Nonparametric Test for
Multivariate Location Translation Alternatives

Jong-Hwa Na

Abstract

In this paper we propose a nonparametric one sided test for location parameters in $p$-variate ($p \geq 2$) location translation model. The exact null distributions of test statistics are calculated by permutation principle in the case of relatively small sample sizes and the asymptotic distributions are also considered. The powers of various tests are compared through computer simulation and the $p$-values with real data are also suggested through example.

Keywords: Nonparametric test, Multivariate location translation, Permutation principle

1. Introduction

Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m \in p$-variate two samples from $X$ and $Y$ populations with continuous distribution function $F$ and $G$, respectively. We assume that for all $x \in R^p$, there is a $\theta \in R^p$ such that

$$G(x) = F(x - \theta)$$

i.e., the $p$-variate location translation model. Sometimes we are interested in testing the following hypotheses:

$$H_0: \theta_1 \leq \theta_{10}, \theta_2 \leq \theta_{20}, \ldots, \theta_p \leq \theta_{p0} \text{ v.s. } H_1: \text{at least one of } \theta_i \text{'s is larger than } \theta_{i0}$$

This is the so-called one sided testing problem for multivariate data. As an example, suppose that a laboratory has developed a medicine which may have effects on two symptoms simultaneously. One can draw a decision that this medicine is acceptable if it become effective for any one of two symptoms or for both. In this problem, the alternative under consideration can be formulated as

$$H_1: \text{at least one symptom may be cured.}$$

In spite of those applicability of one sided test procedure, the developments have not been so

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fruitful. Bhattacharyya and Johnson (1970) considered the one-sided alternatives for the bivariate case based on the concept of two-dimensional layer ranks which were introduced by Barndorff-Nielsen and Sobel (1966). Boyett and Shuster (1977) proposed a nonparametric test procedure which is a maximal $t$-statistics and applied the permutation principle to obtain the null distribution function. However they did not provide the normal approximation for the large sample case. Wei and Knuiman (1987) considered the one-sided alternatives for censored data by specifying the alternatives based on the so-called stochastic ordering of the distribution functions. The test statistic was constructed by defining signum function for the pairs of observation vectors. Therefore the test statistic can be considered as an extension of Gehan test. However even for the small sample case, the exact null permutation distribution of the test statistic cannot be obtained. Therefore the derivation of the large sample approximation to the normal distribution becomes obvious. Up to now, the main obstruction has been the nonexistence of the table for the $p$-variate normal distribution functions. For the bivariate case, Owen (1962) published a book which contains the bivariate distribution functions for varying the values of correlation coefficient. However the tables are not sufficient since they can not contain all the values of the correlation coefficients. Therefore in this paper we propose a test procedure and consider the large sample approximation by obtaining the tail probability of the multivariate normal distribution.

2. Test Statistic and Small Sample Test

Let $T_i$ be a univariate nonparametric test statistic for the $i$-th component for testing $H_0: \theta_i = \theta_0$ for the two-sample problem. Since we are interested in dealing with the locally most powerful test procedures, $T_i$'s are not required to be the same type. For this problem, we use the maximum value among the $p$ univariate test statistics. Therefore we will consider the standardized form for each component. Let $\mu_i(\theta_0) = E_{H_0}(T_i)$ and $\sigma_i^2(\theta_0) = V_{H_0}(T_i)$ be the mean and variance of $T_i$ under $H_0$, respectively. Then we propose a test statistic for testing $H_0$ against $H_1$ in the following way:

$$Q = \max \left\{ \frac{T_1 - E_{H_0}(T_1)}{V_{H_0}^{1/2}(T_1)}, \frac{T_2 - E_{H_0}(T_2)}{V_{H_0}^{1/2}(T_2)}, \ldots, \frac{T_p - E_{H_0}(T_p)}{V_{H_0}^{1/2}(T_p)} \right\}$$

Then the testing rule would be to reject $H_0$ for large values of $Q$. For reasonable sample sizes, we may obtain the null distribution for $Q$ based on the permutation principle. The procedure for obtaining the null distribution function for multivariate data based on the permutation principle is well summarized in Puri and Sen (1971). However, for large sample sizes, we have to consider the large sample approximation.
Example

The following data is a part of the Actual Ordnance Survey (Mardia, 1980).

\[ X = \begin{pmatrix} 257, & 529 \\ 279, & 149 \end{pmatrix} \quad Y = \begin{pmatrix} 292, & 259, & 508 \\ 90, & 665, & 433 \end{pmatrix} \]

Then the corresponding rank matrix of the combined sample is

\[ R_5 = \begin{pmatrix} 1, & 5, & 3, & 2, & 4 \\ 3, & 2, & 1, & 5, & 4 \end{pmatrix} \]

Case I (Wilcoxon rank sum tests for both \( T_1 \) and \( T_2 \))

We deal this problem with Wilcoxon rank sum test. Then for each \( i \), we have

\[ T_1 = 3 + 2 + 4 = 9 \quad T_2 = 1 + 5 + 4 = 10 \]

Since \( E_{H_0}(T_i) = n(m + n + 1)/2 \) and \( V_{H_0}(T_i) = mn(m + n + 1)/12 \), we have

\[ Q = \max \left\{ \frac{T_1 - E_{H_0}(T_1)}{\sqrt{V_{H_0}(T_1)}}, \frac{T_2 - E_{H_0}(T_2)}{\sqrt{V_{H_0}(T_2)}} \right\} \max \{ 0, 1/\sqrt{3} \} = 1/\sqrt{3} \]

In order to perform the test procedure, we need the exact null distribution of \( Q \). This can be obtained from the null permutation distribution of \( (T_1, T_2) \). Then by applying the permutation principle (cf. Puri and Sen, 1971), we obtain that under \( H_0 \),

\[ P(T_1 = 6, T_2 = 9) = 1/10 \]
\[ P(T_1 = 8, T_2 = 8) = 1/10 \]
\[ P(T_1 = 9, T_2 = 6) = 1/10 \]
\[ P(T_1 = 10, T_2 = 8) = 1/10 \]
\[ P(T_1 = 11, T_2 = 11) = 1/10 \]

Then some straightforward calculations show that under \( H_0 \),

\[ P(Q = -1/\sqrt{3}) = 1/10 \]
\[ P(Q = 0) = 2/10 \]
\[ P(Q = 1/\sqrt{3}) = 4/10 \]
\[ P(Q = 2/\sqrt{3}) = 2/10 \]

Since the testing rule is to reject \( H_0 \) for large values of \( Q \), the \( p \)-value would be 7/10.

Case II (Median test for \( T_1 \) and Wilcoxon rank sum test for \( T_2 \))

For the same example, we consider median test for the first component and Wilcoxon rank sum test for the second component. The median test statistic is obtained in the following way: Let \( M = \lceil (m + n)/2 \rceil \) where \( \lceil \cdot \rceil \) is the greatest integer. Then the median test statistic is the number of observations of \( Y \) sample whose values are greater than or equal to \( M \). Then straightforward calculations give the following joint null permutation distribution of \( (T_1, T_2) \)
\[ P(T_1 = 1, T_2 = 9) = \frac{1}{10} P(T_1 = 1, T_2 = 10) = \frac{1}{10} \]
\[ P(T_1 = 1, T_2 = 12) = \frac{1}{10} P(T_1 = 2, T_2 = 6) = \frac{1}{10} \]
\[ P(T_1 = 2, T_2 = 8) = \frac{2}{10} P(T_1 = 2, T_2 = 9) = \frac{1}{10} \]
\[ P(T_1 = 2, T_2 = 10) = \frac{1}{10} P(T_1 = 2, T_2 = 11) = \frac{1}{10} \]
\[ P(T_1 = 3, T_2 = 7) = \frac{1}{10} \]

Thus \( E_{H_0}(T_i) = 1.8 \) and \( V_{H_0}(T_i) = 0.36 \)

\[ Q = \max \left\{ \frac{T_1 - E_{H_0}(T_1)}{\sqrt{V_{H_0}(T_1)}}, \frac{T_2 - E_{H_0}(T_2)}{\sqrt{V_{H_0}(T_2)}} \right\} = \max \left\{ \frac{1}{3}, \frac{1}{\sqrt{3}} \right\} = \frac{1}{\sqrt{3}} \]

Since, under \( H_0 \), the exact distribution of \( Q \) is

\[ P(Q=0) = \frac{1}{10} \]
\[ P(Q=1/3) = \frac{4}{10} \]
\[ P(Q=\sqrt{3}/3) = \frac{2}{10} \]

we see that the p-value for rejecting \( H_0 \) is \( 5/10 \).

3. Large Sample Test and Asymptotic Properties

3.1 Large Sample Test

For this section, we assume that

\[ \lim_{N \to \infty} n/N = \lambda \]

\[ 0 < \lambda < 1 \]

\( (N = m + n) \)

and \( T_i (i = 1, \ldots, p) \)s linear rank statistics. Puri and Sen (1971) showed with above condition that under \( H_0 \), the joint distribution of

\[ Q = \left\{ \frac{T_1 - E_{H_0}(T_1)}{V_{H_0}(T_1)}, \frac{T_2 - E_{H_0}(T_2)}{V_{H_0}(T_2)}, \ldots, \frac{T_p - E_{H_0}(T_p)}{V_{H_0}(T_p)} \right\} \]

converges in distribution to a \( p \)-variate normal distribution with 0 mean vector and covariance matrix \( \Sigma \), where

\[ \Sigma = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{pmatrix} \]
with \( \rho_{ij} = \rho_{H_n}(T_i, T_j) \) the correlation coefficient between \( i \)-th and \( j \)-th components for each \( i \neq j \). We note that for the asymptotic normality, we may use any consistent estimate \( \hat{\rho}_{ij} \) instead of \( \rho_{ij} \).

Since for any real number \( q \),

\[
P(Q \leq q) = P\left\{ \frac{T_1 - E_{H_n}(T_1)}{V_{H_n}(T_1)} \leq q, \ldots, \frac{T_p - E_{H_n}(T_p)}{V_{H_n}(T_p)} \leq q \right\}
\]

the limiting probability of \( P(Q \leq q) \) is the tail probability of the \( p \)-variate normal distribution with 0 mean vector and covariance matrix \( \Sigma \) with the same values of all coordinates. Therefore in order to calculate the limiting probability of \( P(Q \leq q) \), we need the values of \( \rho_{H_n}(T_1, T_j) \)'s. For this, we only consider the bivariate case since the extensions to the multivariate case become straightforward by considering all the possible pairs among the coordinates. Also we note that for the permutation correlation coefficient, \( \rho_{H_n}(T_1, T_2) \) it is enough to consider the permutation covariance, \( \text{Cov}_{H_n}(T_1, T_2) \) Let

\[
\begin{pmatrix}
1 \\
\tau(1) \\
2 \\
\tau(2) \\
3 \\
\tau(3) \\
\vdots \\
N \\
\tau(N)
\end{pmatrix}
\]

be a rank matrix which is obtained from the rank matrix \( R_N \) by permuting its columns. Chatterjee and Sen (1964) provided the formula for \( \text{Cov}_{H_n}(T_1, T_2) \) between Wilcoxon rank sum statistics as follows:

\[
\text{Cov}_{H_n}(T_1, T_2) = \frac{mn}{N(N-1)} \sum_{i=1}^{N} \left( i - \frac{N+1}{2} \right) \left( \tau(i) - \frac{N+1}{2} \right)
\]

For example, when the rank vector is \( \begin{pmatrix} 4 \\ 7 \end{pmatrix} \), \( i = 4 \) and \( \tau(i) = 7 \) in the above formula.

Especially, we note that in case of the independence among all the components,

\[
P(Q \leq q) = \prod_{i=1}^{p} P\left\{ \frac{T_i - E_{H_n}(T_i)}{V_{H_n}^{1/2}(T_i)} \leq q \right\} \approx \Phi^p(q)
\]

where \( \Phi(\cdot) \) stands for the cumulative standard normal distribution function. In this case, the determination of \( q \) becomes easy.

Example (continued)

Case I

We note that for small sample case, we do not need any covariance between \( T_1 \) and \( T_2 \). However for large sample case, we showed that \( Q \) converges in distribution to a bivariate
normal distribution with mean $(0, 0)$ and variance \( \Sigma = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{21} & 1 \end{pmatrix} \) with 
\[ \rho_{12} = \frac{\text{COV}_{H_0}(T_1, T_2)}{\sqrt{V_{H_0}(T_1)} \sqrt{V_{H_0}(T_2)}} \]
Therefore we need to calculate \( \rho_{12} \) with using the null distribution of \( (T_1, T_2) \) under \( H_0 \).
For this we note that 
\[ \frac{E_{H_0}(T_1 T_2)}{\sqrt{V_{H_0}(T_1)} \sqrt{V_{H_0}(T_2)}} = \frac{(54 + 84 + 64 + 80 + 54 + 90 + 80 + 90 + 121 + 84)}{10} = 80.1 \]
Thus we obtain that 
\[ \frac{\text{COV}_{H_0}(T_1, T_2)}{\sqrt{V_{H_0}(T_1)} \sqrt{V_{H_0}(T_2)}} = \frac{E_{H_0}(T_1 T_2) - E_{H_0}(T_1)E_{H_0}(T_2)}{80.1 - 81} = -0.9 \]
Then we obtain that 
\[ \rho_{12} = \frac{\text{COV}_{H_0}(T_1, T_2)}{\sqrt{V_{H_0}(T_1)} \sqrt{V_{H_0}(T_2)}} = -\frac{0.9}{3} = -0.3 \]
(We note that the above results for \( \text{COV}_{H_0}(T_1, T_2) \) and \( \rho_{12} \) can be directly calculated from Chatterjee and Sen's formula described in this section. That is, for the given data, we obtain 
\[ \text{COV}_{H_0}(T_1, T_2) = \frac{6}{5 \times 4} \sum_{i=1}^{5} (i-3)(i(i)-3) = -9/10 \quad \text{and} \quad \rho_{12} = -0.3 \]
These coincide the results calculated by exact null distribution of \( (T_1, T_2) \) Hereafter, including Section 4, we will use the Chatterjee and Sen's formula to obtain the exact null covariance of \( (T_1, T_2) \) because the permutation principle needs very tedious calculations for moderate or large sample size.)
Thus the distribution of \( \left( \frac{(T_1 - E_{H_0}(T_1))}{V_{H_0}^{1/2}(T_2)}, \frac{(T_2 - E_{H_0}(T_2))}{V_{H_0}^{1/2}(T_2)} \right) \) converges in distribution to a bivariate normal distribution with \( (0, 0) \) mean vector and \( \Sigma = \begin{pmatrix} 1 & -0.3 \\ -0.3 & 1 \end{pmatrix} \) Then the \( p \)-value for \( Q \) can be calculated by \( P(Q \geq 1/\sqrt{3}) = 0.5169 \).

**Case II**
By applying similar method to Case I, we can obtain the followings. That is, 
\[ Q = \max \{ 1/3, 1/\sqrt{3} \} = 1/\sqrt{3}, \quad E_{H_0}(T_1 T_2) = 15.6 \quad \text{and} \quad \text{COV}_{H_0}(T_1, T_2) = -0.6 \]. Therefore 
\[ \rho_{12} = -0.6/\sqrt{(0.36\sqrt{3})} = -0.6/(0.6\sqrt{3}) = -1/\sqrt{3} \] 
Thus the distribution of \( \left( \frac{(T_1 - E_{H_0}(T_1))}{V_{H_0}^{1/2}(T_2)}, \frac{(T_2 - E_{H_0}(T_2))}{V_{H_0}^{1/2}(T_2)} \right) \) converges in distribution to a bivariate normal distribution with \( (0, 0) \) mean vector and \( \Sigma = \begin{pmatrix} 1 & -1/\sqrt{3} \\ -1/\sqrt{3} & 1 \end{pmatrix} \) So the \( p \)-value for \( Q \) can be calculated by \( P(Q \geq 1/\sqrt{3}) = 0.5439 \).
The computations of the tail probability for the bivariate normal distribution may be carried out through computer program such as the pmvnorm function which is provided by S-Plus. For \(d\)-variate case with \(d \geq 3\), we may use the \(M_X\) program (Neale, Xie, Hadady and Boker, 1998) to obtain the \(p\)-values. By using this program, we can computer the multiple integrals of the multivariate normal, up to dimension 10. The program and documentation can be downloaded from the website http://www.vipbg.vcu.edu/mxgui.

3.2 Asymptotic Properties

In order to deal with the asymptotic properties for our proposed tests, we note that the test statistic \(Q_N\) consists of \(d\) number of univariate nonparametric test statistics. Therefore some asymptotic properties of \(Q_N\) would be inherited from those of univariate nonparametric tests. We may take the consistency of tests as an example. For each \(i\), let \((T_N)\) be a sequence of \(a\)-level tests of \(H_0: \theta_i \leq \theta_{0i}\) which is consistent against the alternatives \(H_1: \theta_i > \theta_{0i}\). Then the consistency of the tests based on the sequence \((Q_N)\) follows immediately. Also the optimality property such as the locally most powerful test based on \(Q_N\) will follow naturally if a test for each component based on \(T_N\) is locally most powerful.

For the limiting power of our test, we consider the following Pitman translation alternatives: For each \(N\) and for each \(i\), \(i = 1, \ldots, d\) set

\[
H_{1iN}: \theta_{iN} = c_i / \sqrt{N},
\]

where \(c_i\) is a fixed positive real number. We assume that all the univariate test statistics which we consider in this paper satisfy the assumptions and conditions in the section 3.8.3 (pp. 120-121) in Puri and Sen (1971). Then from some straightforward calculations, we have the limiting power of the test as follows:

\[
\lim_{N \to \infty} P_{\theta_0}(Q_N \geq C_N(a)) = 1 - \lim_{N \to \infty} \Phi_\Sigma \left( C_N(a) - \frac{\mu_{1N}(\theta_1)}{\sigma_{1N}(\theta_{01})}, \ldots, C_N(a) - \frac{\mu_{1N}(\theta_d)}{\sigma_{1N}(\theta_{0d})} \right)
\]

\[
= 1 - \Phi_\Sigma(\{a\} - c_1 m_1, \ldots, C(a) - c_d m_d)
\]

where \(\Phi_\Sigma\) is the \(p\)-variate normal cumulative distribution function with 0 mean vector and covariance matrix \(\Sigma\). \(C(a)\) is such that \(\lim_{N \to \infty} P(Q_N \geq C(a)) = a\) and

\[
m_i = \lim_{N \to \infty} \mu_{iN}(\theta_i) / (\sqrt{N} \sigma_{iN}(\theta_i)) \quad \text{and} \quad \mu_{iN}(\theta_i), \sigma_{iN}(\theta_i) \text{ are the expectation and variance of} \ T_{iN} \text{ under} \ H_0: \theta_i = \theta_{0i} \text{ for all} \ i.
\]

Bhattacharyya and Johnson obtained the efficacy for their test statistic \(L_N\) under the
Pitman translation alternatives. We note that the limiting distribution of $L_N$ is univariate normal. However, as we already have seen, the limiting distribution of $Q_N$ is related with the multivariate normal distribution. Therefore the comparisons of the performances between two tests through ARE are not clear without any theoretical results concerned with asymptotic null distribution of $Q_N$. Therefore we compare the power through the computer simulations in the next section.

4. Simulation Results

4.1 Bhattacharyya and Johnson's Test

Let us briefly introduce the nonparametric test proposed by Bhattacharyya and Johnson (1970) for the one-sided bivariate location translation alternatives $K: \theta_1 \geq 0, \theta_2 \geq 0$ ( $\theta \neq 0$ ). Let $\{Z_1, Z_2, \ldots, Z_m, Z_{m+1}, \ldots, Z_{m+n}\}$ be the combined samples of m's X samples and n's Y samples. Let $N = m + n$ and $Z_i = (Z_{R_i}, Z_{Q_i})$. Define

$$L(i, j) = \begin{cases} 1, & Z_{R_i} \geq Z_{R_j}, Z_{Q_i} \geq Z_{Q_j} \\ 0, & \text{otherwise} \end{cases}$$

and

$$L_i = \sum_{j=1}^{N} L(i, j), \ 1 \leq i \leq N, \ 1 \leq j \leq N, \ L = (L_1, \ldots, L_N).$$

Then $L_i$ is called the 3rd quadrant layer rank of $Z_i$ in the combined sample $\{Z_1, \ldots, Z_N\}$. Bhattacharyya and Johnson (1970) suggested a test statistic $L_N$ by

$$L_N = N^{-3}[w \sum_{i=m+1}^{N} L_i - (1-w) \sum_{i=1}^{m} L_i], \quad w = m/N.$$

They also showed that the null distribution of $L_N$ is given by

$$L_N^{*} = L_N \cdot \left[ \frac{mn}{N(N-1)} \sum_{i=1}^{N} (l_i - \bar{l})^2 \right]^{-1/2} \rightarrow N(0, 1),$$

where $l_i$ is the observed value of the 3rd quadrant layer ranks and $\bar{l} = \sum l_i / N$.

4.2 Power Comparisons

We compare the powers of $Q_N$ with those of $L_N$, which is based on the layer ranks through simulation studies involving two different bivariate distributions, the bivariate normal distribution (Table 1) and exponential distribution (Tables 2 and 3). In case of $Q_N$, we only consider the Wilcoxon rank sum statistics for both coordinates. For the normal
distribution, we consider the cases of three different correlation coefficients, \( \rho = 0, 0.2 \) and 0.5. For the exponential distribution, we consider following two different cases: One is that two components are independent (Tables 2) and the other, so-called Marshall–Olkin type bivariate exponential distribution (cf. Barlow and Proschan 1975) (Tables 3). Also we consider two cases for each distribution that the location translation vector, \( \theta = (\theta_1, \theta_2) \) varies with the same values of \( \theta_1 \) and \( \theta_2 \) and \( \theta_1 \) varies while \( \theta_2 \) is fixed as 0.

For each distribution, simulations have been carried out under the nominal significance level \( \alpha = 0.05 \). The results are based on 1000 simulations with sample sizes \( m = 15 \) and \( n = 20 \) for each distribution. For the language, we used S–Plus 4. Especially, for obtaining the quantile points required when we determine the critical values, we used \textit{pbinorm} function of S–Plus.

From Tables 1, 2 and 3, we see that the two tests reveal little difference in powers in case of the same values of \( \theta_1 \) and \( \theta_2 \). However, we note that for the case that one is fixed while the other varies, our procedure is much superior to the that based on \( L_N \) for all distributions. Also we note that in case of both types of the exponential distributions, the powers based on \( Q_N \) and \( L_N \) are much better than those of normal distribution since the both procedures are nonparametric. Therefore our proposed test can be a good alternative to the test by Bhattacharyya and Johnson.

### Table 1. Bivariate normal distribution

<table>
<thead>
<tr>
<th>Test Statistics</th>
<th>( \rho )</th>
<th>((\theta_1, \theta_2)) Location Translation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(0,0)</td>
</tr>
<tr>
<td>(Q_N) 0</td>
<td>0.047</td>
<td>0.194</td>
</tr>
<tr>
<td>(L_N) 0</td>
<td>0.055</td>
<td>0.116</td>
</tr>
<tr>
<td>(Q_N) 0.2</td>
<td>0.048</td>
<td>0.182</td>
</tr>
<tr>
<td>(L_N) 0.2</td>
<td>0.055</td>
<td>0.107</td>
</tr>
<tr>
<td>(Q_N) 0.5</td>
<td>0.047</td>
<td>0.167</td>
</tr>
<tr>
<td>(L_N) 0.5</td>
<td>0.063</td>
<td>0.106</td>
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</tbody>
</table>

### Table 2. Independent exponential distribution

<table>
<thead>
<tr>
<th>Test Statistic</th>
<th>((\theta_1, \theta_2)) Location Translation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0,0)</td>
</tr>
<tr>
<td>(Q_N) 0.048</td>
<td>0.274</td>
</tr>
<tr>
<td>(L_N) 0.048</td>
<td>0.219</td>
</tr>
</tbody>
</table>
Table 3. Marshall-Olkin’s bivariate exponential distribution

<table>
<thead>
<tr>
<th></th>
<th>((0,0))</th>
<th>((0.3,0))</th>
<th>((0.3,0.3))</th>
<th>((0.6,0))</th>
<th>((0.6,0.6))</th>
<th>((0.9,0))</th>
<th>((0.9,0.9))</th>
<th>((1,2,0))</th>
<th>((1.2,1,2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_N)</td>
<td>0.051</td>
<td>0.662</td>
<td>0.843</td>
<td>0.976</td>
<td>0.996</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(L_N)</td>
<td>0.051</td>
<td>0.384</td>
<td>0.873</td>
<td>0.746</td>
<td>0.997</td>
<td>0.898</td>
<td>1</td>
<td>0.962</td>
<td>1</td>
</tr>
</tbody>
</table>

5. Concluding Remarks

In this section, first of all, we consider some variants of the one-sided test and briefly discuss to modify the test procedures for the variants. For the simplicity of arguments, we confine our discussion to the bivariate case. The extensions to the multivariate cases are only notational matters and straightforward. Now we consider the following hypotheses:

(i) \(H_0: \theta_1 \geq \theta_10, \theta_2 \geq \theta_20\) v.s. \(H_1: \) at least one of \(\theta_1\)'s is strictly smaller than \(\theta_10\)

(ii) \(H_0: \theta_1 \leq \theta_10, \theta_2 \geq \theta_20\) v.s. \(H_1: \theta_1 > \theta_10 \) or \(\theta_2 < \theta_20\) or both.

For case (i), by switching the roles of two samples, still we may use the maximum between two univariate test statistics. Or if the roles of two samples are maintained, then we may use the minimum. In any case, we note that we may draw the same conclusions. For case (ii), we may use the reversed rank system such as we assign 1 for the largest observation and \(N\), the smallest one when we use the Wilcoxon rank sum statistic. Or we may count the observations which are less than or equal to a median from the combined sample in case of the median test for the second part of hypotheses. Then the rest of the test procedures remains unchanged.

References


