THE EXISTENCE OF THE RISK-EFFICIENT OPTIONS

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ABSTRACT. We prove the existence of the risk-efficient options proposed by Xu [7]. The proof is given by both indirect and direct ways. Schied [6] showed the existence of the optimal solution of equation (2.1). The one is to use the Schied’s result. The other one is to find the sequences converging to the risk-efficient option.

1. Introduction

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\) be a complete filtered probability space. Let \(S = (S_t)_{0 \leq t \leq T}\) be an adapted positive process which is a semimartingale. It is assumed that the riskless interest rate is zero for simplicity and

\[ \mathcal{M} = \{ Q \mid Q \sim P, S \text{ is a local martingale under } Q \} \neq \emptyset \]

to avoid the arbitrage opportunities [4].

Definition 1.1. A self-financing strategy \((x, \xi)\) is defined as an initial capital \(x \geq 0\) and a predictable process \(\xi_t\) such that the value process (value of the current holdings)

\[ X_t = x + \int_0^t \xi_u dS_u, \quad t \in [0, T] \]

is \(P\)-a.s. well-defined.

The set of admissible self-financing portfolios \(\mathcal{X}(x)\) with initial capital \(x\) is defined as

\[ \mathcal{X}(x) = \left\{ X \mid X_t = x + \int_0^t \xi_u dS_u \geq c, c \text{ is a constant, } t \in [0, T] \right\}. \]

Let \(L^0\) be the set of all measurable functions in the given probability spaces.
**Definition 1.2.** A coherent measure of risk $\rho : L^0 \to \mathbb{R} \cup \{\infty\}$ is a mapping satisfying the following properties for $X, Y \in L^0$

1. $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (subadditivity),
2. $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \geq 0$ (positive homogeneity),
3. $\rho(X) \geq \rho(Y)$ if $X \leq Y$ (monotonicity),
4. $\rho(Y + m) = \rho(Y) - m$ for $m \in \mathbb{R}$ (translation invariance).

The conditions of subadditivity (1) and positive homogeneity (2) in Definition 1.2 can be relaxed to a weaker quantity, i.e., convexity

\[
\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y) \quad \text{for any} \quad \lambda \in [0, 1].
\]

Convexity means that diversification does not increase the risk. Also refer to the papers [1, 3] for coherent or convex risk measures.

**Definition 1.3.** A map $\rho : L^0 \to \mathbb{R}$ is called a convex risk measure if it satisfies the properties of convexity (1.1), monotonicity (3) and translation invariance (4).

**Definition 1.4.** The minimal risk $\rho^x(\cdot)$ with initial capital $x$ is defined as the risk

\[
\rho^x(L) = \inf_{X \in \mathcal{X}(x)} \rho(L - X_T)
\]

where the liability $L$ is a random variable bounded below by a constant at time $T$, $X_T = x + \int_0^T \xi_u dS_u$ and $\rho(L - X_T)$ is a final risk.

**Assumption 1.5.** The convex risk measure $\rho$ satisfies the Fatou property

\[
\rho(X) \leq \lim_{n \to \infty} \inf \rho(X_n) \quad \text{if} \quad X_n \to X \quad \text{a.s. as} \quad n \to \infty.
\]

**Assumption 1.6.** $\rho : L^0 \to \mathbb{R}$ satisfies $\rho(X) = \rho(Y)$ whenever $X = Y$ $P$-a.s. and for the positive payoff function $H$, the bounded conditions

\[
\rho(L + H) < \infty \quad \text{and} \quad -\infty < \rho^0(0).
\]

**Lemma 1.7** ([7]). The minimal risk defined as (1.2) is a convex risk measure. Moreover, the translation invariance property satisfies the following relations

\[
\rho^{x_1}(X - x_2) = \rho^{x_1 + x_2}(X) = \rho^{x_1}(X) - x_2 \quad \text{for any} \quad x_1, x_2 \in \mathbb{R}^+.
\]

**Lemma 1.8** ([7]). Let $L$ be the initial liability bounded below by a constant and $H$ be the positive payoff function. Then for any fixed number $x$

\[
-\infty < \rho(L - H) \leq \rho(L) \leq \rho(L + H) < \infty \quad \text{and}
\]

\[
-\infty < \rho^x(L - H) \leq \rho^x(L) \leq \rho^x(L + H) < \infty.
\]
The risk-efficient options are defined as the options having the same selling price, which minimize the risk. That is, the risk-efficient options are the $H$ that minimizes $\rho^{x_0+\alpha}(L + H)$ with the constraint $p(H) = \alpha$, where $p(H)$ is the selling price of the option $H$, $L$ is the initial liability, $x_0$ is the initial capital, and $\rho^{x_0+\alpha}(L + H)$ is the minimal risk obtained by optimal hedging with capital $x_0 + \alpha$ as defined in (1.2). Here $\rho$ is a risk measure. Xu [7] defined such risk-efficient options and asked a question of their existence. The option seller could get the same minimal risk even though he or she choose any one of available risk-efficient options. Every contingent claim is replicable, i.e., perfectly hedged in a complete market. We should consider risk-efficient options in an incomplete market.

This paper is structured as follows. We prove the existence of risk-efficient options by using Schied’s result in Section 2. We prove it by finding the sequences converging to the risk-efficient option in Section 3.

2. Indirect Proof

In this section, we assume that $\rho$ is convex risk measure satisfying Fatou property and $H$ is $\mathcal{F}_T$-measurable contingent claim which is bounded. Xu [7] treated option $H$ which is positive.

Schied [6] supposes an agent wishes to raise the capital $v(\geq 0)$ by selling a contingent claim and tries to find a contingent claim such that the risk of the terminal liability is minimal among all claims satisfying the issuer’s capital constraints, i.e.,

$$\min_{0 \leq H \leq K} E[\varphi_H] \geq v,$$

where the price density $\varphi$ is a $P$-a.s. strictly positive random variable with $E[\varphi] = 1$.

The problem is called the Neyman-Pearson problem for the risk measure $\rho$.

**Lemma 2.1** ([6]). Assume that the conditions of convexity (1.1), monotonicity in Definition 1.2 and Fatou property (1.3) hold. Then there exists a solution to the Neyman-Pearson problem (2.1).

**Lemma 2.2** ([6]). Any solution $H^*$ of the Neyman-Pearson problem (2.1) with capital constraint $v \in [0, K]$ satisfies $E[\varphi H^*] = v$.

In terms of liabilities $-X$ and $-Y$, the properties of convexity (1.1), monotonicity (3) and translation invariance (4) in Definition 1.2 are respectively expressed as
(2.2) \( \rho(\lambda(-X) + (1 - \lambda)(-Y)) \leq \lambda \rho(-X) + (1 - \lambda)\rho(-Y) \) for \( \lambda \in [0, 1] \),
(2.3) \( \rho(-X) \leq \rho(-Y) \) if \( X \leq Y \),
(2.4) \( \rho(-X + m) = \rho(-X) + m \) for \( m \in \mathbb{R} \).

The properties of (2.2), (2.3) and (2.4) can be easily derived by taking \( \rho(-X) = \psi(X) \) for a convex risk measure \( \psi \).

For the option payoff function \( H \) and an initial capital \( x_0 \), we show that in Theorem 2.4 there exists a risk-efficient option \( H^* \) satisfying
\[
\inf_{\bar{H} \leq K} \rho^{x+x_0}(L + H) = \rho^{x+x_0}(L + H^*),
\]
where \( L \) is the initial liability uniformly bounded below by \( c_L \), and the price density \( \varphi \) is a \( P \)-a.s. strictly positive random variable with \( E[\varphi] = 1 \).

In a term of liability \( -H \), define \( \eta \) as
\[
\eta(-H) := \rho^{x+x_0}(L + H).
\]

Then \( \eta \) is well defined by Assumption 1.6.

Lemma 2.3. \( \eta(-H) \) is a convex risk measure and law-invariant.

Proof. First, let’s prove the convexity. Let \( H_1, H_2 \) and \( H \) be \( \mathcal{F}_T \)-measurable payoff functions and \( \lambda \in [0, 1], m \in \mathbb{R} \).
\[
\eta(\lambda(H_1) + (1 - \lambda)(H_2)) = \rho^{x+x_0}(L + \lambda H_1 + (1 - \lambda)H_2) \leq \lambda \rho^{x+x_0}(L + H_1) + (1 - \lambda)\rho^{x+x_0}(L + H_2) = \lambda \eta(-H_1) + (1 - \lambda)\eta(-H_2).
\]

Secondly, let’s prove the monotonicity. Let \( H_1 \leq H_2 \). Then
\[
\eta(-H_1) = \rho^{x+x_0}(L + H_1) = \inf_{X \in \mathcal{X}(x+x_0)} \rho(L + H_1 - X_T) \leq \inf_{X \in \mathcal{X}(x+x_0)} \rho(L + H_2 - X_T) = \eta(-H_2).
\]

Thirdly, let’s prove the translation invariance.
\[
\eta(-H + m) = \rho^{x+x_0}(L - (-H + m)) = \inf_{X \in \mathcal{X}(x+x_0)} \rho(L + H - X_T - m) = \inf_{X \in \mathcal{X}(x+x_0)} \rho(L + H - X_T) + m = \rho^{x+x_0}(L + H) + m = \eta(-H) + m.
\]
So $\eta$ is a convex risk measure.

Last, let’s prove $\eta(-H_1) = \eta(-H_2)$ whenever $H_1 = H_2$ $P$–a.s.. Let $H_1 = H_2$ $P$–a.s.. Then we have $L + H_1 = L + H_2$ $P$–a.s.. Since $\rho(L + H_1) = \rho(L + H_2)$, we get

$$\eta(-H_1) = \rho^{x+\rho_0}(L + H_1) = \rho^{x+\rho_0}(L + H_2) = \eta(-H_2).$$

□

**Theorem 2.4.** If $x \in (0, K)$, then there exists $H^* \in [0, K]$, $E[\varphi H^*] = x$ such that

$$\inf_{0 \leq H \leq K} \inf_{E[\varphi H] \geq x} \eta(-H) = \eta(-H^*) \iff \inf_{0 \leq H \leq K} \rho^{x+\rho_0}(L + H) = \rho^{x+\rho_0}(L + H^*).$$

**Proof.** $\eta(H)$ is a convex risk measure by Lemma 2.3. By Lemmas 2.1 and 2.2, it is proved. □

Now we give bounded conditions to $x$ for the $E[\varphi H^*] = x$ to be a no-arbitrage price. Xu [7] defined the selling price $SP$ and the buying price $BP$ of the option $H(\geq 0)$ as

$$SP(H) = \min\{x : \rho^{x_0+x}(L + H) \leq \rho^{x_0}(L)\},$$
$$BP(H) = \max\{x : \rho^{x_0-x}(L - H) \leq \rho^{x_0}(L)\}$$

respectively.

By the translation invariance relation (1.5), the equations (2.6) and (2.7) become

$$SP(H) = \min\{x : \rho^{x_0}(L + H) - \rho^{x_0}(L) \leq x\}$$
$$= \rho^{x_0}(L + H) - \rho^{x_0}(L),$$
$$BP(H) = \max\{x : x \leq \rho^{x_0}(L) - \rho^{x_0}(L - H)\}$$
$$= \rho^{x_0}(L) - \rho^{x_0}(L - H)$$

respectively. Since the final risk exposure both $\rho^{x_0+x}(L + H)$ and $\rho^{x_0-x}(L - H)$ do not exceed the initial risk $\rho^{x_0}(L)$, i.e.,

$$\rho^{x_0}(L + H) - x = \rho^{x_0+x}(L + H) \leq \rho^{x_0}(L),$$
$$\rho^{x_0}(L - H) + x = \rho^{x_0-x}(L - H) \leq \rho^{x_0}(L),$$

we have

$$SP(H) = \rho^{x_0}(L + H) - \rho^{x_0}(L) \leq x \leq \rho^{x_0}(L) - \rho^{x_0}(L - H) = BP(H).$$

(2.8)
Thus for the $E[\varphi H^*] = x$ to be a no-arbitrage price of $H^*$, it should satisfy the inequalities

$$SP(H) \leq E[\varphi H^*] = x \leq BP(H).$$

3. DIRECT PROOF

In this section, we find the sequences converging to the risk-efficient option for the proof of its existence.

**Lemma 3.1** (Föllmer and Schied [5]). Let $(\xi_n)_{n \geq 1}$ be a sequence in $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$ such that $\sup_n |\xi_n| < +\infty$ $P$-a.s.. Then there exists a sequence of convex combinations

$$\eta_n \in \text{conv}\{\xi_n, \xi_{n+1}, \ldots\}$$

which converges $P$-a.s. to some $\eta \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$.

Define

$$\mathcal{X}(x, b) = \{X | X \in \mathcal{X}(x) \text{ and } X_T \geq x - b\}. $$

Then we have

$$\mathcal{X}(x) = \bigcup_{b \in \mathbb{R}^+} \mathcal{X}(x, b). $$

**Theorem 3.2** ([7]). Under two assumptions (1.3) and (1.4) and $\mathcal{M} \neq \emptyset$, there exists an optimal admissible hedging portfolio $X^* \in \mathcal{X}(x, b)$ which is the solution of the minimal risk problem

$$\rho^Q_x(L) := \inf_{X \in \mathcal{X}(x, b)} \rho(L - X_T) = \rho(L - X^*_T), $$

for any $b \in \mathbb{R}^+$ and $x \in \mathbb{R}$.

Let $H$ be a payoff function of an option, $x \in \mathbb{R}^+$, and let $Q \in \mathcal{M}$ be fixed.

**Lemma 3.3.** There exists $\mathcal{F}$–measurable $H^*$ and $X^b_{T,H} \in \mathcal{X}(x, b)$, depending on $H^*$ such that $E^Q[H^*] = x,$

$$\inf_{E^Q[H]=x} \rho^Q_x(L + H) = \rho(L + H^* - X^b_{T,H}) := \rho^Q_x(L + H^*).$$

**Proof.** By Theorem 3.2, for each $H$ there exists $X^b_{T,H} \in \mathcal{X}(x, b)$ such that

$$\rho^Q_x(L + H) := \inf_{X \in \mathcal{X}(x, b)} \rho(L + H - X_T) = \rho(L + H - X^b_{T,H}).$$
Choose the sequences $H_n$ and $X^n_T \in \mathcal{X}(x, b)$ satisfying
\[
E^Q[H_n] = x, \quad \rho(L + H_n - X^n_T) \xrightarrow{\inf} \rho^x(Q(L + H)).
\]

Then Lemma 3.1 implies that there exist the sequences $\tilde{X}^n_T \in \text{conv}\{X^n_T, X^{n+1}_T, \ldots\}$ such that
\[
\tilde{X}^n_T \to X^{x*}_T \in \mathcal{X}(x, b) \quad \text{as } n \to \infty.
\]

The sequence $\tilde{X}^n_T$ can be expressed as the convex combination
\[
\tilde{X}^n_T = \sum_{i=k_1}^{k_m} \lambda_i^n X^i_T, \quad n \leq k_1 < \cdots < k_m, \quad \sum_{i=k_1}^{k_m} \lambda_i^n = 1, \quad \lambda_i^n \geq 0.
\]

Set $\tilde{H}_n = \sum_{i=k_1}^{k_m} \lambda_i^n H_i$, in which is the sequence $H_i$ in the chosen pair $H_i$ and $X^i_T \in \mathcal{X}(x, b)$.

It is easy to see
\[
E^Q[\tilde{H}_n] = \sum_{i=k_1}^{k_m} \lambda_i^n E^Q[H_i] = x. \tag{3.2}
\]

If we apply the Lebesgue Dominated Convergence Theorem to the equation (3.2), then there exists $H^*$ such that $\lim_{n \to \infty} \tilde{H}_n = H^*$ $Q$-a.s., and $E^Q[H^*] = x$.

So we have
\[
\rho(L + \tilde{H}_n - \tilde{X}^n_T) = \rho\left(L + \sum_{i=k_1}^{k_m} \lambda_i^n H_i - \sum_{i=k_1}^{k_m} \lambda_i^n X^i_T\right)
\]
\[
= \rho\left(\sum_{i=k_1}^{k_m} \lambda_i^n (L + H_i - X^i_T)\right) \leq \sum_{i=k_1}^{k_m} \lambda_i^n \rho(L + H_i - X^i_T)
\]
\[
\leq \rho(L + H_n - X^n_T) \sum_{i=k_1}^{k_m} \lambda_i^n = \rho(L + H_n - X^n_T)
\]
\[
\leq \sup_{m \geq n} \rho(L + H_m - X^m_T). \tag{3.3}
\]

By applying the Fatou property to $\rho(L + \tilde{H}^n - \tilde{X}^n_T)$ and also using the inequality (3.3), we have
\[ \begin{align*}
\rho(L + H^* - X_{T}^{b,*}) & \leq \liminf_{n \to \infty} \rho(L + \tilde{H}_n - \tilde{X}_T^n) \\
& \leq \lim_{n \to \infty} \sup_{m \geq n} \rho(L + H_m - X_T^m) \\
& = \inf_{E^Q[H]=x} \rho_b^*(L + H).
\end{align*} \]

Since \( E^Q[H^*] = x \) and \( X_{T}^{b,*} \in \mathcal{X}(x, b) \), we have
\[ \rho(L + H^* - X_{T}^{b,*}) = \inf_{E^Q[H]=x} \rho_b^*(L + H). \]

Theorem 3.4. Let \( p(H) = E^Q[H] \) be the pricing rule of the option \( H \) for a fixed \( Q \in \mathcal{M} \). Let \( x_0 \) be an initial capital. Then there exists a risk-efficient option \( H^* \) satisfying
\[ \inf_{p(H)=x} \rho^{x_0+x}(L + H) = \rho^{x_0+x}(L + H^*), \]
where \( L \) is the initial liability uniformly bounded below by \( c_L \).

Proof. Let \( Q \in \mathcal{M} \) be fixed. Since \( \rho^{x_0+x}(L + H) = \rho^{x_0}(L + H) - x_0 \), we need only to consider
\[ \rho^{x_0}(L + H). \]

For \( X \in \mathcal{X}(0) \), by Assumption 1.6 and translation invariance property, the following both inequality and equality
\[ \rho(L + H - X_T) \geq \rho(c_L + 0 - X_T) \geq c_L + \rho(-X_T) \]
\[ \geq c_L + \rho^0(0) > -\infty, \quad \text{and} \]
\[ \rho^x(L + H) = \rho^0(L + H) - x \]
imply that \( \rho^x(L + H) \) is well-defined for all \( X \in \mathcal{X}(x) \).

By Theorem 3.2, for each \( H \) there exists \( X_{T}^{b,H} \in \mathcal{X}(x, b) \) such that
\[ \rho_b^x(L + H) := \inf_{X \in \mathcal{X}(x, b)} \rho(L + H - X_T) = \rho(L + H - X_{T}^{b,H}). \]

Let \( \epsilon > 0 \). Then since
\[ \rho_b^x(L + H) \searrow \rho^x(L + H) \text{ as } b \nearrow \infty, \]
there exists a large nonnegative integer \( N \in \mathbb{Z}^+ \) satisfying
\[ (3.4) \quad b > N \implies \rho^x(L + H) + \epsilon > \rho_b^x(L + H). \]

The equation (3.4) and Lemma 3.3 imply the following inequality
\[
\inf_{E^Q[H]=x} \rho^x(L + H) + \epsilon > \inf_{E^Q[H]=x} \rho^x_b(L + H) = \rho^x_b(L + H^*). 
\]
So we have
\[
\inf_{E^Q[H]=x} \rho^x(L + H) \geq \rho^x_b(L + H^*), 
\]
and so
\[
(3.5) \quad \inf_{E^Q[H]=x} \rho^x(L + H) \geq \lim_{b \to \infty} \rho^x_b(L + H^*). 
\]
On the other hand, since \( \rho^x(L + H) < \rho^x_b(L + H) \) we have the inequality
\[
\inf_{E^Q[H]=x} \rho^x(L + H) \leq \inf_{E^Q[H]=x} \rho^x_b(L + H) = \rho^x_b(L + H^*) 
\]
and by letting \( b \) go to infinity we get
\[
(3.6) \quad \inf_{E^Q[H]=x} \rho^x(L + H) \leq \lim_{b \to \infty} \rho^x_b(L + H^*). 
\]
By the inequalities (3.5) and (3.6), we get
\[
\inf_{E^Q[H]=x} \rho^x(L + H) = \rho^x(L + H^*). 
\]
The theorem has been proved. \( \square \)

For the pricing rule \( E^Q[H] = x \) of the option \( H \) to be an no-arbitrage price, it should also satisfy
\[
SP(H) \leq x \leq BP(H), 
\]
as we showed the reason in Section 2.

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