HUGE CONTRACTION ON PARTIALLY ORDERED METRIC SPACES

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Abstract. We establish coincidence point theorem for $g$-nondecreasing mappings satisfying generalized nonlinear contraction on partially ordered metric spaces. We also obtain the coupled coincidence point theorem for generalized compatible pair of mappings $F, G : X^2 \to X$ by using obtained coincidence point results. Furthermore, an example is also given to demonstrate the degree of validity of our hypothesis. Our results generalize, modify, improve and sharpen several well-known results.

1. Introduction and Preliminaries

In the sequel, we denote by $X$ a non-empty set and $\preceq$ will represent a partial order on $X$. Given $n \in \mathbb{N}$ with $n \geq 2$, let $X^n$ be the $n$th Cartesian product $X \times X \times \ldots \times X$ (n times). For simplicity, if $x \in X$, we denote $g(x)$ by $gx$.

The idea of the coupled fixed point was initiated by Guo and Lakshmikantham [9] in 1987.

Definition 1 ([9]). Let $F : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a coupled fixed point of $F$ if

(1) $F(x, y) = x$ and $F(y, x) = y$.

Following this paper, Bhaskar and Lakshmikantham [2] where the authors introduced the notion of mixed monotone property for $F : X^2 \to X$ (wherein $X$ is an ordered metric space) and utilized the same to prove some theorems on the existence and uniqueness of coupled fixed points.

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Definition 2 ([2]). Let \((X, \preceq)\) be a partially ordered set. Suppose \(F : X^2 \to X\) be a given mapping. We say that \(F\) has the mixed monotone property if for all \(x, y \in X\), we have

$$x_1, x_2 \in X, \ x_1 \preceq x_2 \implies F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, \ y_1 \preceq y_2 \implies F(x, y_1) \succeq F(x, y_2).$$

In 2009, Lakshmikantham and Ciric [15] generalized these results for nonlinear contraction mappings by introducing the notions of coupled coincidence point and mixed \(g\)-monotone property.

Definition 3 ([15]). Let \(F : X^2 \to X\) and \(g : X \to X\) be given mappings. An element \((x, y) \in X^2\) is called a coupled coincidence point of the mappings \(F\) and \(g\) if

$$F(x, y) = gx \text{ and } F(y, x) = gy.$$  

Definition 4 ([15]). Let \(F : X^2 \to X\) and \(g : X \to X\) be given mappings. An element \((x, y) \in X^2\) is called a common coupled fixed point of the mappings \(F\) and \(g\) if

$$x = F(x, y) = gx \text{ and } y = F(y, x) = gy.$$  

Definition 5 ([15]). The mappings \(F : X^2 \to X\) and \(g : X \to X\) are said to be commutative if

$$gF(x, y) = F(gx, gy), \text{ for all } (x, y) \in X^2.$$  

Definition 6 ([15]). Let \((X, \preceq)\) be a partially ordered set. Suppose \(F : X^2 \to X\) and \(g : X \to X\) are given mappings. We say that \(F\) has the mixed \(g\)-monotone property if for all \(x, y \in X\), we have

$$x_1, x_2 \in X, \ gx_1 \preceq gx_2 \implies F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, \ gy_1 \preceq gy_2 \implies F(x, y_1) \succeq F(x, y_2).$$

If \(g\) is the identity mapping on \(X\), then \(F\) satisfies the mixed monotone property.

Subsequently, Choudhury and Kundu [3] introduced the notion of compatibility and by using this notion to improve the results of Lakshmikantham and Ciric [15],
thenafter several authors established coupled fixed/coincidence point theorems by using this notion.

**Definition 7 ([3]).** The mappings \( F : X^2 \to X \) and \( g : X \to X \) are said to be *compatible* if

\[
\lim_{n \to \infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0,
\]

whenever \( \{x_n\} \) and \( \{y_n\} \) are sequences in \( X \) such that

\[
\lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} gx_n = x,
\]

\[
\lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} gy_n = y, \text{ for some } x, y \in X.
\]

A great deal of these studies investigate contractions on partially ordered metric spaces because of their applicability to initial value problems defined by differential or integral equations.

Hussain et al. [11] introduced the notion of generalized compatibility of a pair \( \{F, G\} \), of mappings \( F, G : X \times X \to X \), then the authors employed this notion to obtained coupled coincidence point results for such a pair of mappings involving \((\varphi, \psi)\)-contractive condition without mixed \(G\)-monotone property of \( F \).

**Definition 8 ([11]).** Suppose that \( F, G : X^2 \to X \) are two mappings. The mapping \( F \) is said to be \( G \)-*increasing* with respect to \( \preceq \) if for all \( x, y, u, v \in X \) with \( G(x, y) \preceq G(u, v) \) we have \( F(x, y) \preceq F(u, v) \).

**Definition 9 ([11]).** Let \( F, G : X^2 \to X \) be two mappings. We say that the pair \( \{F, G\} \) is *commuting* if

\[
F(G(x, y), G(y, x)) = G(F(x, y), F(y, x)), \text{ for all } x, y \in X.
\]

**Definition 10 ([11]).** Suppose that \( F, G : X^2 \to X \) are two mappings. An element \((x, y) \in X^2 \) is called a *coupled coincidence point* of mappings \( F \) and \( G \) if

\[
F(x, y) = G(x, y) \text{ and } F(y, x) = G(y, x).
\]

**Definition 11 ([11]).** Let \((X, \preceq)\) be a partially ordered set, \( F : X^2 \to X \) and \( g : X \to X \) are two mappings. We say that \( F \) is \( g \)-*increasing with respect to \( \preceq \) if for any \( x, y \in X \),

\[
gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y),
\]
and

(14) \( gy_1 \leq gy_2 \) implies \( F(x, y_1) \leq F(x, y_2) \).

**Definition 12 ([11]).** Let \((X, \preceq)\) be a partially ordered set, \(F : X^2 \rightarrow X\) be a mapping. We say that \(F\) is *increasing with respect to* \(\preceq\) if for any \(x, y \in X\),

(15) \( x_1 \preceq x_2 \) implies \( F(x_1, y) \preceq F(x_2, y) \),

and\[ y_1 \preceq y_2 \) implies \( F(x, y_1) \preceq F(x, y_2) \).

**Definition 13** ([11]). Let \(F, G : X^2 \rightarrow X\) are two mappings. We say that the pair \(\{F, G\}\) is *generalized compatible* if

\[
\lim_{n \to \infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) = 0,
\]

\[
\lim_{n \to \infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) = 0,
\]

whenever \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(X\) such that

(17) \( \lim_{n \to \infty} G(x_n, y_n) = \lim_{n \to \infty} F(x_n, y_n) = x \),

\( \lim_{n \to \infty} G(y_n, x_n) = \lim_{n \to \infty} F(y_n, x_n) = y \), for some \(x, y \in X\).

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Erhan et al. [7], announced that the results established in Hussain et al. [11] can be easily derived from the coincidence point results in the literature.

In [7], Erhan et al. recalled the following basic definitions:

**Definition 14 ([1, 8]).** A coincidence point of two mappings \(T, g : X \rightarrow X\) is a point \(x \in X\) such that \(Tx = gx\).

**Definition 15** ([7]). An ordered metric space \((X, d, \preceq)\) is a metric space \((X, d)\) provided with a partial order \(\preceq\).

**Definition 16 ([2, 11]).** An ordered metric space \((X, d, \preceq)\) is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for every sequence \(\{x_n\} \subseteq X\) such that \(\{x_n\} \rightarrow x\) and \(x_n \preceq x_{n+1}\) (respectively, \(x_n \geq x_{n+1}\)) for all \(n\), we have that \(x_n \preceq x\) (respectively, \(x_n \geq x\)) for all \(n\). \((X, d, \preceq)\) is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular.
Definition 17 ([7]). Let \((X, \preceq)\) be a partially ordered set and let \(T, g : X \to X\) be two mappings. We say that \(T\) is \((g, \preceq)\)-non-decreasing if \(Tx \preceq Ty\) for all \(x, y \in X\) such that \(gx \preceq gy\). If \(g\) is the identity mapping on \(X\), we say that \(T\) is \(\preceq\)-non-decreasing.

Remark 18 ([7]). If \(T\) is \((g, \preceq)\)-non-decreasing and \(gx = gy\), then \(Tx = Ty\). It follows that
\[
(18) \quad gx = gy \Rightarrow \left\{ \begin{array}{l} gx \preceq gy, \\ gy \preceq gx \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} Tx \preceq Ty, \\ Ty \preceq Tx \end{array} \right\} \Rightarrow Tx = Ty.
\]

Definition 19 ([18]). Let \((X, \preceq)\) be a partially ordered set and endow the product space \(X^2\) with the following partial order:
\[
(19) \quad (u, v) \sqsubseteq (x, y) \Leftrightarrow x \succeq u \text{ and } y \preceq v, \text{ for all } (u, v), (x, y) \in X^2.
\]

Definition 20 ([3, 10, 17, 18]). Let \((X, d, \preceq)\) be an ordered metric space. Two mappings \(T, g : X \to X\) are said to be \(O\)-compatible if
\[
(20) \quad \lim_{n \to \infty} d(gTx_n, Tgx_n) = 0,
\]
provided that \(\{x_n\}\) is a sequence in \(X\) such that \(\{gx_n\}\) is \(\preceq\)-monotone, that is, it is either non-increasing or non-decreasing with respect to \(\preceq\) and
\[
\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} gx_n \in X.
\]

Samet et al. [20] declared that most of the coupled fixed point theorems for single-valued mappings on ordered metric spaces can be derived from well-known fixed point theorems.

On the other hand, Ding et al. [6] proved coupled coincidence and common coupled fixed point theorems for generalized nonlinear contraction on partially ordered metric spaces which generalize the results of Lakshmikantham and Ciric [15]. Our fundamental sources are [4-7, 11-14, 16, 18-20].

In this paper, we obtain a coincidence point theorem for \(g\)-non-decreasing mappings satisfying generalized nonlinear contraction on partially ordered metric spaces. With the help of our result, we derive a coupled coincidence point theorem of generalized compatible pair of mappings \(F, G : X^2 \to X\). We also give an example and an application to integral equation to support our results. Our results generalize, extend, modify, improve and sharpen the results of Bhaskar and Lakshmikantham [2], Ding et al. [6] and Lakshmikantham and Ciric [15].
2. Main Results

Lemma 21. Let \((X, d)\) be a metric space. Suppose \(Y = X^2\) and define \(\delta : Y \times Y \to [0, +\infty)\) by

\[
\delta((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}, \quad \text{for all } (x, y), (u, v) \in Y.
\]

Then \(\delta\) is metric on \(Y\) and \((X, d)\) is complete if and only if \((Y, \delta)\) is complete.

Let \(\Phi\) denote the set of all functions \(\varphi : [0, +\infty) \to [0, +\infty)\) satisfying

(i) \(\varphi\) is non-decreasing,
(ii) \(\lim_{n \to \infty} \varphi^n(t) = 0\) for all \(t > 0\), where \(\varphi^{n+1}(t) = \varphi^n(\varphi(t))\).

It is clear that \(\varphi(t) < t\) for each \(t > 0\). In fact, if \(\varphi(t_0) \geq t_0\) for some \(t_0 > 0\), then, since \(\varphi\) is non-decreasing, \(\varphi^n(t_0) \geq t_0\) for all \(n \in \mathbb{N}\), which contradicts with \(\lim_{n \to \infty} \varphi^n(t_0) = 0\). In addition, it is easy to see that \(\varphi(0) = 0\).

Theorem 22. Let \((X, d, \preceq)\) be a partially ordered metric space and let \(T, g : X \to X\) be two mappings such that the following properties are fulfilled:

(i) \(T(X) \subseteq g(X)\),
(ii) \(T\) is \((g, \preceq)\)-non-decreasing,
(iii) there exists \(x_0 \in X\) such that \(gx_0 \preceq Tx_0\),
(iv) there exists \(\varphi \in \Phi\) such that

\[
d(Tx, Ty) \leq \varphi(M(x, y)),
\]

where

\[
M(x, y) = \max\left\{\frac{d(gx, gy), d(gx, Tx), d(gy, Ty)}{d(gx, Ty) + d(gy, Tx)}\right\},
\]

for all \(x, y \in X\) such that \(gx \preceq gy\). Also assume that, at least, one of the following conditions holds:

(a) \((X, d)\) is complete, \(T\) and \(g\) are continuous and the pair \((T, g)\) is \(O\)-compatible,
(b) \((X, d)\) is complete, \(T\) and \(g\) are continuous and commuting,
(c) \((g(X), d)\) is complete and \((X, d, \preceq)\) is non-decreasing-regular,
(d) \((X, d)\) is complete, \(g(X)\) is closed and \((X, d, \preceq)\) is non-decreasing-regular,
(e) \((X, d)\) is complete, \(g\) is continuous, the pair \((T, g)\) is \(O\)-compatible and \((X, d, \preceq)\) is non-decreasing-regular.

Then \(T\) and \(g\) have, at least, a coincidence point.

Proof. We divide the proof into four steps.

Step 1. We claim that there exists a sequence \(\{x_n\} \subseteq X\) such that \(\{gx_n\}\) is \(\preceq\)-non-decreasing and \(gx_{n+1} = Tx_n\), for all \(n \geq 0\). Let \(x_0 \in X\) be arbitrary. Since
$T x_0 \in T(X) \subseteq g(X)$, therefore there exists $x_1 \in X$ such that $T x_0 = g x_1$. Then $g x_0 \preceq T x_0 = g x_1$. Since $T$ is $(g, \preceq)$-non-decreasing, therefore $T x_0 \preceq T x_1$. Again, since $T x_1 \in T(X) \subseteq g(X)$, therefore there exists $x_2 \in X$ such that $T x_1 = g x_2$. Then $g x_1 = T x_0 \preceq T x_1 = g x_2$. Since $T$ is $(g, \preceq)$-non-decreasing, therefore $T x_1 \preceq T x_2$.

Repeating this argument, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that $\{g x_n\}$ is $\preceq$-non-decreasing, $g x_{n+1} = T x_n \preceq T x_{n+1} = g x_{n+2}$ and

\[(22) \quad g x_{n+1} = T x_n \text{ for all } n \geq 0.\]

Step 2. We claim that $\{g x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $X$. Now, by contractive condition (iv), we have

\[(23) \quad d(g x_{n+1}, g x_{n+2}) = d(T x_n, T x_{n+1}) \leq \varphi (M(x_n, x_{n+1})) ,\]

where

\[
M(x_n, x_{n+1}) = \max \left\{ d(g x_n, g x_{n+1}), d(g x_n, T x_n), d(g x_{n+1}, T x_{n+1}) \right\} \frac{1}{2} (d(g x_n, T x_n) + d(g x_{n+1}, T x_{n+1})) \leq \max \{d(g x_n, g x_{n+1}), d(g x_{n+1}, g x_{n+2})\}.
\]

If $d(g x_{n+1}, g x_{n+2}) \geq d(g x_n, g x_{n+1})$. Then

\[(24) \quad M(x_n, x_{n+1}) \leq d(g x_{n+1}, g x_{n+2}).\]

From (23), (24) and by the fact that $\varphi(t) < t$ for all $t > 0$, we get

\[d(g x_{n+1}, g x_{n+2}) \leq \varphi (d(g x_{n+1}, g x_{n+2})) < d(g x_{n+1}, g x_{n+2}),\]

which is a contradiction. Hence, $d(g x_n, g x_{n+1}) \geq d(g x_{n+1}, g x_{n+2})$. Then

\[(25) \quad M(x_n, x_{n+1}) \leq d(g x_n, g x_{n+1}).\]

Thus, by (23) and (25), we have for all $n \in \mathbb{N}$,

\[(26) \quad d(g x_{n+1}, g x_{n+2}) \leq \varphi (d(g x_n, g x_{n+1})) \leq \varphi^n (d(g x_0, g x_1)) \leq \varphi^n (\delta),\]

where

\[
\delta = d(g x_0, g x_1).
\]

Without loss of generality, we can assume that $d(g x_0, g x_1) \neq 0$. In fact, if this is not true, then $g x_0 = g x_1 = T x_0$, that is, $x_0$ is a coincidence point of $g$ and $T$. 

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Thus, for \( m, n \in \mathbb{N} \) with \( m > n \), by triangle inequality and (26), we get

\[
d(g_{x_n}, g_{x_{m+n}}) 
\leq d(g_{x_n}, g_{x_{n+1}}) + d(g_{x_{n+1}}, g_{x_{n+2}}) + \ldots + d(g_{x_{n+m-1}}, g_{x_{m+n}}) 
\leq \varphi^n(\delta) + \varphi^{n+1}(\delta) + \ldots + \varphi^{n+m-1}(\delta) 
\leq \sum_{i=n}^{n+m-1} \varphi^i(\delta),
\]

which implies, by (\( ii_{\varphi} \)), that \( \{g_{x_n}\} \) is a Cauchy sequence in \( X \).

Step 3. We claim that \( T \) and \( g \) have a coincidence point distinguishing between cases (a) – (e).

Suppose now that (a) holds, that is, \((X, d)\) is complete, \( T \) and \( g \) are continuous and the pair \((T, g)\) is \( O \)-compatible. Since \((X, d)\) is complete, therefore there exists \( z \in X \) such that \( \{g_{x_n}\} \to z \) and \( \{T_{x_n}\} \to z \). Since \( T \) and \( g \) are continuous, therefore \( \{T_{gx_n}\} \to Tz \) and \( \{ggx_n\} \to gz \). Since the pair \((T, g)\) is \( O \)-compatible, therefore

\[
\lim_{n \to \infty} d(g_{Tx_n}, Tg_{x_n}) = 0.
\]

Thus, we conclude that

\[
d(gz, Tz) = \lim_{n \to \infty} d(g_{gx_{n+1}}, Tgx_n) = \lim_{n \to \infty} d(g_{Tx_n}, Tgx_n) = 0,
\]

that is, \( z \) is a coincidence point of \( T \) and \( g \).

Suppose now that (b) holds, that is, \((X, d)\) is complete, \( T \) and \( g \) are continuous and commuting. It is evident that (b) implies (a).

Suppose now that (c) holds, that is, \((g(X), d)\) is complete and \((X, d, \preceq)\) is non-decreasing-regular. As \( \{g_{x_n}\} \) is a Cauchy sequence in the complete space \((g(X), d)\), so there exists \( y \in g(X) \) such that \( \{g_{x_n}\} \to y \). Let \( z \in X \) be any point such that \( y = gz \), then \( \{g_{x_n}\} \to gz \). Indeed, as \((X, d, \preceq)\) is non-decreasing-regular and \( \{g_{x_n}\} \) is \( \preceq \)-non-decreasing and converging to \( gz \), we deduce that \( g_{x_n} \preceq gz \) for all \( n \geq 0 \).

Applying the contractive condition (iv), we get

\[
d(g_{x_{n+1}}, Tz) = d(Tx_n, Tz) \leq \varphi(M(x_n, z)),
\]

where

\[
M(x_n, z) = \max \left\{ \frac{d(g_{x_n}, gz)}, d(g_{x_n}, Tz) + d(gz, Tz), \right\}
\]

\[
= \max \left\{ \frac{d(g_{x_n}, gz)}, d(g_{x_{n+1}}, Tz) + d(gz, Tz), \right\}.
\]

Since \( \{g_{x_n}\} \to gz \), therefore there exists \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \),

\[
M(x_n, z) = d(gz, Tz).
\]
By (27) and (28), we get
\[ d(gx_{n+1}, Tz) \leq \varphi(d(gz, Tz)). \]

Now, we claim that \( d(gz, Tz) = 0 \). If this is not true, then \( d(gz, Tz) > 0 \), which, by the fact that \( \varphi(t) < t \) for all \( t > 0 \), implies
\[ d(gx_{n+1}, Tz) < d(gz, Tz). \]

Letting \( n \to \infty \) in the above inequality and using \( \lim_{n \to \infty} gx_n = gz \), we get
\[ d(gz, Tz) < d(gz, Tz), \]
which is a contradiction. Hence we must have \( d(gz, Tz) = 0 \), that is, \( z \) is a coincidence point of \( T \) and \( g \).

Suppose now that \( (d) \) holds, that is, \((X, d)\) is complete, \( g(X) \) is closed and \((X, d, \preceq)\) is non-decreasing-regular. It follows from the fact that a closed subset of a complete metric space is also complete. Then, \((g(X), d)\) is complete and \((X, d, \preceq)\) is non-decreasing-regular. Thus \( (d) \) implies \( (c) \).

Suppose now that \( (e) \) holds, that is, \((X, d)\) is complete, \( g \) is continuous, the pair \((T, g)\) is \( O \)-compatible and \((X, d, \preceq)\) is non-decreasing-regular. As \((X, d)\) is complete, so there exists \( z \in X \) such that \( \{gx_n\} \to z \). Since \( Tx_n = gx_{n+1} \) for all \( n \), we also have that \( \{Tx_n\} \to z \). As \( g \) is continuous, then \( \{ggx_n\} \to gz \). Furthermore, since the pair \((T, g)\) is \( O \)-compatible, we have \( \lim_{n \to \infty} d(ggx_{n+1}, Tgx_n) = \lim_{n \to \infty} d(gT x_n, Tgx_n) = 0 \). As \( \{ggx_n\} \to gz \) the previous property means that \( \{Tgx_n\} \to gz \).

Indeed, as \((X, d, \preceq)\) is non-decreasing-regular and \( \{gx_n\} \) is \( \preceq \)-non-decreasing and converging to \( z \), we deduce that \( gx_n \preceq z \) for all \( n \geq 0 \). Applying the contractive condition \((iv)\), we get
\[ d(Tgx_n, Tz) \leq \varphi(M(gx_n, z)), \]
where
\[ M(gx_n, z) = \max \left\{ d(ggx_n, gz), \frac{d(ggx_n, Tgx_n)}{d(ggx_n, Tz)}, \frac{d(gz, Tz)}{2d(gz, Tgx_n)} \right\}. \]

Since \( \{ggx_n\} \to gz \), therefore there exists \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \),
\[ M(gx_n, z) = d(gz, Tz). \]

By (29) and (30), we get
\[ d(Tgx_n, Tz) \leq \varphi(d(gz, Tz)), \]
Now, we claim that \( d(gz, Tz) = 0 \). If this is not true, then \( d(gz, Tz) > 0 \), which, by the fact that \( \varphi(t) < t \) for all \( t > 0 \), implies
\[
d(Tgx_n, Tz) < d(gz, Tz).
\]
Letting \( n \to \infty \) in the above inequality and using \( \{Tgx_n\} \to gz \), we get
\[
d(gz, Tz) < d(gz, Tz),
\]
which is a contradiction. Hence we must have \( d(gz, Tz) = 0 \), that is, \( z \) is a coincidence point of \( T \) and \( g \).
\[
\square
\]
Next, we derive the two dimensional version of Theorem 22. For the ordered metric space \((X, d, \preceq)\), let us consider the ordered metric space \((X^2, \delta, \sqsubseteq)\), where \( \delta \) was defined in Lemma 21 and \( \sqsubseteq \) was introduced in (19). Define the mappings \( T_F, T_G : X^2 \to X^2 \), for all \((x, y) \in X^2\), by,
\[
(31) \quad T_F(x, y) = (F(x, y), F(y, x)) \text{ and } T_G(x, y) = (G(x, y), G(y, x)).
\]
Under these conditions, the following properties hold:

**Lemma 23.** Let \((X, d, \preceq)\) be a partially ordered metric space and let \( F, G : X^2 \to X \) be two mappings. Then

1. \((X, d)\) is complete if and only if \((X^2, \delta)\) is complete.
2. If \((X, d, \preceq)\) is regular, then \((X^2, \delta, \sqsubseteq)\) is also regular.
3. If \( F \) is \( d \)-continuous, then \( T_F \) is \( \delta \)-continuous.
4. If \( F \) is \( G \)-increasing with respect to \( \preceq \), then \( T_F \) is \((T_G, \sqsubseteq)\)-nondecreasing.
5. If there exist two elements \( x_0, y_0 \in X \) with \( G(x_0, y_0) \preceq F(x_0, y_0) \) and \( G(y_0, x_0) \preceq F(y_0, x_0) \), then there exists a point \((x_0, y_0) \in X^2\) such that \( T_G(x_0, y_0) \sqsubseteq T_F(x_0, y_0) \).
6. For any \( x, y \in X \), there exist \( u, v \in X \) such that \( F(x, y) = G(u, v) \) and \( F(y, x) = G(v, u) \), then \( T_F(X^2) \subseteq T_G(X^2) \).
7. Assume there exists \( \varphi \in \Phi \) such that
\[
(32) \quad d(F(x, y), F(u, v)) \leq \varphi(M(x, u, v), v),
\]
where
\[
M(x, u, v) = \max \left\{ \frac{d(G(x, y), G(u, v))}{2}, \frac{d(G(x, y), F(x, y))}{2}, \frac{d(G(x, y), F(u, v))}{2}, \frac{d(G(u, v), G(y, x))}{2}, \frac{d(G(u, v), F(y, x))}{2}, \frac{d(G(y, x), G(v, u))}{2}, \frac{d(G(y, x), F(v, u))}{2}, \frac{d(G(v, u), G(y, x))}{2}, \frac{d(G(v, u), F(v, u))}{2} \right\}.
\]
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for all $x, y, u, v \in X$, where $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$, then

$$\delta(T_F(x, y), T_F(u, v)) \leq \varphi(M_\delta((x, y), (u, v)))$$

where

$$M_\delta((x, y), (u, v)) = \max \left\{ \begin{array}{l} \delta(T_G(x, y), T_G(u, v)), \\
\delta(T_G(x, y), T_F(x, y)), \\
\delta(T_G(u, v), T_F(u, v)), \\
\frac{\delta(T_G(x, y), T_F(x, y)) + \delta(T_G(u, v), T_F(u, v))}{2} \end{array} \right\},$$

for all $(x, y), (u, v) \in X^2$, where $T_G(x, y) \sqsubseteq T_G(u, v)$.

(8) If the pair $\{F, G\}$ is generalized compatible, then the mappings $T_F$ and $T_G$ are $O$-compatible in $(X^2, \delta, \sqsubseteq)$.

(9) A point $(x, y) \in X^2$ is a coupled coincidence point of $F$ and $G$ if and only if it is a coincidence point of $T_F$ and $T_G$.

Proof. Statement (1) follows from Lemma 21 and (2), (3), (5), (6) and (9) are obvious.

(4) Assume that $F$ is $G$-increasing with respect to $\preceq$ and let $(x, y), (u, v) \in X^2$ be such that $T_G(x, y) \sqsubseteq T_G(u, v)$. Then $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$. Since $F$ is $G$-increasing with respect to $\preceq$, we have that $F(x, y) \preceq F(u, v)$ and $F(y, x) \succeq F(v, u)$. Therefore $T_F(x, y) \sqsubseteq T_F(u, v)$ which shows that $T_F$ is $(T_G, \sqsubseteq)$-non-decreasing.

(7) Let $(x, y), (u, v) \in X^2$ be such that $T_G(x, y) \sqsubseteq T_G(u, v)$. Therefore $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$. From (32), we have

$$d(F(x, y), F(u, v)) \leq \varphi(M(x, y, u, v)).$$

Furthermore $G(y, x) \succeq G(v, u)$ and $G(x, y) \preceq G(u, v)$, the contractive condition (32) implies that

$$d(F(y, x), F(v, u)) \leq \varphi(M(x, y, u, v)).$$

Combining (33) and (34), we get

$$\max \{d(F(x, y), F(u, v)), d(F(y, x), F(v, u))\} \leq \varphi(M(x, y, u, v)).$$
Hence, the mappings

\[ \delta(T_F(x, y), T_F(u, v)) \]

\[ = \delta((F(x, y), F(y, x)), (F(u, v), F(v, u))) \]

\[ = \max \{d(F(x, y), F(u, v)), d(F(y, x), F(v, u))\} \]

\[ \leq \varphi(M(x, y, u, v)) \]

\[ \leq \varphi(M_5((x, y), (u, v))). \]

(8) Let \( \{x_n, y_n\} \subseteq X^2 \) be any sequence such that \( T_F(x_n, y_n) \xrightarrow{\delta} (x, y) \) and \( T_G(x_n, y_n) \xrightarrow{\delta} (x, y) \) (Note that it is not require to suppose that \( \{T_G(x_n, y_n)\} \) is \( \sqsubseteq \)-monotone). Thus

\[ (F(x_n, y_n), F(y_n, x_n)) \xrightarrow{\delta} (x, y) \]

\[ \Rightarrow F(x_n, y_n) \xrightarrow{d} x \text{ and } F(y_n, x_n) \xrightarrow{d} y, \]

and

\[ (G(x_n, y_n), G(y_n, x_n)) \xrightarrow{\delta} (x, y) \]

\[ \Rightarrow G(x_n, y_n) \xrightarrow{d} x \text{ and } G(y_n, x_n) \xrightarrow{d} y. \]

Therefore

\[ \lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} G(x_n, y_n) = x \in X, \]

\[ \lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} G(y_n, x_n) = y \in X. \]

Since the pair \( \{F, G\} \) is generalized compatible, therefore

\[ \lim_{n \to \infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) = 0, \]

\[ \lim_{n \to \infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) = 0. \]

In particular,

\[ \lim_{n \to \infty} \delta(T_GT_F(x_n, y_n), T_FT_G(x_n, y_n)) \]

\[ = \lim_{n \to \infty} \delta(T_G(F(x_n, y_n), F(y_n, x_n)), T_F(G(x_n, y_n), G(y_n, x_n))) \]

\[ = \lim_{n \to \infty} \delta \left( \left\{ \begin{array}{l} G(F(x_n, y_n), F(y_n, x_n)), G(F(y_n, x_n), F(x_n, y_n)) \end{array} \right\} \right) \]

\[ = \lim_{n \to \infty} \max \left\{ \begin{array}{l} d(G(F(x_n, y_n), F(y_n, x_n)), G(F(x_n, y_n), G(y_n, x_n))) \end{array} \right\} \]

\[ = 0. \]

Hence, the mappings \( T_F \text{ and } T_G \) are \( \mathcal{O} \)-compatible in \( (X^2, \delta, \sqsubseteq) \). \qed
Theorem 24. Let \((X, \preceq)\) be a partially ordered set such that there exists a complete metric \(d\) on \(X\). Assume \(F, G : X^2 \to X\) be two generalized compatible mappings such that \(F\) is \(G\)-increasing with respect to \(\preceq\), \(G\) is continuous and there exist two elements \(x_0, y_0 \in X\) with

\[
G(x_0, y_0) \preceq F(x_0, y_0) \text{ and } G(y_0, x_0) \succeq F(y_0, x_0).
\]

Suppose that there exists \(\varphi \in \Phi\) satisfying (32) and for any \(x, y \in X\), there exist \(u, v \in X\) such that

\[
(36) \quad F(x, y) = G(u, v) \text{ and } F(y, x) = G(v, u).
\]

Also suppose that either

(a) \(F\) is continuous or
(b) \((X, d, \preceq)\) is regular.

Then \(F\) and \(G\) have a coupled coincidence point.

Proof. It is only require to use Theorem 22 to the mappings \(T = TF\) and \(g = TG\) in the ordered metric space \((X^2, \delta, \sqsubseteq)\) with Lemma 23. \(\square\)

Corollary 25. Let \((X, \preceq)\) be a partially ordered set such that there exists a complete metric \(d\) on \(X\). Assume \(F, G : X^2 \to X\) be two commuting mappings satisfying (32) and (36) such that \(F\) is \(G\)-increasing with respect to \(\preceq\), \(G\) is continuous and there exist two elements \(x_0, y_0 \in X\) with

\[
G(x_0, y_0) \preceq F(x_0, y_0) \text{ and } G(y_0, x_0) \succeq F(y_0, x_0).
\]

Also suppose that either

(a) \(F\) is continuous or
(b) \((X, d, \preceq)\) is regular.

Then \(F\) and \(G\) have a coupled coincidence point.

Next, we deduce results without \(g\)-mixed monotone property of \(F\).

Corollary 26. Let \((X, \preceq)\) be a partially ordered set such that there exists a complete metric \(d\) on \(X\), \(F : X \times X \to X\) and \(g : X \to X\) be two compatible mappings such that \(F\) is \(g\)-increasing with respect to \(\preceq\). Assume there exists \(\varphi \in \Phi\) such that

\[
(37) \quad d(F(x, y), F(u, v)) \leq \varphi(M_g(x, y, u, v)),
\]
where
\[
M_g(x, y, u, v) = \max \left\{ \frac{d(gx, gu) + d(gx, F(x, y)), d(gu, F(u, v)), d(gx, F(u, v)) + d(gu, F(x, y))}{2}, \frac{d(gy, gv) + d(gy, F(y, x)), d(gv, F(v, u)), d(gy, F(v, u)) + d(gv, F(y, x))}{2} \right\},
\]
for all \(x, y, u, v \in X\), where \(gx \preceq gu\) and \(gy \succeq gv\). Furthermore \(F(X \times X) \subseteq g(X)\), \(g\) is continuous and monotone increasing with respect to \(\preceq\). Also suppose that either

(a) \(F\) is continuous or
(b) \((X, d, \preceq)\) is regular.

If there exist two elements \(x_0, y_0 \in X\) with
\[ gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0). \]
Then \(F\) and \(g\) have a coupled coincidence point.

**Corollary 27.** Let \((X, \preceq)\) be a partially ordered set such that there exists a complete metric \(d\) on \(X\). Assume \(F : X \times X \to X\) and \(g : X \to X\) be two commuting mappings satisfying (37) such that \(F\) is \(g\)-increasing with respect to \(\preceq\). Furthermore \(F(X \times X) \subseteq g(X)\), \(g\) is continuous and monotone increasing with respect to \(\preceq\). Also suppose that either

(a) \(F\) is continuous or
(b) \((X, d, \preceq)\) is regular.

If there exist two elements \(x_0, y_0 \in X\) with
\[ gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0). \]
Then \(F\) and \(g\) have a coupled coincidence point.

Now, we deduce result without mixed monotone property of \(F\).

**Corollary 28.** Let \((X, \preceq)\) be a partially ordered set such that there exists a complete metric \(d\) on \(X\). Assume \(F : X \times X \to X\) be an increasing mapping with respect to \(\preceq\) and there exists \(\varphi \in \Phi\) such that
\[ d(F(x, y), F(u, v)) \leq \varphi(m(x, y, u, v)), \]
where
\[
m(x, y, u, v) = \max \left\{ \frac{d(x, u), d(x, F(x, y)), d(u, F(u, v)), d(x, F(u, v)) + d(u, F(x, y))}{2}, \frac{d(y, v), d(y, F(y, x)), d(v, F(v, u)), d(y, F(v, u)) + d(v, F(y, x))}{2} \right\},
\]
for all \( x, y, u, v \in X \), where \( x \preceq u \) and \( y \succeq v \). Also suppose that either

(a) \( F \) is continuous or

(b) \( (X, d, \preceq) \) is regular.

If there exist two elements \( x_0, y_0 \in X \) with

\[ x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0). \]

Then \( F \) has a coupled fixed point.

**Example 29.** Suppose that \( X = [0, 1] \), equipped with the usual metric \( d : X \times X \to [0, +\infty) \) with the natural ordering of real numbers \( \leq \). Let \( F, G : X \times X \to X \) be defined as

\[
F(x, y) = \begin{cases} 
\frac{x^2 - y^2}{3}, & \text{if } x \geq y, \\
0, & \text{if } x < y,
\end{cases}
\]

and

\[
G(x, y) = \begin{cases} 
\frac{x^2 - y^2}{3}, & \text{if } x \geq y, \\
0, & \text{if } x < y.
\end{cases}
\]

Define \( \varphi : [0, +\infty) \to [0, +\infty) \) as follows

\[
\varphi(t) = \begin{cases} 
t^3, & \text{for } t \neq 1, \\
1, & \text{for } t = 1.
\end{cases}
\]

First, we shall show that the contractive condition (32) holds for the mappings \( F \) and \( G \). Let \( x, y, u, v \in X \) such that \( G(x, y) \preceq G(u, v) \) and \( G(y, x) \succeq G(v, u) \), we have

\[
d(F(x, y), F(u, v)) = \left| \frac{x^2 - y^2}{3} - \frac{u^2 - v^2}{3} \right|
\]

\[
= \frac{1}{3} |G(x, y) - G(u, v)|
\]

\[
= \frac{1}{3} d(G(x, y), G(u, v))
\]

\[
\leq \frac{1}{3} M(x, y, u, v)
\]

\[
\leq \varphi(M(x, y, u, v)).
\]

Thus the contractive condition (32) holds for all \( x, y, u, v \in X \). In addition, like in [11], all the other conditions of Theorem 24 are satisfied and \( z = (0, 0) \) is a coincidence point of \( F \) and \( G \).

**Remark 30.** Using the same technique that can be used in [12 − 14, 18, 19, 20] it is possible to derive tripled, quadruple and in general, multidimensional coincidence point theorems from Theorem 22.
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