GLOBAL SOLUTIONS OF THE COOPERATIVE CROSS-DIFFUSION SYSTEMS

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ABSTRACT. In this paper the existence of global solutions of the parabolic cross-diffusion systems with cooperative reactions is obtained under certain conditions. The uniform boundedness of $W_{1,2}$ norms of the local maximal solution is obtained by using interpolation inequalities and comparison results on differential inequalities.

1. INTRODUCTION

This article deals with the following quasilinear parabolic system in population dynamics which is called cooperative cross-diffusion system.

\begin{equation}
\begin{aligned}
& u_t = (d_1 u + \alpha_{11} u^2 + \alpha_{12} uv)_{xx} + u(a_1 - b_1 u + c_1 v) & \text{in } [0, 1] \times (0, \infty), \\
& v_t = (d_2 v + \alpha_{21} uv + \alpha_{22} v^2)_{xx} + v(a_2 + b_2 u - c_2 v) & \text{in } [0, 1] \times (0, \infty), \\
& u_x(x, t) = v_x(x, t) = 0 & \text{at } x = 0, 1, \\
& u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 & \text{in } [0, 1],
\end{aligned}
\end{equation}

where $\alpha_{12}$, $\alpha_{21}$, $d$, $a_i$, $b_i$, $c_i$ are positive constants for $i = 1, 2$. The initial functions $u_0$, $v_0$ are not constantly zero. In the system (1.1) $u$ and $v$ are nonnegative functions which represent the population densities of two species in a cooperative relationship. $d_1$ and $d_2$ are the diffusion rates of the two species, respectively. $a_1$ and $a_2$ denote the intrinsic growth rates, $b_1$ and $c_1$ account for intra-specific cooperative pressures, $b_2$ and $c_2$ are the coefficients for inter-specific competitions. $\alpha_{11}$ and $\alpha_{22}$ are usually referred as self-diffusion, and $\alpha_{12}$, $\alpha_{21}$ are cross-diffusion pressures. By adopting the coefficients $\alpha_{ij}$ ($i, j = 1, 2$) the system (1.1) takes into account the pressures created by mutually interacting species. For more details on the backgrounds of this model, the readers are referred to Okubo and Levin[7].

Pao[8] in 2005, and Delgado et al.[4] in 2008 have obtained some results on the existence of global solutions of the elliptic cross-diffusion systems with cooperative...
reactions. In this paper the existence of global solutions of the parabolic cross-diffusion systems with cooperative reactions is obtained under certain conditions. To state results on the system (1.1) we use the following notation throughout this paper.

**Notations.** Let $\Omega$ be a region in $\mathbb{R}^n$. The norm in $L_p(\Omega)$ is denoted by $\| \cdot \|_{L_p(\Omega)}$, $1 \leq p \leq \infty$, where $|f|_{L_p(\Omega)} = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p}$, if $1 \leq p < \infty$, and $|f|_{L_\infty(\Omega)} = \sup \{|f(x)| : x \in \Omega\}$. The usual Sobolev spaces of real valued functions in $\Omega$ with exponent $k \geq 0$ are denoted by $W^k_p(\Omega)$, $1 \leq p < \infty$. And $\| \cdot \|_{W^k_p(\Omega)}$ represents the norm in the Sobolev space $W^k_p(\Omega)$. For $\Omega = [0, 1] \subset \mathbb{R}^1$ we shall use the simplified notation $\| \cdot \|_{k,p}$ for $\| \cdot \|_{W^k_p(\Omega)}$ and $\| \cdot \|_p$ for $\| \cdot \|_{L_p(\Omega)}$.

The local existence of solutions to (1.1) was established by Amann [1], [2], [3]. According to his results the system (1.1) has a unique nonnegative solution $u(\cdot, t), v(\cdot, t)$ in $C([0, T), W^1_p(\Omega)) \cap C^\infty((0, T), C^\infty(\Omega))$, where $T \in (0, \infty]$ is the maximal existence time for the solution $u, v$. The following result is also due to Amann [2].

**Theorem 1.1.** Let $u_0$ and $v_0$ be in $W^1_p(\Omega)$. The system (1.1) possesses a unique nonnegative maximal smooth solution $u(x, t), v(x, t) \in C([0, T), W^1_p(\Omega)) \cap C^\infty(\bar{\Omega} \times (0, T))$ for $0 \leq t < T$, where $p > n$ and $0 < T \leq \infty$. If the solution satisfies the estimates $\sup_{0 < t < T} \|u(\cdot, t)\|_{W^1_p(\Omega)} < \infty$, $\sup_{0 < t < T} \|v(\cdot, t)\|_{W^1_p(\Omega)} < \infty$, then $T = +\infty$. If, in addition, $u_0$ and $v_0$ are in $W^2_p(\Omega)$ then $u(x, t), v(x, t) \in C([0, \infty), W^2_p(\Omega))$, and $\sup_{0 \leq t < \infty} \|u(\cdot, t)\|_{W^2_p(\Omega)} < \infty$, $\sup_{0 \leq t < \infty} \|v(\cdot, t)\|_{W^2_p(\Omega)} < \infty$.

Here we state the main results of this paper. Throughout this this paper we assume the condition

$$b_1 c_2 > b_2 c_1$$

which means the inter-specific competition pressures are greater than the intra-specific cooperative pressures.

**Theorem 1.2.** Suppose that the initial functions $u_0$, $v_0$ are in $W^2_p([0, 1])$. Also assume the condition (1.2). Let $(u(x, t), v(x, t))$ be the maximal solution to the system (1.1) as in Theorem 1.1. Then there exist positive constant $M_0 = M_0(\|u_0\|_1, \|v_0\|_1, a_1, a_2, b_1, b_2, c_1, c_2)$ such that

$$\sup\{\|u(\cdot, t)\|_1, \|v(\cdot, t)\|_1 : t \in [0, T)\} \leq M_0$$
For the boundedness results of $L_2$ and $W_{1,2}$ norms of the maximal solution to the system (1.1) we assume the following condition in Theorem 1.3, Theorem 1.4

(1.3) \[ \alpha_{12}^2 < 8\alpha_{11}\alpha_{21} \quad \text{and} \quad \alpha_{21}^2 < 8\alpha_{12}\alpha_{22}. \]

**Theorem 1.3.** Suppose that the initial functions $u_0$, $v_0$ are in $W_2^2([0,1])$. Also assume the conditions (1.2) and (1.3). Let $(u(x,t), v(x,t))$ be the maximal solution to the system (1.1) as in Theorem 1.1. Then there exists a positive constant $M_1 = M_1(\|u_0\|_1, \|v_0\|_1, d_i, a_i, b_i, c_i, i = 1, 2)$ such that

\[ \sup \{ \|u(\cdot, t)\|_2, \|v(\cdot, t)\|_2 : t \in [0, T) \} \leq M_1. \]

**Theorem 1.4.** Suppose that the initial functions $u_0$, $v_0$ are in $W_2^2([0,1])$. Also assume the conditions (1.2) and (1.3). Let $(u(x,t), v(x,t))$ be the maximal solution to the system (1.1) as in Theorem 1.1. Then there exists a positive constant $M_2 = M_2(\|u_0\|_1, \|v_0\|_1, d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$ such that

\[ \sup \{ \|u(\cdot, t)\|_{1,2}, \|v(\cdot, t)\|_{1,2} : t \in [0, T) \} \leq M_2. \]

From the results of Theorems 1.2, 1.3 and 1.4 and the Sobolev embedding inequality we have positive constants $M' = M'(d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$ and $M = M(d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$ such that for the maximal solution $(u, v)$ of (1.1) with the conditions (1.2), (1.3)

(1.4) \[ \sup \{ \|u(\cdot, t)\|_{1,2}, \|v(\cdot, t)\|_{1,2} : t \in [0, T) \} \leq M', \]
\[ \sup \{ u(x, t), v(x, t) : (x, t) \in [0, 1] \times [0, T) \} \leq M. \]

We also conclude that $T = +\infty$ from Theorem 1.1.

This paper is organized as follows. Section 2 provides preliminaries on differential equations and a few consequences of Gagliardo-Nirenberg interpolation inequality which are necessary for the proofs of Theorems 1.2, 1.3, and 1.4. And Sections 3, 4, and 5 present the proofs of Theorems 1.2, 1.3, and 1.4, respectively.

2. Preliminaries

This section introduce the Gagliardo-Nirenberg interpolation inequality and its consequences. Also some preliminary results on the bounds and comparisons of differential equations and inequalities are provided.

**Theorem 2.1** (Gagliardo-Nirenberg interpolation inequality). Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial \Omega$ in $C^m$. For every function $u$ in $W^{m,r}(\Omega)$, $1 \leq q, r \leq \infty$
the derivative $D^j u$, $0 \leq j < m$, satisfies the inequality
\begin{equation}
|D^j u|_p \leq C(|D^m u|_r^a |u|_q^{1-a} + |u|_q),
\end{equation}
where $\frac{1}{p} = \frac{j}{n} + a \left( \frac{1}{r} - \frac{m}{n} \right) + (1-a) \frac{1}{q}$ for all $a$ in the interval $\frac{j}{m} \leq a < 1$, provided one of the following three conditions:

(i) $r \leq q$,
(ii) $0 < \frac{n(r-q)}{mrq} < 1$, or
(iii) $\frac{n(r-q)}{mrq} = 1$ and $m - \frac{n}{q}$ is not a nonnegative integer.

(The positive constant $C$ depends only on $n$, $m$, $j$, $q$, $r$, $a$.)

\textbf{Proof.} We refer the reader to A. Friedman [5] or L. Nirenberg [6] for the proof of this well-known calculus inequality. \hfill \Box

\textbf{Corollary 2.1.} There exist positive constants $C$, $\check{C}$, and $\hat{C}$ such that for every function $u$ in $W^{1}_{2}([0,1])$

\begin{equation}
|u|_{4} \leq C(|ux|_{\frac{1}{2}} |u|_{\frac{1}{2}}^\frac{1}{3} + |u|_1).
\end{equation}

\begin{equation}
|u|_{\frac{5}{2}} \leq \check{C}(|ux|_{\frac{5}{2}} |u|_{\frac{5}{2}}^\frac{2}{3} + |u|_1).
\end{equation}

\begin{equation}
|u|_{2} \leq \hat{C}(|ux|_{\frac{1}{2}} |u|_{\frac{1}{2}}^\frac{2}{3} + |u|_1),
\end{equation}

\textbf{Proof.} $n = 1$, $m = 1$, $r = 2$, $q = 1$ satisfy the condition (ii) in Theorem 2.1. Letting $j = 0$ in this case the necessary condition on $p$, $a$ for inequality (2.1) becomes

\begin{equation}
\frac{1}{p} = \frac{j}{n} + a \left( \frac{1}{r} - \frac{m}{n} \right) + (1-a) \frac{1}{q} = 1 - \frac{2}{3}a
\end{equation}

From equation (2.5) if $p = 4$, then $a = \frac{1}{2}$, if $p = \frac{5}{2}$, then $a = \frac{2}{5}$, and if $p = 2$, then $a = 13$. Therefore we have inequalities (2.2), (2.3), (2.4). \hfill \Box

\textbf{Corollary 2.2.} For every function $u$ in $W^{2}_{2}([0,1])$

\begin{equation}
|ux|_{2} \leq C(|uxx|_{\frac{1}{2}} |u|_{\frac{1}{2}}^{\frac{3}{2}} + |u|_1).
\end{equation}

\textbf{Proof.} $m = 2$, $r = 2$, $q = 1$ satisfy the condition (ii) in Theorem 2.1. \hfill \Box

\textbf{Theorem 2.2} (Young’s Inequality). If $a$ and $b$ are nonnegative real numbers and $p$ and $q$ are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$ 

The equality hold if and only if $a^p = b^q$. 

Theorem 2.3 (Hölder’s Inequality). If \( f, g : \mathbb{R}^n \to \mathbb{R} \) are Lebesgue measurable and \( p, q \in [1, \infty] \) are real numbers such that \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[
|fg|_1 \leq |f|_p|g|_q.
\]

Lemma 2.1 below presents a few basic inequalities that will be used for the computations in this paper.

**Lemma 2.1.** Let \( x \geq 0, y \geq 0 \). Then

\[
\begin{align*}
&(x + y)^2 \geq \frac{1}{2}x^2 - y^2 \quad (2.7) \\
x^k \leq x^s + x^t, & \quad \text{if } 0 < t \leq k \leq s \quad (2.8) \\
(x + y)^k \leq 2^{k-1}(x^k + y^k), & \quad \text{if } k \geq 1 \quad (2.9) \\
x^k + y^k \leq 2^{1-k}(x + y)^k, & \quad \text{if } 0 < k \leq 1 \quad (2.10)
\end{align*}
\]

**Proof.** Inequalities (2.7), (2.8), (2.9) are simply proved. To show inequalities (2.10), (2.11), let \( g : [0, \infty) \to \mathbb{R}, \quad g(x) = k2^{k-1}x^{k-1} - k(x + y)^{k-1} \). Then

\[
g'(x) = k2^{k-1}x^{k-2} - k(x + y)^{k-2}.
\]

Hence the function \( g(x) \) has the critical value 0 at \( x = y \) which is the minimum value if \( k \geq 1 \), and the maximum value if \( 0 < k \leq 1 \). Thus we obtain inequalities (2.10) and (2.11). \( \square \)

**Theorem 2.4** (Picard’s local existence and uniqueness theorem). If \( f(x, t) \) is a continuous real-valued function that satisfies the Lipschitz condition

\[
|f(x, t) - f(y, t)| \leq L|x - y|
\]

in some open rectangle \( R = \{(x, t) \mid a < x < b, c < t < d\} \) that contains the point \((x_0, t_0)\), then the initial value problem

\[
x' = f(x, t), \quad x(t_0) = x_0
\]

has a unique solution in some closed interval \( I = [t_0 - \epsilon, t_0 + \epsilon] \), where \( \epsilon > 0 \).

**Theorem 2.5.** Let \( f(x) \) be a real-valued differentiable functions defined on an open interval \((a, b)\). Then for every initial point \( x_0 \) in \((a, b)\) a solution of the initial value problem
\[ x' = f(x), \quad x(0) = x_0 \]

is either constant or strictly monotone.

**Proof.** The conclusion follow from the fact that \( f(x(t)) \) never changes sign for the solution \( x(t) \) of the given initial value problem. To see why this is so, suppose that \( x(t) \) is not a constant solution, and \( f(x(t)) \) changes sign. Then it would have to be \( f(x(t_1)) = 0 \) at some \( t_1 > 0 \) and \( f(x(t)) \neq 0 \) for \( t \) in the left of \( t_1 \) or right of \( t_1 \). But it contradict the fact that from Theorem 2.4 the constant solution \( y(t) \equiv x(t_1) \) is a unique solution in some closed interval \([t_1 - \epsilon, t_1 + \epsilon]\), where \( \epsilon > 0 \). \( \square \)

**Corollary 2.3.** Let \( c_1 > 0 \), \( p > 1 \), and \( c_2, c_3 \) be any real numbers. Then there exists a positive constant \( M = M(x_0, p, c_1, c_2, c_3) \) such that the solution of the initial value problem

\[ x' = -c_1 x^p + c_2 x + c_3, \quad x(0) = x_0 \geq 0 \]

satisfies that

\[ x(t) \leq M \quad \text{for all } t \geq 0. \]

**Proof.** The function \( f(x) = -c_1 x^p + c_2 x + c_3 \) is differentiable functions on \( \mathbb{R} \) and falls in either of the two cases:

- **case(a)** \( f(x) \leq 0 \) for all \( x \geq 0 \)
- **case(b)** there exist a positive constant \( m = m(p, c_1, c_2, c_3) \) such that \( f(m) = 0 \), \( f(x) > 0 \) for \( x \) in some interval on the left of \( m \), and \( f(x) < 0 \) for all \( x > M \).

In case (a) \( x'(0) = f(x_0) \leq 0 \), and thus by Theorem 2.5 \( x'(t) \leq 0 \) for all \( t \geq 0 \). Hence \( x(t) \leq x_0 \) for all \( t \geq 0 \). In case (b) if \( 0 < x_0 < m \) then the solution \( x(t) \) cannot cross the constant solution \( y(t) \equiv m \) by Theorem 2.5, and thus \( x(t) \leq m \) for all \( t \geq 0 \). If \( x_0 \geq m \) then \( x'(0) = f(x_0) \leq 0 \), and thus by Theorem 2.5 \( x'(t) \leq 0 \) for all \( t \geq 0 \). Hence \( x(t) \leq x_0 \) for all \( t \geq 0 \). Therefore in any case there exists a positive constant \( M = M(x_0, p, c_1, c_2, c_3) \) such that \( x(t) \leq M \) for all \( t \geq 0 \). \( \square \)

**Lemma 2.2** (Gronwall’s inequality and the Comparison Principle for differential equations). Let \( a < b \leq \infty \), and \( \xi(t) \) and \( \beta(t) \) be real-valued continuous functions defined on the interval \([a, b]\). If \( \xi(t) \) is differentiable in \((a, b)\) and satisfies the differential inequality

\[ \xi'(t) \leq \beta(t)\xi(t), \quad t \in (a, b), \]

then \( \xi(t) \) is bounded by the solution of the corresponding differential equation \( y'(t) = \beta(t)y(t), \ y(a) = \xi(a) \), that is,
\[
\xi(t) \leq \xi(a) \exp \left( \int_a^t \beta(s) \, ds \right)
\]
for all \( t \in [a, b] \). And it follows that if in addition \( \xi(a) \leq 0 \), then \( \xi(t) \leq 0 \) for all \( t \in [a, b] \).

**Proof.** We refer the reader to [2]. \( \square \)

**Lemma 2.3.** Let \( c_1 > 0, \ p > 1 \), and \( c_2, c_3 \) be any real numbers. Suppose that two differentiable functions \( \phi(t) \) and \( x(t) \) satisfy

\[
\begin{align*}
\phi' &\leq -c_1\phi^p + c_2\phi + c_3, & \phi(0) = \phi_0 \\
x' &\leq -c_1x^p + c_2x + c_3, & x(0) = \phi_0.
\end{align*}
\]

Then

\[
\phi(t) \leq x(t) \quad \text{for all } t \geq 0.
\]

And especially, if \( \phi_0 \geq 0 \) then there exists a positive constant \( M = M(\phi_0, p, c_1, c_2, c_3) \) such that

\[
\phi(t) \leq M \quad \text{for all } t \geq 0.
\]

**Proof.** Let \( \xi = \phi - x \). Then

\[
\begin{align*}
\xi' &= \phi' - x' \\
&\leq -c_1(\phi^p - x^p) + c_2(\phi - x) \\
&= \xi(-c_1\eta + c_2),
\end{align*}
\]

where

\[
\eta(t) = \begin{cases} 
\frac{\phi(t)^p - x(t)^p}{\phi(t)^p - x(t)^p}, & \text{if } \phi(t) \neq x(t) \\
\frac{p\phi(t)^{p-1}}{\phi(t)^p - x(t)^p}, & \text{if } \phi(t) = x(t)
\end{cases}
\]

Here notice that \( \eta(t) \) is a continuous function using the mean value theorem and the discontinuities of \( \phi(t) \) and \( x(t) \). Now, since \( \xi(0) = 0 \) we conclude that \( \xi(t) = \phi(t) - x(t) \leq 0 \) for all \( t \geq 0 \) from Lemma 2.2. And if \( \phi_0 \geq 0 \), from Corollary 2.3 there exists a positive constant \( M = M(\phi_0, p, c_1, c_2, c_3) \) such that \( x(t) \leq M \) for all \( t \geq 0 \). Thus \( \phi(t) \leq M \) for all \( t \geq 0 \). \( \square \)

3. **L_1-bound of Solutions to (1.1)**

**Proof of Theorem 1.2.** By taking integration over the interval \([0, 1]\) for the first and second equations in (1.1) we have that

\[
\begin{align*}
\frac{d}{dt} \int_0^1 u(x, t) \, dx &= \int_0^1 (a_1u - b_1u^2 + c_1uv) \, dx \\
\frac{d}{dt} \int_0^1 v(x, t) \, dx &= \int_0^1 (a_2v - b_2uv + c_2v^2) \, dx,
\end{align*}
\]
\[ \frac{d}{dt} \int_0^1 (b_2u + c_1 v) \, dx = \int_0^1 (a_1 b_2 u + a_2 c_1 v) \, dx - \int_0^1 (b_1 b_2 u^2 - 2b_2 c_1 u v + c_1 c_2 v^2) \, dx. \]

Let \( \delta = \frac{b_2 c_1 (b_1 c_2 - b_2 c_1)}{b_1 b_2 + c_1 c_2} \). The condition \( b_1 c_2 > b_2 c_1 \) implies \( \delta > 0 \). It also holds that

\[ \delta = \frac{b_2 c_1 (b_1 c_2 - b_2 c_1)}{b_1 b_2 + c_1 c_2} < \min \{ b_1 b_2, c_1 c_2 \}. \]

Thus it is shown that

\[ \int_0^1 (b_1 b_2 u^2 - 2b_2 c_1 u v + c_1 c_2 v^2) \, dx - \delta \int_0^1 (u^2 + v^2) \, dx \geq 0 \]

from the facts

\[ (b_2 c_1)^2 - (b_1 b_2 - \delta) (c_1 c_2 - \delta) = -\delta^2 + (b_1 b_2 + c_1 c_2) \delta - b_2 c_1 (b_1 c_2 - b_2 c_1) < 0. \]

Using (3.1) we have

\[ \frac{d}{dt} \int_0^1 (b_2 u + c_1 v) \, dx \leq \int_0^1 (a_1 b_2 u + a_2 c_1 v) \, dx - \delta \int_0^1 (u^2 + v^2) \, dx, \]

and thus

\[ \frac{d}{dt} \int_0^1 (u + v) \, dx \leq C_1 \int_0^1 (u + v) \, dx - C_0^\prime \int_0^1 (u^2 + v^2) \, dx, \]

where \( C_1 = \max \{ a_1 b_2, a_2 c_1 \} / \min \{ b_2, c_1 \} \), \( C_0^\prime = \delta / \min \{ b_2, c_1 \} \). From Hölder’s inequality

\[ \int_0^1 u \, dx \leq \left( \int_0^1 u^2 \, dx \right)^{\frac{1}{2}} \], \quad \int_0^1 v \, dx \leq \left( \int_0^1 v^2 \, dx \right)^{\frac{1}{2}}, \]

it follows that

\[ \frac{d}{dt} \int_0^1 (u + v) \, dx \leq C_1 \int_0^1 (u + v) \, dx - C_0 \left\{ \left( \int_0^1 u \, dx \right)^2 + \left( \int_0^1 v \, dx \right)^2 \right\}, \]

and thus

\[ (3.2) \quad \frac{d}{dt} \int_0^1 (u + v) \, dx \leq C_1 \int_0^1 (u + v) \, dx - C_0 \left\{ \int_0^1 (u + v) \, dx \right\}^2, \]

where \( C_0 = C_0^\prime / 2 \). Hence by the Gronwall’s type inequality in Lemma 2.3 there exists positive constant \( M_0 = M_0 (\| u_0 \|_1, \| v_0 \|_1, a_1, a_2, b_1, b_2, c_1, c_2) \) satisfying

\[ \int_0^1 u(x, t) \, dx \leq M_0, \quad \int_0^1 v(x, t) \, dx \leq M_0 \]

for all \( t \geq 0 \). \qed
4. \textit{L}_2\text{-bound of Solutions to (1.1)}

\textit{Proof of Theorem 1.3.} Multiplying the first and second equations in (1.1) by \( u = u(x, t) \) and \( v = v(x, t) \), respectively, and taking integrations over \([0, 1]\) we have that

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 u^2 \, dx = \int_0^1 u (d_1 u + \alpha_{11} u^2 + \alpha_{12} uv)_{xx} \, dx + \int_0^1 u^2 (a_1 - b_1 u + c_1 v) \, dx
\]

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 v^2 \, dx = \int_0^1 v (d_2 v + \alpha_{21} uv + \alpha_{22} v^2)_{xx} \, dx + \int_0^1 v^2 (a_2 - b_2 u + c_2 v) \, dx.
\]

Using Neumann boundary conditions

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 u^2 \, dx = \int_0^1 u (d_1 u + \alpha_{11} u^2 + \alpha_{12} uv)_{xx} \, dx + \int_0^1 u^2 (a_1 - b_1 u + c_1 v) \, dx
\]

\[
= -\int_0^1 u_x (d_1 u + \alpha_{11} u^2 + \alpha_{12} uv) \, dx + \int_0^1 u^2 (a_1 - b_1 u + c_1 v) \, dx
\]

\[
= -\int_0^1 u_x (d_1 u + 2\alpha_{11} uv_x + \alpha_{12} uv_x) \, dx + \int_0^1 \alpha_{12} uu_x v_x \, dx
\]

\[
+ \int_0^1 u^2 (a_1 - b_1 u + c_1 v) \, dx,
\]

and similarly

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 v^2 \, dx = -\int_0^1 (d_2 v + 2\alpha_{21} uv + \alpha_{22} v^2) v_x^2 \, dx - \int_0^1 \alpha_{21} uu_x v_x \, dx
\]

\[
+ \int_0^1 v^2 (a_2 + b_2 u - c_2 v) \, dx.
\]

(4.1)

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (u^2 + v^2) \, dx = -d_1 \int_0^1 u_x^2 \, dx - d_2 \int_0^1 v_x^2 \, dx
\]

\[
- \int_0^1 \left((2\alpha_{11} u + \alpha_{12} v)u_x^2 + (\alpha_{12} u + \alpha_{21} v)u_x v_x + (2\alpha_{21} u + \alpha_{22} v)v_x^2\right) \, dx
\]

\[
+ \int_0^1 u^2 (a_1 - b_1 u + c_1 v) \, dx + \int_0^1 v^2 (a_2 + b_2 u - c_2 v) \, dx,
\]
Using condition (1.3) that \( \alpha_{12}^2 < 8 \alpha_{11} \alpha_{21} \) and \( \alpha_{21}^2 < 8 \alpha_{11} \alpha_{22} \), we have
\[
(\alpha_{12} u + \alpha_{21} v)^2 - 4(2 \alpha_{11} u + \alpha_{12} v)(2 \alpha_{21} u + \alpha_{22} v) = (\alpha_{12}^2 - 8 \alpha_{11} \alpha_{21}) u^2 - 2(\alpha_{12} \alpha_{21} + 8 \alpha_{11} \alpha_{22}) uv + (\alpha_{21}^2 - 8 \alpha_{11} \alpha_{22}) v^2 \leq 0.
\]
Thus it follows from (4.1) that
\[
(4.2)
\]
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (u^2 + v^2) \, dx \leq -d_1 \int_0^1 u_x^2 \, dx - d_2 \int_0^1 v_x^2 \, dx + \int_0^1 u^2 (a_1 - b_1 u + c_1 v) \, dx + \int_0^1 v^2 (a_2 + b_2 u - c_2 v) \, dx,
\]
\[
\leq -d_1 \int_0^1 u_x^2 \, dx - d_2 \int_0^1 v_x^2 \, dx + a_1 \int_0^1 u^2 \, dx + a_2 \int_0^1 v^2 \, dx + c_1 \int_0^1 u^2 v \, dx + b_2 \int_0^1 w^2 \, dx.
\]
By Young’s inequality
\[
\int_0^1 u^2 v \, dx \leq \int_0^1 \frac{1}{2} \left( \epsilon u^4 + \frac{1}{\epsilon} v^2 \right) \, dx,
\]
\[
\int_0^1 v^2 \, dx \leq \int_0^1 \frac{1}{2} \left( \epsilon v^4 + \frac{1}{\epsilon} u^2 \right) \, dx
\]
holds for any \( \epsilon > 0 \). And by applying Lemma 2.1 to inequality (2.2) and using the uniform \( L_1 \)-boundedness of \( u \) and \( v \) from Step 1, we have
\[
\int_0^1 u^4 \, dx \leq C \left( \int_0^1 u_x^2 \, dx + 1 \right),
\]
\[
\int_0^1 v^4 \, dx \leq C \left( \int_0^1 v_x^2 \, dx + 1 \right),
\]
where \( C \) is a positive constant depending only on \( a_i, b_i, c_i \ (i, j = 1, 2) \). Thus (4.2) becomes
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (u^2 + v^2) \, dx
\]
\[
\leq -d_1 \int_0^1 u_x^2 \, dx - d_2 \int_0^1 v_x^2 \, dx + a_1 \int_0^1 u^2 \, dx + a_2 \int_0^1 v^2 \, dx + c_1 \frac{1}{2 \epsilon} \int_0^1 v^2 \, dx
\]
\[
+ b_2 \epsilon C \left( \int_0^1 v_x^2 \, dx + 1 \right) + b_2 \epsilon C \int_0^1 u^2 \, dx
\]
\[
\leq -d_1 \int_0^1 u_x^2 \, dx - d_2 \int_0^1 v_x^2 \, dx + c_1 \epsilon \int_0^1 u^2 v \, dx + b_2 \epsilon C \int_0^1 u^2 \, dx + C'_1,
\]
where \( \epsilon = \frac{1}{C} \min\{ \frac{a_i}{2 \epsilon}, \frac{d_i}{2 \epsilon} \} \) and the constants \( C'_0 \) and \( C'_1 \) are depending on \( d_i, a_i, b_i, c_i \ (i, j = 1, 2) \). And by applying Lemma 2.1 to inequality (2.4) and using the
uniform $L_1$-boundedness of $u$ and $v$ from Step 1, we have
\[
\left( \int_0^1 u_x^2 \, dx \right)^3 \leq \hat{C} \left( \int_0^2 u_x^2 \, dx + 1 \right) \quad \text{and} \quad \left( \int_0^1 v_x^2 \, dx \right)^3 \leq \hat{C} \left( \int_0^2 v_x^2 \, dx + 1 \right),
\]
where $\hat{C}$ is a positive constant depending only on $a_i, b_i, c_i \ (i, j = 1, 2)$. And thus
\[
- \int_0^1 u_x^2 \, dx \leq 1 - C' \left( \int_0^1 u^2 \, dx \right)^3 \quad \text{and} \quad - \int_0^1 v_x^2 \, dx \leq 1 - C' \left( \int_0^1 v^2 \, dx \right)^3,
\]
where $C'$ is a positive constant depending only on $a_i, b_i, c_i \ (i, j = 1, 2)$. Thus we have
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (u^2 + v^2) \, dx \leq \left[ -C'_2 \left( \int_0^1 u^2 \, dx \right)^3 - C'_2 \left( \int_0^1 v^2 \, dx \right)^3 + C'_1 \int_0^1 (u^2 + v^2) \, dx + C_0'' \right]
\]
by Lemma 2.1, where $C_0, C_1, C_2$ are positive constants $d_i, a_i, b_i, c_i \ (i, j = 1, 2)$. Hence by the Gronwall’s type inequality in Lemma 2.3 we obtain the following $L_2$-bound of $u$ and $v$ such that
\[
\int_0^1 (u^2 + v^2) \, dx \leq M_1 \quad \text{for all} \quad t \geq 0,
\]
where $M_1$ is a positive constant depending on $\|u_0\|_2, \|v_0\|_2, d_i, a_i, b_i, c_i \ (i, j = 1, 2)$. □

5. $W_{1,2}$-bound of Solutions to (1.1)

Proof of Theorem 1.4. To obtain uniform bounds of $|u_x|_2$ and $|v_x|_2$ for the solution of (1.1) let us denote that
\[
P = d_1 u + \alpha_{11} u^2 + \alpha_{12} u v, \quad Q = d_2 v + \alpha_{21} u v + \alpha_{22} v^2.
\]
We would show the uniform boundedness of $|P_x|_2$ and $|Q_x|_2$ and then obtain the uniform bounds of $|u_x|_2$ and $|v_x|_2$ from it. Here we note from Theorem 1.1 that $P, Q \in C([0, T), W^2_1([0, 1])) \cap C^\infty([0, 1] \times (0, T))$ for $0 \leq t < T$, and
\[
\int_0^1 P_t u_x \, dx = \int_0^1 \left[ (d_1 + 2\alpha_{11} u + \alpha_{12} v) u_x^2 + \alpha_{12} u_x v_x \right] \, dx
\]
\[
\int_0^1 Q_t v_t \, dx = \int_0^1 (d_2 v_t + \alpha_{21} u_t v + \alpha_{21} u v_t + 2 \alpha_{22} u v_t) v_t \, dx
\]
\[
= \int_0^1 [(d_2 + \alpha_{21} u + 2 \alpha_{22} v) v_t^2 + \alpha_{21} u v_t v_t] \, dx,
\]
\[
\int_0^1 P_t P_{xx} \, dx = - \int_0^1 P_{xt} P_x \, dx = - \frac{1}{2} \frac{d}{dt} \int_0^1 P_x^2 \, dx,
\]
\[
\int_0^1 Q_t Q_{xx} \, dx = - \int_0^1 Q_{xt} Q_x \, dx = - \frac{1}{2} \frac{d}{dt} \int_0^1 Q_x^2 \, dx
\]
from the Neumann boundary conditions on \( u, v \). Now, multiplying the first equation in (1.1) by \( P_t \) and the second equation by \( Q_t \), we have
\[
\int_0^1 \left[ (d_1 + 2 \alpha_{11} u + \alpha_{12} v) u_t^2 + \alpha_{12} uu v_t \right] \, dx
\]
\[
= - \frac{1}{2} \frac{d}{dt} \int_0^1 P_x^2 \, dx
\]
\[
+ \int_0^1 \left[ u (d_1 + 2 \alpha_{11} u + \alpha_{12} v) (a_1 \cdot b_1 u + c_1 v) u_t + \alpha_{12} u^2 (a_1 \cdot b_1 u + c_1 v) v_t \right] \, dx,
\]
\[
\int_0^1 \left[ (d_2 + \alpha_{21} u + 2 \alpha_{22} v) v_t^2 + \alpha_{21} u v_t v_t \right] \, dx
\]
\[
= - \frac{1}{2} \frac{d}{dt} \int_0^1 Q_x^2 \, dx
\]
\[
+ \int_0^1 \left[ v (d_2 + \alpha_{21} u + 2 \alpha_{22} v) (a_2 + b_2 u - c_2 v) v_t + \alpha_{21} v^2 (a_2 + b_2 u - c_2 v) u_t \right] \, dx,
\]
and thus
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (P_x^2 + Q_x^2) \, dx \leq -d_1 \int_0^1 u_t^2 \, dx - d_2 \int_0^1 v_t^2 \, dx
\]
\[
- \int_0^1 \left[ (2 \alpha_{11} u + \alpha_{12} v) u_t^2 + (\alpha_{12} u + \alpha_{21} v) u_t v_t + (\alpha_{21} u + 2 \alpha_{22} v) v_t^2 \right] \, dx
\]
\[
+C_1 \int_0^1 \left( u + v + u^2 + uv + v^2 + u^3 + u^2 v + uv^2 + v^3 \right) (|u_t| + |v_t|) \, dx,
\]
since
\[
(\alpha_{12}u + \alpha_{21}v)^2 - 4(2\alpha_{11}u + \alpha_{12}v - \lambda u - \lambda v)(\alpha_{21}u + 2\alpha_{22}v - \lambda u - \lambda v)
\]
\[
= (\alpha_{21}^2 - 8\alpha_{11}\alpha_{21})u^2 - 2(\alpha_{12}\alpha_{21} + 8\alpha_{11}\alpha_{22})uv + (\alpha_{22}^2 - 8\alpha_{12}\alpha_{22})v^2
\]
\[
+ 4\lambda [(2\alpha_{11} + \alpha_{21})u^2 + (2\alpha_{11} + \alpha_{12} + \alpha_{21} + 2\alpha_{22})uv + (\alpha_{12} + 2\alpha_{22})v^2]
\]
\[-4\lambda^2(u + v)^2 \leq 0
\]

for all \( u \geq 0, v \geq 0 \), if \( \lambda = \lambda(\alpha_{ij}, i, j = 1, 2) > 0 \) is small enough.

The terms \( \int_0^1 u t^2 \, dx \), \( \int_0^1 v t^2 \, dx \) in (5.1) are estimated in terms of \( P \) and \( Q \) from inequality (2.7) in lemma 2.1.

\[
\begin{align*}
- \int_0^1 u t^2 \, dx &= - \int_0^1 \left[ P_{xx} + u(a_1 - b_1 + c_1 v)^2 \right] \, dx \\
&\leq - \frac{1}{2} \int_0^1 P_{xx}^2 \, dx + \int_0^1 u^2 (a_1 - b_1 + c_1 v)^2 \, dx, \\
- \int_0^1 v t^2 \, dx &= - \int_0^1 \left[ Q_{xx} + v(a_2 + b_2 u - c_2 v)^2 \right] \, dx \\
&\leq - \frac{1}{2} \int_0^1 Q_{xx}^2 \, dx + \int_0^1 v^2 (a_2 + b_2 u - c_2 v)^2 \, dx.
\end{align*}
\]

Now we observe using Young's inequality that

\[
\left| \int_0^1 uu_t \, dx \right| = \left| \int_0^1 a^2 \left( u^2 u_t \right) \, dx \right| \leq \frac{1}{2\epsilon} \int_0^1 u \, dx + \epsilon \int_0^1 uu_t^2 \, dx
\]

hold for any \( \epsilon > 0 \). Similar estimates are applied to the terms \( \int_0^1 uu_t \, dx \), \( \int_0^1 vu_t \, dx \), \( \int_0^1 u^2 u_t \, dx \), \( \int_0^1 v^2 u_t \, dx \), and so on. Using these inequalities and inequalities (2.8), (2.9) in lemma 2.1 we obtain that

\[
C_1 \int_0^1 \left( u + v + u^2 + uv + v^2 + u^3 + u^2 v + uv^2 + v^3 \right) (|u_t| + |v_t|) \, dx \\
+ d_1 \int_0^1 u^2(a_1 - b_1 + c_1 v)^2 \, dx + d_2 \int_0^1 v^2(a_2 + b_2 u - c_2 v)^2 \, dx \\
\leq C_2 \int_0^1 (u + v + u^2 + v^2 + u^3 + u^2 v + uv^2 + v^3) (|u_t| + |v_t|) \, dx \\
+ d_1 \int_0^1 u^2(a_1 - b_1 + c_1 v)^2 \, dx + d_2 \int_0^1 v^2(a_2 + b_2 u - c_2 v)^2 \, dx
\]

\[
\leq \frac{C_3}{2\epsilon} \int_0^1 (u + v + u^5 + v^5) \, dx + \frac{\epsilon C_3}{2} \int_0^1 (u + v) (u_t^2 + v_t^2) \, dx,
\]

\[
\leq \left( \frac{C_3}{2\epsilon} + C_4 \right) \int_0^1 (1 + u^5 + v^5) \, dx + \frac{\epsilon C_3}{2} \int_0^1 (u + v) (u_t^2 + v_t^2) \, dx,
\]
where $C_2$, $C_3$, $C_4$ are positive constant depending on $d_i$, $\alpha_{ij}$, $a_i$, $b_i$, $c_i$ ($i, j = 1, 2$). Thus we have
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (P_x^2 + Q_x^2) \, dx \\
\leq - \frac{d_1}{2} \int_0^1 P_{xx}^2 \, dx - \frac{d_2}{2} \int_0^1 Q_{xx}^2 \, dx - \lambda \int_0^1 (u + v)(u_i^2 + v_i^2) \, dx \\
+ \left( \frac{C_3}{2\epsilon} + C_4 \right) \int_0^1 (1 + u^5 + v^5) \, dx + \frac{\epsilon C_5}{2} \int_0^1 (u + v) \left( u_i^2 + v_i^2 \right) \, dx
\]
for any $\epsilon > 0$. Here we choose a small $\epsilon > 0$ so that $\frac{C_3}{2\epsilon} \leq \lambda$, and thus
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (P_x^2 + Q_x^2) \, dx \leq - \frac{d_1}{2} \int_0^1 P_{xx}^2 \, dx - \frac{d_2}{2} \int_0^1 Q_{xx}^2 \, dx + C_5 \int_0^1 (1 + u^5 + v^5) \, dx
\]
where $C_5$ is a positive constants depending on $d_i$, $\alpha_{ij}$, $a_i$, $b_i$, $c_i$ ($i, j = 1, 2$). Now we observe that
\[
P = d_1 u + \alpha_{11} u^2 + \alpha_{12} uv \geq \alpha_{11} u^2, \quad Q = d_2 v + \alpha_{21} uv + \alpha_{22} v^2 \geq \alpha_{22} v^2,
\]
and thus
\[
\int_0^1 (u^5 + v^5) \, dx \leq C_0 \int_0^1 (P_x^2 + Q_x^2) \, dx,
\]
where $C_0$ is a positive constant depending only on $d_i$, $\alpha_{ij}$, $a_i$, $b_i$, $c_i$ ($i, j = 1, 2$).

Applying the inequalities (2.6) and (2.3) to the function $P = d_1 u + \alpha_{11} u^2 + \alpha_{12} uv$ we have
\[
|P_x|_2 \leq \tilde{C} \left( |P_{xx}|_{L^2} \right) \leq C_1 |P|_{L^2} \left( |P_{xx}|_{L^2} + |P|_{L^2} \right),
\]
\[
|P|_{L^2} \leq \tilde{C} \left( |P_x|_{L^2} \right).
\]
Here using the uniform boundedness of the $L_1$ norm of $P$, we have
\[
\int_0^1 P_{xx}^2 \, dx \leq C_7 - C_8 \left( \int_0^1 P_x^2 \, dx \right)^{\frac{2}{5}},
\]
\[
\int_0^1 P_x^2 \, dx \leq C_9 \left( \int_0^1 P_x^2 \, dx \right)^{\frac{1}{5}} + C_{10}
\]
where $C_7$, $C_8$, $C_9$, $C_{10}$ are positive constants depending on $d_i$, $\alpha_{ij}$, $a_i$, $b_i$, $c_i$ ($i, j = 1, 2$). Similar estimates are obtained also for $Q$. Hence we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_0^1 (P_x^2 + Q_x^2) \, dx \leq -C_{11} \left( \int_0^1 (P_x^2 + Q_x^2) \, dx \right)^{\frac{1}{2}} + C_{12} \left( \int_0^1 (P_x^2 + Q_x^2) \, dx \right)^{\frac{1}{2}} + C_{13},
\end{equation}

where $C_{11}, C_{12}, C_{13}$ are positive constants depending on $d_i, \alpha_{ij}, a_i, b_i, c_i$ ($i, j = 1, 2$). Hence by the Gronwall's type inequality in Lemma 2.3 we obtain the following $W_{1,2}$-bound of $P$ and $Q$ such that

\begin{equation}
\int_0^1 P_x^2 \, dx < \tilde{M}_2, \quad \int_0^1 Q_x^2 \, dx < \tilde{M}_2 \quad \text{for all} \quad t \geq 0,
\end{equation}

where $\tilde{M}_2$ is a positive constant depending on $\|u_0\|_2, \|v_0\|_2, d_i, \alpha_{ij}, a_i, b_i, c_i$ ($i, j = 1, 2$).

In order to obtain estimates for $u_x$ and $v_x$, we notice that

\[
\begin{pmatrix}
u_x \\
v_x
\end{pmatrix} = \begin{pmatrix}
u_u & P_v \\
P_u & \nu_v
\end{pmatrix}^{-1} \begin{pmatrix}
u_x \\
Q_x
\end{pmatrix} = A^{-1} \begin{pmatrix}
u_x \\
Q_x
\end{pmatrix},
\]

where

\[
A = \begin{pmatrix}
d_1 + 2\alpha_{11}u + \alpha_{12}v & \alpha_{12}u \\
\alpha_{21}v & d_2 + \alpha_{21}u + 2\alpha_{22}v
\end{pmatrix}.
\]

Here we note that $|A|$, the determinant of $A$, is bounded below by the positive constant $d_1d_2$, and $|A|$ is of class $O(u^2 + v^2)$ as $u \to \infty$ and $v \to \infty$, we have the inequality

\[
|u_x| + |v_x| \leq C_{14} (|P_x| + |Q_x|) \quad \text{for every} \quad x \in [0,1] \times [0, \infty)
\]

for some constant $C_{14}$ depending only on $d_i, \alpha_{ij}, (i, j = 1, 2)$. Therefore we obtain the following $W_{1,2}$-bound of $u$ and $v$ such that

\[
\int_0^1 u_x^2 \, dx < M_2, \quad \int_0^1 v_x^2 \, dx < M_2 \quad \text{for all} \quad t \geq 0,
\]

where $M_2$ is a positive constant depending on $\|u_0\|_2, \|v_0\|_2, d_i, \alpha_{ij}, a_i, b_i, c_i$ ($i, j = 1, 2$). \hfill $\square$

\section*{References}


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