QUADRATIC $\rho$-FUNCTIONAL INEQUALITIES

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Abstract. In this paper, we solve the quadratic $\rho$-functional inequalities

\begin{align}
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \\
\leq \|\rho \left( 4f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) - 2f(y) \right) \|,
\end{align}

where $\rho$ is a fixed complex number with $|\rho| < 1$, and

\begin{align}
\|4f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) - 2f(y)\| \\
\leq \|\rho(f(x + y) + f(x - y) - 2f(x) - 2f(y))\|,
\end{align}

where $\rho$ is a fixed complex number with $|\rho| < \frac{1}{2}$.

Furthermore, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and Preliminaries


The functional equation $f(x + y) = f(x) + f(y)$ is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Gavrut [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

The functional equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \]
is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a \textit{quadratic mapping}. The stability of quadratic functional equation was proved by Skof \cite{10} for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa \cite{3} noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group.

The functional equation

$$4f\left(\frac{x+y}{2}\right) + (x - y) = 2f(x) + 2f(y)$$

is called a \textit{Jensen type quadratic equation}. See \cite{2, 4, 7, 9, 12} for more information on the stability problems of functional equations.

In Section 2, we solve the quadratic $\rho$-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (0.1) in complex Banach spaces.

In Section 3, we solve the quadratic $\rho$-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, let $G$ be a 2-divisible abelian group. Assume that $X$ is a real or complex normed space with norm $\| \cdot \|$ and that $Y$ is a complex Banach space with norm $\| \cdot \|$.

2. QUADRATIC $\rho$-FUNCTIONAL INEQUALITY (0.1)

Throughout this section, assume that $\rho$ is a fixed complex number with $|\rho| < 1$.

In this section, we solve and investigate the quadratic $\rho$-functional inequality (0.1) in complex Banach spaces.

\textbf{Lemma 2.1.} If a mapping $f : G \to Y$ satisfies

\begin{equation}
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| 
\leq \left\| \rho \left(4f\left(\frac{x+y}{2}\right) + f(x - y) - 2f(x) - 2f(y)\right) \right\|
\end{equation}

for all $x, y \in G$, then $f : G \to Y$ is quadratic.

\textbf{Proof.} Assume that $f : G \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$. 

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in G$. Thus
\[
(2.2) \quad f \left( \frac{x}{2} \right) = \frac{1}{4} f(x)
\]
for all $x \in G$.

It follows from (2.1) and (2.2) that
\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \\
\leq \left\| \rho \left( 4f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) - 2f(y) \right) \right\| \\
= |\rho| \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|
\]
and so
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]
for all $x, y \in G$. □

We prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (2.1) in complex Banach spaces.

**Theorem 2.2.** Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying
\[
(2.3) \quad \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \\
\leq \left\| \rho \left( 4f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) - 2f(y) \right) \right\| + \theta(\|x\|^r + \|y\|^r)
\]
for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that
\[
(2.4) \quad \|f(x) - h(x)\| \leq \frac{2\theta}{2^r - 4} \|x\|^r
\]
for all $x \in X$.

**Proof.** Letting $x = y = 0$ in (2.3), we get $\|2f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.3), we get
\[
(2.5) \quad \|f(2x) - 4f(x)\| \leq 2\theta\|x\|^r
\]
for all $x \in X$. So
\[
\left\| f(x) - 4f \left( \frac{x}{2} \right) \right\| \leq \frac{2}{2^r}\theta\|x\|^r
\]
for all $x \in X$. Hence

\[
\left\| 4^l f \left( \frac{x}{2^l} \right) - 4^m f \left( \frac{x}{2^m} \right) \right\| \leq \sum_{j=l}^{m-1} \left\| 4^j f \left( \frac{x}{2^j} \right) - 4^{j+1} f \left( \frac{x}{2^{j+1}} \right) \right\| 
\]

(2.6)

\[
\leq \frac{2}{2^r} \sum_{j=1}^{m-1} 4^j \theta \|x\|^r
\]

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{4^n f \left( \frac{x}{2^n} \right) \}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f \left( \frac{x}{2^n} \right) \}$ converges. So one can define the mapping $h : X \to Y$ by

\[
h(x) := \lim_{n \to \infty} 4^n f \left( \frac{x}{2^n} \right)
\]

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

\[
\| h(x + y) + h(x - y) - 2h(x) - 2h(y) \| 
\]

\[
= \lim_{n \to \infty} 4^n \left\| f \left( \frac{x + y}{2^n} \right) + f \left( \frac{x - y}{2^n} \right) - 2f \left( \frac{x}{2^n} \right) - 2f \left( \frac{y}{2^n} \right) \right\| 
\]

\[
\leq \lim_{n \to \infty} 4^n |\rho| \left( \left\| 4f \left( \frac{x + y}{2^n + 1} \right) + f \left( \frac{x - y}{2^n} \right) - 2f \left( \frac{x}{2^n} \right) - 2f \left( \frac{y}{2^n} \right) \right\| 
\]

\[
+ \lim_{n \to \infty} 4^n \theta \left( \|x\|^r + \|y\|^r \right) 
\]

\[
= |\rho| \left\| 4h \left( \frac{x + y}{2} \right) + h(x) - y - 2h(x) - 2h(y) \right\|
\]

for all $x, y \in X$. So

\[
\| h(x + y) + h(x - y) - 2h(x) - 2h(y) \| \leq \left\| \rho \left( 4h \left( \frac{x + y}{2} \right) + h(x) - y - 2h(x) - 2h(y) \right) \right\|
\]

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.4). Then we have

\[
\| h(x) - T(x) \| = 4^n \left\| h \left( \frac{x}{2^n} \right) - T \left( \frac{x}{2^n} \right) \right\|
\]

\[
\leq 4^n \left( \left\| h \left( \frac{x}{2^n} \right) - f \left( \frac{x}{2^n} \right) \right\| + \left\| T \left( \frac{x}{2^n} \right) - f \left( \frac{x}{2^n} \right) \right\| \right)
\]

\[
\leq \frac{4 \cdot 4^n}{(2^r - 4)2^{nr} \theta \|x\|^r},
\]
which tends to zero as \( n \to \infty \) for all \( x \in X \). So we can conclude that \( h(x) = T(x) \) for all \( x \in X \). This proves the uniqueness of \( h \). Thus the mapping \( h : X \to Y \) is a unique quadratic mapping satisfying (2.4). □

**Theorem 2.3.** Let \( r < 2 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (2.3). Then there exists a unique quadratic mapping \( h : X \to Y \) such that

\[
\|f(x) - h(x)\| \leq \frac{2\theta}{4 - 2^r} \|x\|^r
\]

for all \( x \in X \).

**Proof.** It follows from (2.5) that

\[
\left\|f(x) - \frac{1}{4} f(2x)\right\| \leq \frac{\theta}{2} \|x\|^r
\]

for all \( x \in X \). Hence

\[
\left\|\frac{1}{4l} f(2^lx) - \frac{1}{4m} f(2^mx)\right\| \leq \sum_{j=l}^{m-1} \left\|\frac{1}{4j} f(2^jx) - \frac{1}{4j+1} f(2^{j+1}x)\right\|
\]

\[
\leq \sum_{j=l}^{m-1} \frac{2^rj \theta}{4^j 2} \|x\|^r
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.8) that the sequence \( \{\frac{1}{4^n} f(2^nx)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{\frac{1}{4^n} f(2^nx)\} \) converges. So one can define the mapping \( h : X \to Y \) by

\[
h(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^nx)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.8), we get (2.7).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Remark 2.4.** If \( \rho \) is a real number such that \(-1 < \rho < 1\) and \( Y \) is a real Banach space, then all the assertions in this section remain valid.

3. Quadratic \( \rho \)-Functional Inequality (0.2)

Throughout this section, assume that \( \rho \) is a fixed complex number with \( |\rho| < \frac{1}{2} \).

In this section, we solve and investigate the quadratic \( \rho \)-functional inequality (0.2) in complex Banach spaces.
Lemma 3.1. If a mapping $f : G \to Y$ satisfies
\begin{align}
\|4f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) - 2f(y)\| \\
\leq \|\rho(f(x + y) + f(x - y) - 2f(x) - 2f(y))\|
\end{align}
for all $x, y \in G$, then $f : G \to Y$ is quadratic.

Proof. Assume that $f : G \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get
\[
\|4f \left( \frac{x}{2} \right) - f(x)\| \leq 0
\]
and so
\[
f \left( \frac{x}{2} \right) = f(x)
\]
for all $x \in G$.

It follows from (3.1) and (3.2) that
\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|
\leq \|\rho(f(x + y) + f(x - y) - 2f(x) - 2f(y))\|
\]
and so
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]
for all $x, y \in G$. \hfill \Box

We prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (3.1) in complex Banach spaces.

Theorem 3.2. Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that
\begin{align}
\|4f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) - 2f(y)\| \\
\leq \|\rho(f(x + y) + f(x - y) - 2f(x) - 2f(y))\| + \theta(\|x\|^r + \|y\|^r)
\end{align}
for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that
\begin{align}
\|f(x) - h(x)\| \leq \frac{2^r \theta}{2^r - 4} \|x\|^r
\end{align}
for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.3), we get $\|f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.3), we get
(3.5) \[ \|4f\left(\frac{x}{2}\right) - f(x)\| \leq \theta \|x\|^r \]
for all \( x \in X \). So
\[ \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \]
(3.6)
\[ \leq \sum_{j=l}^{m-1} 4^j \theta \|x\|^r \]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.6) that the sequence \( \{4^n f\left(\frac{x}{2^n}\right)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{4^n f\left(\frac{x}{2^n}\right)\} \) converges. So one can define the mapping \( h : X \to Y \) by
\[ h(x) := \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right) \]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.6), we get (3.4).

It follows from (3.3) that
\[ \left\|4h\left(\frac{x + y}{2}\right) + h(x - y) - 2h(x) - 2h(y)\right\| = \lim_{n \to \infty} 4^n \left\|4f\left(\frac{x + y}{2^n}\right) + f\left(\frac{x - y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\| \]
\[ \leq \lim_{n \to \infty} 4^n \left\|\rho\left(f\left(\frac{x + y}{2^n}\right) + f\left(\frac{x - y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right\| \]
\[ + \lim_{n \to \infty} \frac{4^n \theta}{2^{nr}} (\|x\|^r + \|y\|^r) \]
\[ = \|\rho(h(x + y) + h(x - y) - 2h(x) - 2h(y))\| \]
for all \( x, y \in X \). So
\[ \left\|4h\left(\frac{x + y}{2}\right) + h(x - y) - 2h(x) - 2h(y)\right\| \leq \|\rho(h(x + y) + h(x - y) - 2h(x) - 2h(y))\| \]
for all \( x, y \in X \). By Lemma 3.1, the mapping \( h : X \to Y \) is quadratic.

Now, let \( T : X \to Y \) be another quadratic mapping satisfying (3.4). Then we have
\[ \|h(x) - T(x)\| = 4^n \|h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\| \]
\[ \leq 4^n \left(\|h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\| + \|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\|\right) \]
\[ \leq \frac{2 \cdot 4^n \cdot 2^r}{(2^r - 4)2^{nr}} \theta \|x\|^r, \]
which tends to zero as \( n \to \infty \) for all \( x \in X \). So we can conclude that \( h(x) = T(x) \) for all \( x \in X \). This proves the uniqueness of \( h \). Thus the mapping \( h : X \to Y \) is a unique quadratic mapping satisfying (3.4).

**Theorem 3.3.** Let \( r < 2 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (3.3). Then there exists a unique quadratic mapping \( h : X \to Y \) such that

\[
\|f(x) - h(x)\| \leq \frac{2^r \theta}{4 - 2^r} \|x\|^r
\]

for all \( x \in X \).

**Proof.** It follows from (3.5) that

\[
\left\| f(x) - \frac{1}{4^r} f(2^r x) \right\| \leq \frac{2^r \theta}{4} \|x\|^r
\]

for all \( x \in X \). Hence

\[
\left\| \frac{1}{4^r} f(2^r x) - \frac{1}{4^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\|
\]

\[
\leq \frac{2^r \theta}{4} \sum_{j=l}^{m-1} \frac{2^r j}{4^j} \|x\|^r
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.8) that the sequence \( \left\{ \frac{1}{4^n} f(2^n x) \right\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \left\{ \frac{1}{4^n} f(2^n x) \right\} \) converges. So one can define the mapping \( h : X \to Y \) by

\[
h(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.8), we get (3.7).

The rest of the proof is similar to the proof of Theorem 3.2. \( \square \)

**Remark 3.4.** If \( \rho \) is a real number such that \(- \frac{1}{2} < \rho < \frac{1}{2}\) and \( Y \) is a real Banach space, then all the assertions in this section remain valid.

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REFERENCES


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