DYNAMIC RISK MEASURES AND G-EXPECTATION

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Abstract. A standard deviation has been a starting point for a mathematical
definition of risk. As a remedy for drawbacks such as subadditivity property
discouraging the diversification, coherent and convex risk measures are introduced in
an axiomatic approach. Choquet expectation and $g$-expectations, which general-
ize mathematical expectations, are widely used in hedging and pricing contingent
claims in incomplete markets. The each risk measure or expectation give rise to its
own pricing rules. In this paper we investigate relationships among dynamic risk
measures, Choquet expectation and dynamic $g$-expectations in the framework of the
continuous-time asset pricing.

1. Introduction

Various kinds of risk measures have been proposed and discussed to measure or
quantify the market risks in theoretical and practical perspectives. A starting point
for a mathematical definition of risk is simply as standard deviation. Markowitz [19]
used the standard deviation to measure the market risk in his portfolio theory but his
method doesn’t tell the difference between the positive and the negative deviation.
Artzner et al. [2, 3] proposed a coherent risk measure in an axiomatic approach,
and formulated the representation theorems. Frittelli [12] proposed sublinear risk
measures to weaken coherent axioms. Heath [16] firstly studied the convex risk mea-
sures and Föllmer & Schied [9, 10, 11] and Frittelli & Rosazza Gianin [13] extended
them to general probability spaces. They had weakened the conditions of positive
homogeneity and subadditivity by replacing them with convexity.

There exist stochastic phenomena like Allais paradox [1] and Ellsberg paradox [8]
which can not be dealt with linear mathematical expectation in economics. Cho-
quet [6] introduced a nonlinear expectation called Choquet expectation which ap-
plied to many areas such as statistics, economics and finance. But Choquet expect-
ation has a difficulty in defining a conditional expectation. Peng [21] introduced

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a nonlinear expectation, \(g\)-expectation which is a solution of a nonlinear backward stochastic differential equation. It’s easy to define conditional expectation with Peng’s \(g\)-expectation. In this paper, we show that Choquet expectation is equal to \(g\)-expectation under some conditions via \(\{F_t\}_{t \in [0,T]}\)-consistent expectation \(\mathcal{E}\) satisfying \(\mathcal{E}^\mu\)-domination and translability condition.

The coherent (or convex) risk measure which is a static risk measures is defined in section 2. Peng’s \(g\)-expectation, Choquet expectation and dynamic risk measure are introduced in section 3. The relationships between Choquet expectation and \(g\)-expectation are given as in the literature in section 4. It is shown that Choquet expectation is equal to \(g\)-expectation under some conditions via \(\{F_t\}_{t \in [0,T]}\)-consistent expectation \(\mathcal{E}\) in section 5.

2. Static Risk Measures

Risk measures are introduced to measure or quantify investors’ risky positions such as financial contracts or contingent claims. Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(T\) be a fixed horizon time. Assume that \(\mathcal{X} = L^p(\Omega, \mathcal{F}, P)\), with \(1 \leq p \leq +\infty\) is the space of financial positions to be quantified or measured. \(L^p(\Omega, \mathcal{F}, P)\) is endowed with its norm topology for \(p \in (1, +\infty)\) and with the weak topology \(\sigma(L^\infty, L^1)\) for \(p = +\infty\).

**Definition 2.1.** A coherent risk measure \(\rho: \mathcal{X} \to \mathbb{R}\) is a mapping satisfying for \(X, Y \in \mathcal{X}\)

1. \(\rho(X) \geq \rho(Y)\) if \(X \leq Y\) (monotonicity),
2. \(\rho(X + m) = \rho(X) - m\) for \(m \in \mathbb{R}\) (translation invariance),
3. \(\rho(X + Y) \leq \rho(X) + \rho(Y)\) (subadditivity),
4. \(\rho(\lambda X) = \lambda \rho(X)\) for \(\lambda \geq 0\) (positive homogeneity).

The subadditivity and the positive homogeneity can be relaxed to a weaker quantity, i.e. convexity

\[
\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y) \quad \forall \lambda \in [0, 1],
\]

which means diversification should not increase the risk.

3. Peng’s \(g\)-expectation and Choquet Expectation

Let \((W_t)_{t \geq 0}\) a standard \(d\)-dimensional Brownian motion and \((\mathcal{F}_t)_{t \geq 0}\) the augmented filtration associated with the one generated by \((W_t)_{t \geq 0}\). Let \(L^2_\mathcal{F}(T; \mathbb{R}^n)\) be
the space of the adapted processes \( (\xi_t)_{t \in [0, T]} \) such that
\[
E \left[ \int_0^T \| \xi_s \|^2 ds \right] < +\infty.
\]
where \( \| \cdot \| \) represents the Euclidean norm on \( \mathbb{R}^n \).

Suppose that for \( t \in [0, T] \), \( L^2(\mathcal{F}_t) := L^2(\Omega, \mathcal{F}_t, P) \) is the space of real-valued, \( \mathcal{F}_t \)-measurable and square integrable random variables endowed with the \( L^2 \)-norm \( \| \cdot \|_2 \) topology.

Let \( g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) a function that \( g \mapsto g(t, y, z) \) is measurable for each \((y, z) \in \mathbb{R} \times \mathbb{R}^n \) and satisfy the following conditions
\[
|g(t, y, z) - g(t, \bar{y}, \bar{z})| \leq K(|y - \bar{y}| + |z - \bar{z}|) \quad \forall t \in [0, T], \forall (y, z), (\bar{y}, \bar{z}) \in \mathbb{R} \times \mathbb{R}^n, \text{ for some } K > 0,
\]
\[
\int_0^T |g(t, 0, 0)|^2 dt < \infty,
\]
\[
\text{For each } (t, y) \in [0, T] \times \mathbb{R}, g(t, y, 0) = 0.
\]

**Theorem 3.1** ([20]). For every terminal condition \( X \in L^2(\mathcal{F}_T) \) the following backward stochastic differential equation
\[
- \frac{dy_t}{dt} = g(t, y_t, z_t) dt - z_t dW_t, \quad 0 \leq t \leq T
\]
\[
y_T = X
\]
has a unique solution \((y_t, z_t)_{t \in [0, T]} \in L^2(\mathbb{T}; \mathbb{R}) \times L^2(\mathbb{T}; \mathbb{R}^n)\).

**Definition 3.2.** For each \( X \in L^2(\mathcal{F}_T) \) and for each \( t \in [0, T] \) \( g \)-expectation of \( X \) and the conditional \( g \)-expectation of \( X \) under \( \mathcal{F}_t \) is respectively defined by
\[
\mathcal{E}_g[X] := y_0, \quad \mathcal{E}_g[X | \mathcal{F}_t] := y_t,
\]
where \( y_t \) is the solution of the BSDE (3.2).

Since \( g \)-expectation and conditional \( g \)-expectation can be considered as the extension of classic mathematical expectation and conditional mathematical expectation, they preserve most properties of classic mathematical expectation and conditional mathematical expectation except the linearity.

**Definition 3.3.** A real-valued set function \( c : \mathcal{F} \to [0, 1] \) is called capacity if it satisfies (1) \( c(A) \leq c(B) \) for \( A \subset B \), (2) \( c(\emptyset) = 0 \) and \( c(\Omega) = 1 \).

**Definition 3.4.** A capacity is called submodular or 2-alternating if
\(c(A \cup B) + c(A \cap B) \leq c(A) + c(B).\)

**Definition 3.5.** Two measurable functions \(X\) and \(Y\) on \((\Omega, \mathcal{F})\) are called *comonotone* if there exists a measurable function \(Z\) on \((\Omega, \mathcal{F})\) and increasing functions \(f\) and \(g\) on \(\mathbb{R}\) such that
\[X = f(Z)\text{ and }Y = g(Z).\]

A risk measure \(\rho\) on \(L^p(\mathcal{F}_T)\) is called *comonotonic* if
\[\rho(X + Y) = \rho(X) + \rho(Y)\]
whenever \(X\) and \(Y\) are comonotonic.

Define the Choquet integral of the loss as
\[\rho(X) := \int (-X) \, dc.\]

Then \(\rho : \mathcal{X} \to \mathbb{R}\) satisfies monotonicity, translation invariance and positive homogeneity, and other properties according to the given conditions.

1. (Constant preserving) \(\int \lambda dc = \lambda\) for constant \(\lambda\).
2. (Monotonicity) If \(X \leq Y\), then \(\int (-X) dc \geq \int (-Y) dc\).
3. (Positive homogeneity) For \(\lambda \geq 0\), \(\int \lambda (-X) dc = \lambda \int (-X) dc\).
4. (Translation invariance) \(\int (-X + m) \, dc = \int (-X) \, dc + m, \ m \in \mathbb{R}\).
5. (Comonotone additivity) If \(X\) and \(Y\) are comonotone functions, then
\[\int [(-X) + (-Y)] \, dc = \int (-X) \, dc + \int (-Y) \, dc.\]
6. (Subadditivity) If \(c\) is submodular or concave function, then
\[\int (X + Y) \, dc \leq \int X \, dc + \int Y \, dc.\]

The static risk measures do not account for payoffs or new information according to the time evolution (refer to [25, 26]).

**Definition 3.6.** A dynamic risk measures are defined as the mappings \((\rho_t)_{t \in [0, T]}\) satisfying

1. \(\rho_t : L^p(\mathcal{F}_T) \to L^0(\Omega, \mathcal{F}_t, P)\), for all \(t \in [0, T]\),
2. \(\rho_0\) is a static risk measure,
3. \(\rho_T(X) = -X\quad P\text{-a.s.},\) for all \(X \in L^p(\mathcal{F}_T)\).
4. Nonlinear Expectations and Nonlinear Pricing

To quantify riskiness of financial positions, coherent (or convex) risk measures, Choquet expectation and $g$-expectation are widely used. It depends on practitioner’s appropriate choices. The paper [5] shows that the pricing with the coherent risk measure is less than one with the Choquet expectation.

Denote the Choquet expectation $C(\cdot)$ as $C_g(\cdot)$ with respect to the capacity $V_g$ defined as

$$V_g(A) := E_g[I_A] \quad \forall A \in \mathcal{F}_T.$$  

**Theorem 4.1** ([5]). If $E_g[\cdot]$ is a coherent risk measure, then $E_g[\cdot]$ is bounded by the Choquet expectation $C_g(\cdot)$, that is

$$E_g[X] \leq C_g(X), \quad X \in L^2(\Omega, \mathcal{F}, P).$$

But if $E_g[\cdot]$ is a convex risk measure, then the above inequality does not hold generally.

**Theorem 4.2** ([15]). Let $g$ be convex function with respect to $z$, independent of $y$ and deterministic. Let $g$ also satisfy (3.1). Then $\rho^g(X) \leq C_g[-X]$ for $X \in L^2(\mathcal{F}_T)$ if and only if $\rho^g$ is a coherent risk measure. Here $\rho^g(X)$ is defined as $\rho^g(X) := E_g[-X]$ for $X \in L^2(\mathcal{F}_T)$.

Note that $\rho^g : L^2(\mathcal{F}_T) \mapsto \mathbb{R}$ is a coherent (or convex) risk measure if and only if $g$ is independent of $y$ and is positively homogeneous and subadditive (or convex) with respect to $z$ (see [23, 14, 22]).

The positive homogeneity and comonotonic additivity hold in the Choquet expectation. The time consistency holds in the $g$-expectation.

$$E[\xi + \eta] = E[\xi] + E[\eta] \quad \forall \xi, \eta \in L^2(\Omega, \mathcal{F}, P).$$

The above equality holds for the Choquet expectation if $\xi$ and $\eta$ are comonotonic. But if $g$ is nonlinear, the above equality does not hold for the $g$-expectation even if $\xi$ and $\eta$ are comonotonic. These facts means that $g$-expectation is more nonlinear than the Choquet expectation on $L^2(\Omega, \mathcal{F}, P)$ [15].

The following Lemmas (4.3) and (4.6), Proposition (4.4), and Theorem (4.5) are from the paper [5].

**Lemma 4.3.** For any $X \in L^2(\Omega, \mathcal{F}_T, P)$, there exists unique $\eta \in L^2(\Omega, \mathcal{F}_t, P)$ such that

$$E_g[I_A X] = E_g[I_A \eta] \quad \forall A \in \mathcal{F}_t.$$
The $\eta$ is called the conditional $g$-expectation of $X$ and it is written as $\mathcal{E}_g[X|\mathcal{F}_t]$. This $\mathcal{E}_g[X|\mathcal{F}_t]$ is exactly the $y_t$ which is the solution of BSDE (3.2).

**Proposition 4.4.** Let $\mu = \{\mu_t\}_{t \in [0,T]}$ be a continuous functions. Suppose that $g(t,y,z) = \mu_t |z_t|$ and the process $(z_t)_{t \in [0,T]}$ is one dimensional. Then for any $\xi \in L^2(\Omega, \mathcal{F}, P)$, the conditional $g$-expectation satisfies

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = \operatorname{ess sup}_{Q \in \mathcal{Q}} E_Q[\xi|\mathcal{F}_t]$$

for $\mu > 0$.

**Theorem 4.5** ([5]). Suppose that $g$ satisfies the given Hypotheses. Then there exists a Choquet expectation whose restriction to $L^2(\Omega, \mathcal{F}, P)$ is equal to a $g$-expectation if and only if $g$ is independent of $y$ and is linear in $z$, i.e. there exists a continuous function $\nu(t)$ such that

$$g(y, z, t) = \nu(t)z.$$

**Lemma 4.6.** Suppose that $g$ is a convex (or concave) function. If $\mathcal{E}_g[\cdot]$ is comonotonic additive on $L^2_+(\Omega, \mathcal{F}, P)$ (or $L^2_-(\Omega, \mathcal{F}, P)$), then $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is also comonotonic additive on $L^2_+(\Omega, \mathcal{F}, P)$ (or $L^2_-(\Omega, \mathcal{F}, P)$) for any $t \in [0,T)$.

**Corollary 4.7.** Suppose that $g$ is a convex (or concave) function. If $\mathcal{E}_g[\cdot]$ is a Choquet expectation on $L^2_+(\Omega, \mathcal{F}, P)$ (or $L^2_-(\Omega, \mathcal{F}, P)$), then $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is also a Choquet expectation on $L^2_+(\Omega, \mathcal{F}, P)$ (or $L^2_-(\Omega, \mathcal{F}, P)$) for any $t \in [0,T)$.

5. $\mathcal{F}_t$-consistent Expectation

In this section, an $\{\mathcal{F}_t\}_{t \in [0,T]}$-consistent expectation $\mathcal{E}$ is defined as a nonlinear functional on $L^2(\mathcal{F}_T)$. We’ll show that Choquet expectation is an $\{\mathcal{F}_t\}_{t \in [0,T]}$-consistent expectation $\mathcal{E}$ under some conditions.

**Definition 5.1.** A nonlinear expectation is defined as a functional $\mathcal{E} : L^2(\mathcal{F}_T) \to \mathbb{R}$ satisfying

1. (Monotonicity) If $X \geq Y$ $P$-a.s., then $\mathcal{E}(X) \geq \mathcal{E}(Y)$. Moreover, under the inequality $X \geq Y$, $\mathcal{E}(X) = \mathcal{E}(Y)$ if and only if $X = Y$ $P$-a.s..

2. (Constancy) $\mathcal{E}(c) = c$ $\forall c \in \mathbb{R}$. 
Definition 5.2. An \( \{ \mathcal{F}_t \}_{t \in [0,T]} \)-consistent expectation \( \mathcal{E} \) is defined as the nonlinear expectation \( \mathcal{E} \) such that if for any \( X \in L^2(\mathcal{F}_T) \) and any \( t \in [0,T] \) there exists \( \eta \in L^2(\mathcal{F}_t) \) satisfying
\[
\mathcal{E}[1_A X] = \mathcal{E}[1_A \eta] \quad \forall A \in \mathcal{F}_t.
\]
(5.1)

The \( \eta \) satisfying (5.1) is called conditional \( \{ \mathcal{F}_t \}_{t \in [0,T]} \)-consistent expectation of \( X \) under \( \mathcal{F}_t \) and denoted by \( \mathcal{E}[X | \mathcal{F}_t] \).

Definition 5.3. It is called that \( \{ \mathcal{F}_t \}_{t \in [0,T]} \)-consistent expectation \( \mathcal{E} \) is dominated by \( \mathcal{E}^\mu \) (\( \mu > 0 \)) if
\[
\mathcal{E}[X + Y] - \mathcal{E}[X] \leq \mathcal{E}^\mu[Y] \quad \forall X, Y \in L^2(\mathcal{F}_T)
\]
where \( \mathcal{E}^\mu \) is \( g \)-expectation with \( g(t, y, z) = \mu |z| \).

An \( \{ \mathcal{F}_t \}_{t \in [0,T]} \)-consistent expectation \( \mathcal{E} \) is called to satisfy the translability condition if
\[
\mathcal{E}[X + \beta | \mathcal{F}_t] = \mathcal{E}[X | \mathcal{F}_t] + \beta \quad \forall X \in L^2(\mathcal{F}_T), \forall \beta \in L^2(\mathcal{F}_t).
\]
(5.2)

The following theorem tells us the relationships between conditional \( g \)-expectation and \( \{ \mathcal{F}_t \}_{t \in [0,T]} \)-consistent expectation.

Theorem 5.4 ([7]). Let \( \mathcal{E} : L^2(\mathcal{F}_T) \rightarrow \mathbb{R} \) be a \( \{ \mathcal{F}_t \}_{t \in [0,T]} \)-consistent expectation. If \( \mathcal{E} \) is \( \mathcal{E}^\mu \)-dominated for some \( \mu > 0 \) and if it satisfies translability condition (5.2), then there exists a unique \( g \) which is independent of \( y \), satisfies the assumptions (3.1) and \( |g(t, z)| \leq \mu |z| \) such that
\[
\mathcal{E}[X] = \mathcal{E}_g[X] \quad \text{and} \quad \mathcal{E}[X | \mathcal{F}_t] = \mathcal{E}_g[X | \mathcal{F}_t] \quad \forall X \in L^2(\mathcal{F}_T).
\]

Theorem 5.5 ([11]). For the Choquet integral with respect to a capacity \( c \), the following are equivalent.

1. \( \rho_0(X) := \int (-X) \, dc \) is a convex risk measure on \( L^2(\mathcal{F}_T) \).
2. \( \rho_0(X) := \int (-X) \, dc \) is a coherent risk measure on \( L^2(\mathcal{F}_T) \).
3. For \( Q_c := \{ Q \in \mathcal{M}_{1,f} \mid Q[A] \leq c(A) \ \forall A \in \mathcal{F}_T \} \),
\[
\int X \, dc = \sup_{Q \in Q_c} E_Q[X] \quad \text{for} \ X \in L^2(\mathcal{F}_T).
\]
(5.3)
4. The set function \( c \) is submodular. In this case, \( Q_c = Q_{\max} \).

The set \( \mathcal{M}_{1,f} = \mathcal{M}_{1,f}(\Omega, \mathcal{F}) \) in Theorem (5.3) is the one of all finitely additive set functions \( Q : \mathcal{F} \rightarrow [0,1] \) which is normalized to \( Q[\Omega] = 1 \). The \( Q_{\max} \) is defined
as

\[ Q_{\text{max}} := \left\{ Q \in \mathcal{M}_{1,f} \mid \sup_{X \in A} E_Q[-X] = 0 \right\} \]

where \( A_\rho \) is defined as

\[ A_\rho := \{ X \in L^2(\mathcal{F}_T) \mid \rho(X) \leq 0 \}. \]

From the viewpoint of Proposition (4.4) and Theorem (4.5), the set \( Q_c \) of (5.3) is unnecessarily too large so that it could be reduced to a suitable set of probability measures for consistency, i.e.

\[ Q_c := \left\{ Q \in \mathcal{M}_{1,f} \mid \rho(X) \leq 0 \right\}. \]

It can be shown that \( Q_c \) is indeed the set of equivalent martingale measures by the following Proposition (5.6).

**Proposition 5.6 ([11]).** If \( Q \ll P \) on \( \mathcal{F} \), then \( Q \) is equivalent to \( P \) if and only if \( \frac{dQ}{dP} > 0 \) \( P \)-a.s.

Assume that the capacity \( c \) is submodular. Under the new set \( Q_c \) as in (5.4), we define a nonlinear expectation \( \mathcal{E} : L^2(\mathcal{F}_T) \rightarrow \mathbb{R} \) as

\[ \mathcal{E}[X] := \int X \, dc = \text{ess sup}_{Q \in Q_c} E_Q[X], \quad X \in L^2(\mathcal{F}_T). \]

We will show that the above \( \mathcal{E}[X] \) satisfies all the assumptions of Theorem (5.4). It is easy to show that \( \mathcal{E}[X] \) satisfies the monotonicity and constancy in the Definition (5.1) but if \( X \geq Y \), \( \mathcal{E}[X] = \mathcal{E}[Y] \) if and only if \( X = Y \) \( P \)-a.s. Suppose that \( X \geq Y \) and \( \mathcal{E}[X] = \mathcal{E}[Y] \). We prove it contrapositively. Suppose \( X = Y \) \( P \)-a.s. does not hold. Let \( A = \{ w \in \Omega \mid X \neq Y \} \in \mathcal{F} \). Then \( E_Q[1_A X] > E_Q[1_A Y] \) for each \( Q \in Q_c \) and there exists a \( r \in \mathbb{R} \) such that \( E_Q[1_A X] > r > E_Q[1_A Y] \). By taking supremum on the left hand side first, we have \( \text{ess sup}_{Q \in Q_c} E_Q[1_A X] > r \) for \( Q \in Q_c \) and so \( \text{ess sup}_{Q \in Q_c} E_Q[1_A X] > r \). By definition, it’s a contradiction.

We need the stability property of a set \( Q_c \) to show that \( \mathcal{E}[X] \) is a \( \{ \mathcal{F}_t \}_{t \in [0,T]} \) consistent expectation. In the following definitions, the stopping times \( \sigma \) and \( \tau \) can be replaced by \( t \in [0,T] \) without any loss.
Definition 5.7. Let $Q_1$ and $Q_2$ be two equivalent probability measures and $\sigma$ be a stopping time. The probability measure

$$\tilde{Q}[A] := E_{Q_1} [Q_2[A|F_{\sigma}]], \quad A \in \mathcal{F}_T,$$

is called the pasting of $Q_1$ and $Q_2$ in $\sigma$.

Note that by the monotone convergence theorem for conditional expectation $\tilde{Q}$ is a probability measure and

$$E_{\tilde{Q}}[Y] := E_{Q_1} [E_{Q_2}[Y|F_{\sigma}]], \quad \forall Y \in L^2(\mathcal{F}_T), Y \geq 0.$$

Definition 5.8. A set $Q$ of equivalent probability measures on $(\Omega, \mathcal{F})$ is called stable if, for any $Q_1, Q_2 \in Q$ and the stopping time $\sigma$, also their pasting in $\sigma$ is contained in $Q$.

Proposition 5.9 ([11]). The set $Q_c$ of equivalent martingale measures is stable.

Theorem 5.10 ([11]). Let $Q$ be a set of equivalent probability measures. If $Q$ is stable, then the following holds for $X \in L^2(\mathcal{F}_T)$

$$\text{ess sup}_{Q \in Q} E_Q[X|\mathcal{F}_t] = \text{sup}_{\mathcal{Q} \in Q} \text{ess sup}_{Q' \in \mathcal{Q}} E_{Q'}[X|\mathcal{F}_s]|\mathcal{F}_t] \quad \forall t, s \in [0, T] \text{ with } t \leq s.$$

From the Theorem (5.10), we can easily see that $\mathcal{E}[X]$ is a $\{\mathcal{F}_t\}_{t \in [0, T]}$-consistent expectation condition (5.1), $\mathcal{E}[1_A X] = \mathcal{E}[1_A \cdot \mathcal{E}[X|\mathcal{F}_t]] \forall A \in \mathcal{F}_t$.

Let us show that $\{\mathcal{F}_t\}_{t \in [0, T]}$-consistent expectation $\mathcal{E}$ is dominated by $\mathcal{E}^\mu (\mu > 0)$. Since $\mathcal{E}[X + Y] - \mathcal{E}[X] \leq \text{ess sup}_{Q \in Q_c} E_Q[Y]$ and there exists $g$-expectation $\mathcal{E}^\mu$ with $g(t, y, z) = \mu z$ satisfying $\mathcal{E}^\mu[X] = \text{ess sup}_{Q \in Q_c} E_Q[Y]$ by Theorem (4.5), $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$. Note that $\mathcal{E}^\mu$-dominated nonlinear expectation $\mathcal{E}$ implies that $\mathcal{E}$ is lower semi-continuous [7].

Finally we show that $\{\mathcal{F}_t\}_{t \in [0, T]}$-consistent expectation $\mathcal{E}$ satisfies the translability condition. Let $X \in L^2(\mathcal{F}_T)$ and $\beta \in L^2(\mathcal{F}_t)$. Then by the definition of $\mathcal{E}$ we have

$$\mathcal{E}[(X + \beta)|\mathcal{F}_t] = \text{ess sup}_{Q \in Q_c} E_Q[(X + \beta)|\mathcal{F}_t] = \text{ess sup}_{Q \in Q_c} E_Q[X|\mathcal{F}_t] + \beta = \mathcal{E}[X|\mathcal{F}_t] + \beta.$$

Therefore, the nonlinear expectation $\mathcal{E}$ defined as (5.5) satisfies the all the conditions of Theorem (5.4). Thus the results so far can be summarized in the following

Theorem 5.11. Let the nonlinear expectation $\mathcal{E}$ be defined as (5.5). Then there exists a unique $g$ which is independent of $y$, satisfies the assumptions (3.1) and
\[ |g(t, z)| \leq \mu |z| \] such that

\[ \mathcal{E}[X] := \int X \, dc = \mathcal{E}_g[X] \quad \text{and} \quad \mathcal{E}[X | \mathcal{F}_t] := \int X | \mathcal{F}_t \, dc = \mathcal{E}_g[X | \mathcal{F}_t] \quad \forall X \in L^2(\mathcal{F}_T). \]

Note that the generator \( g \) in Theorem (5.11) should be the form of \( g(t, y, z) = \mu t z \) which is linear in \( z \) and so \( \mathcal{E}_g = \mathcal{E}^\mu \) to be consistent to the results of Theorem (4.5).

In fact, for \( g(t, y, z) = \mu t z \), let us consider the BSDE

\[ Y_t = X + \int_t^T \mu_s z_s \, ds - \int_t^T z_s \, dW_s, \quad X \in L^2(\mathcal{F}_T). \]

The above differential equation (5.6) is reduced to

\[ Y_t = X - \int_t^T z_s \, d\tilde{W}_s, \quad \tilde{W}_t = W_t - \int_0^t \mu_s \, ds. \]

By Girsanov’s Theorem, \((\tilde{W}_t)_{0 \leq t \leq T}\) is a \( Q \)-Brownian motion under \( Q \) defined as

\[ \frac{dQ}{dP} = \exp \left[ - \frac{1}{2} \int_0^T \mu_s^2 \, ds + \int_0^T \mu_s \, dW_s \right]. \]

Therefore we have the relations

\[ \mathcal{E}_g[X] = E_Q[X], \quad \mathcal{E}_g[X | \mathcal{F}_t] = E_Q[X | \mathcal{F}_t] \]

which means that \( g \)-expectation is a classical mathematical expectation.

**Proposition 5.12** ([23]). *Let the risk measure \( \rho^g_t(X) \) be defined as

\[ \rho^g_t(X) := \mathcal{E}_g[-X | \mathcal{F}_t], \quad \forall X \in L^2(\mathcal{F}_T), \quad \forall t \in [0, T] \]

where \( g \) satisfies the conditions (3.1). Moreover, if \( g \) is sublinear in \((y, z)\), i.e. positively homogeneous in \((y, z)\) and subadditive in \((y, z)\), then \((\rho^g_t)_{t \in [0, T]} \) is a dynamic coherent and time-consistent risk measure.*

Note that if \( g \) satisfies both positive homogeneity and subadditivity, \( g \) is independent of \( y \). The proposition (5.12) and Theorem (4.2) tells us that for Theorem (5.11) to hold the linearity of \( g \) is necessary.

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