.
Using the previous lemma, we deduce the following result. Its proof is inspired from [10, Theorem 3.2].

**Theorem 2.6.** Let $A$ be a complex algebra satisfying the condition $[[A, A], A] = 0$ and let $D$ be an algebraic nil derivation on $A$. Then $D(A)$ is contained in $N(A)$.

**Proof.** By the previous lemma, $A \oplus A$ is a commutative algebra. Now we define the linear mapping $\overline{D} : A \oplus A \to A \oplus A$ by

$$\overline{D}(a, b) = (D(a), D(b)).$$

Since $D$ is an algebraic nil derivation on $A$, it is not hard to prove that $\overline{D}$ is an algebraic nil derivation on $A \oplus A$. By Theorem 1, we have $\overline{D}(A \oplus A) \subset N(A \oplus A) = N(A) \oplus N(A)$. Therefore $D(A) \subset N(A)$. $\square$

**Corollary 2.7.** Let $A$ be a complex algebra satisfying the condition $[[A, A], A] = 0$ and let $D$ be a nilpotent derivation on $A$. Then $D(A)$ is contained in $N(A)$.

Next, we will be interested with the range of derivations on noncommutative algebra $A$ satisfying the following condition:

$$(\chi) \quad a[A, A]b = 0$$

for all $a, b \in A$.

**Theorem 2.8.** Let $A$ be a complex algebra satisfying the condition $(\chi)$ and let $D$ be an algebraic nil derivation on $A$. Then $D(A)$ is contained in $N(A)$.

**Proof.** Let $x \in A$. Then there exists $n = n(x) \in \mathbb{N}$ such that $D^n(x) = D^n(x^2) = D^n(x^n) = 0$. Let $a, b \in A$. It follows that

$$(4) \quad aD^n(x^2) b = a \left( \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(x) \right) b = 0.$$ 

Moreover, let $n_1, n_2 \in \mathbb{N}$ such that $n_1 + n_2 = n(x)$, then

$$(5) \quad aD^n(x^n) b = a \left( \sum_{k=0}^{n} \binom{n}{k} D^k(x^{n_1}) D^{n-k}(x^{n_2}) \right) b = 0.$$ 

By using the Leibnitz rule for $aD^k(x^{n_1}) b$ and $aD^k(x^{n_2}) b$ in Equality (5), by using Equality (4) and taking into account that $D^n(x) = 0$, we deduce that

$$a(D(x))^n b = 0$$

for all $a, b \in A$. Consequently $(D(x))^{n+2} = 0$. Therefore $D(A) \subset N(A)$. $\square$
Corollary 2.9. Let \(A\) be a complex algebra satisfying the condition \((\chi)\) and let \(D\) be a nilpotent derivation on \(A\), then \(D(A)\) is contained in \(N(A)\).

In the following lines, we recall definitions and some basic facts about lattice-ordered algebras. For more information about this field, one can refer to [1,3]. A (real) algebra \(A\) which is simultaneously a vector lattice such that the partial ordering and the multiplication in \(A\) are compatible, that is \(a, b \in A^+\) implies \(ab \in A^+\) is called lattice-ordered algebra (briefly \(\ell\)-algebra). The \(\ell\)-algebra \(A\) is said to be a \(d\)-algebra whenever \(a \wedge b = 0\) in \(A\) implies \(ac \wedge bc = ca \wedge cb = 0\), for all \(0 \leq c \in A\).

In general, \(d\)-algebras are not commutative, see [3]. Since any Archimedean \(d\)-algebra satisfies the condition \((\chi)\), see [3, Corollary 5.7], we deduce the following result:

Corollary 2.10. Let \(A\) be an Archimedean \(d\)-algebra and let \(D\) be an algebraic nil derivation on \(A\). Then \(D(A)\) is contained in \(N(A)\).

Definition 2.11. Let \(A\) be an algebra. For a fixed \(a \in A\), define \(D : A \to A\) by \(D(x) = [x,a] = xa - ax\), for all \(x \in A\). Then \(D\) is called inner derivation of \(A\) associated with \(a\) and is generally denoted by \(D_a\).

Theorem 2.12. Let \(A\) be an Archimedean \(d\)-algebra with the condition \(Z(A) = \{0\}\), where \(Z(A)\) denotes the center of \(A\) and let \(D\) be an inner derivation on \(A\). Then the following assertions are equivalent:

i) \(D\) is nilpotent;

ii) \(D^3 = 0\);

iii) \(D\) is induced by a nilpotent element.

Proof. i) \(\Rightarrow\) ii) Let \(a \in A\) such that \(D = D_a\). Since any Archimedean \(d\)-algebra satisfies the condition \((\chi)\), then for all \(k \in \mathbb{N}\), we have

\[
D_a^{2k+1}(x) = xa^{2k+1} - a^{2k+1}x
\]

for all \(x \in A\). Since \(D_a\) is nilpotent, there exists \(n \in \mathbb{N}\) such that \(D_n^a = 0\). Therefore

\[
D_a^{2n+1}(x) = xa^{2n+1} - a^{2n+1}x = 0
\]

for all \(x \in A\). Consequently \(a^{2n+1} \in Z(A) = \{0\}\). Hence \(a^{2n+1} = 0\). By [3, Theorem 5.5], we deduce that \(a^3 = 0\). It follows that \(D_a^3 = 0\).

ii) \(\Rightarrow\) iii) \(D^3 = D_a^3 = 0\) means that \(a^3 = 0\). Therefore \(a \in N(A)\).

iii) \(\Rightarrow\) i) This path is obvious.
Remark 2.13. It is obvious that algebraic nil derivations are nil derivations. The simple-minded attempt to extend Theorem 1, 2 and 3 to nil derivations obviously fails. This is illustrated in the following example.

Example 2.14. Let $A = \mathbb{C}[X]$ and $D : A \to A$ defined by

$$D \left( \sum_{i=1}^{n} a_i X^i \right) = a_1 + 2a_2 X + \ldots + na_n X^{n-1}.$$ 

It is not hard to prove that $D$ is a nil derivation but not an algebraic nil derivation, whereas $D(A) = A \neq N(A)$.

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References


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