Generalized Multi-Phase Multivariate Ratio Estimators for Partial Information Case Using Multi-Auxiliary Variables

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Abstract

In this paper we propose generalized multi-phase multivariate ratio estimators in the presence of multi-auxiliary variables for estimating population mean vector of variables of interest. Some special cases have been deduced from the suggested estimator in the form of remarks. The expressions for mean square errors of proposed estimators have also been derived. The suggested estimators are theoretically compared and an empirical study has also been conducted.

Keywords: Multi-phase sampling, multivariate ratio estimator, multi-auxiliary variables.

1. Introduction

The estimation of the population mean is constant issue in sampling theory and several efforts have been made to improve the precision of the estimators in the presence of multi-auxiliary variables. A variety of estimators have been proposed following different ideas of ratio, regression and product estimators.

Olkin (1958) was the first author to deal with the problem of estimating the mean of a survey variable when auxiliary variables are made available. John (1969) proposed two multivariate generalizations of ratio and product estimators which actually reduce to the Olkin’s (1958) and Singh’s (1967a) estimators. Srivastava (1971) proposed a general ratio-type estimator which generates a large class of estimators including most of the estimators up to that time proposed. Sen (1972) developed a multivariate ratio estimator under two-phase sampling using multi-auxiliary variables. Singh and Namjoshi (1988) discussed a class of multivariate regression estimators of population mean of study variable in two-phase sampling.


In multipurpose surveys, the problem is to estimate population means of several variables simultaneously (Swain, 2000). Tripathi and Khattree (1989) estimated means of several variables of interest, using multi-auxiliary variables, under simple random sampling. Further Tripathi (1989) extended the results to the case of two phase sampling.

We suggest general classes of ratio estimators for estimating the population mean of study variable for two-phase and multi-phase sampling using multi-auxiliary variables when information on all...
Similarly, we define $x_{i(h)}$ and $x_{i(k)}$ denote the $i^h$ auxiliary variables from $h^{th}$ and $k^{th}$ phase samples respectively and $y_i$ denote the variable of interest from the $k^{th}$ phase. Let, $X_i$, $C_x$ and $\rho_{x_i}$ denote the population mean, coefficient of variation of $i^{th}$ auxiliary variables respectively and the population correlation coefficient of $Y$ and $X_i$. Further let $\theta_h = 1/n_h - 1/N$, $\theta_k = 1/n_k - 1/N$. Also, $Y_{i(h)} = Y + e_{y_{i(h)}}$, $X_{i(h)} = X_i + e_{x_{i(h)}}$ and $X_{i(k)} = X_i + e_{x_{i(k)}}$ ($i = 1, 2, \ldots, k$), where $e_{y_{i(h)}}$, $e_{x_{i(h)}}$, and $e_{x_{i(k)}}$ are sampling errors. We assume that $E_h(e_{y_{i(h)}}) = E_h(e_{x_{i(h)}}) = E_k(e_{x_{i(k)}}) = 0$ where $E_h$ and $E_k$ denote the expectations of errors of $h^{th}$ and $k^{th}$ phase sampling respectively. Then for simple random sampling without replacement for both first and second phases, we write by using phase wise operation of expectations as:

\[
E_h\left(\bar{y}_{i(h)} - \bar{y}_{i(k)}\right) = \frac{1}{N} \left( n_k - n_h \right) \sigma_{y_{i(h)}}
\]

Similarly,

\[
E_h\left(\bar{x}_{i(h)} - \bar{x}_{i(k)}\right) = \left( \theta_h - \theta_k \right) \sigma_{x_{i(h)}}
\]

\[
E_k\left(\bar{y}_{i(k)} - \bar{y}_{i(h)}\right) = \left( \theta_h - \theta_k \right) \sigma_{y_{i(k)}}
\]

\[
E_k\left(\bar{x}_{i(k)} - \bar{x}_{i(h)}\right) = \left( \theta_h - \theta_k \right) \sigma_{x_{i(k)}}
\]

and

\[
E_h\left(\bar{y}_{i(h)} - \bar{y}_{i(k)}\right) = \left( \theta_h - \theta_k \right) \sigma_{y_{i(h)}}
\]

\[
E_k\left(\bar{x}_{i(k)} - \bar{x}_{i(h)}\right) = \left( \theta_h - \theta_k \right) \sigma_{x_{i(k)}}
\]

2. Multi-Phase Sampling Using Multi-Auxiliary Variables

Consider a population of $N$ units. Let $Y$ be the variable of interest and $X_1, X_2, \ldots, X_q$ are $q$ auxiliary variables. For multi-phase sampling design let $n_h$ and $n_k$ ($n_h < n_k$) be sample sizes for $h^{th}$ and $k^{th}$ phase respectively. $x_{i(h)}$ and $x_{i(k)}$ denote the $i^h$ auxiliary variables from $h^{th}$ and $k^{th}$ phase samples respectively and $y_i$ denote the variable of interest from the $k^{th}$ phase. Let, $\bar{X}_i$, $C_x$ and $\rho_{x_i}$ denote the population mean, coefficient of variation of $i^{th}$ auxiliary variables respectively and the population correlation coefficient of $Y$ and $X_i$. Further let $\theta_h = 1/n_h - 1/N$, $\theta_k = 1/n_k - 1/N$. Also, $Y_{i(h)} = Y + e_{y_{i(h)}}$, $X_{i(h)} = X_i + e_{x_{i(h)}}$ and $X_{i(k)} = X_i + e_{x_{i(k)}}$ ($i = 1, 2, \ldots, k$), where $e_{y_{i(h)}}$, $e_{x_{i(h)}}$, and $e_{x_{i(k)}}$ are sampling errors. We assume that $E_h(e_{y_{i(h)}}) = E_h(e_{x_{i(h)}}) = E_k(e_{x_{i(k)}}) = 0$ where $E_h$ and $E_k$ denote the expectations of errors of $h^{th}$ and $k^{th}$ phase sampling respectively. Then for simple random sampling without replacement for both first and second phases, we write by using phase wise operation of expectations as:

\[
E_h\left(\bar{y}_{i(h)} - \bar{y}_{i(k)}\right) = \frac{1}{N} \left( n_k - n_h \right) \sigma_{y_{i(h)}}
\]

Similarly,

\[
E_h\left(\bar{x}_{i(h)} - \bar{x}_{i(k)}\right) = \left( \theta_h - \theta_k \right) \sigma_{x_{i(h)}}
\]

\[
E_k\left(\bar{y}_{i(k)} - \bar{y}_{i(h)}\right) = \left( \theta_h - \theta_k \right) \sigma_{y_{i(k)}}
\]

\[
E_k\left(\bar{x}_{i(k)} - \bar{x}_{i(h)}\right) = \left( \theta_h - \theta_k \right) \sigma_{x_{i(k)}}
\]

and

\[
E_h\left(\bar{y}_{i(h)} - \bar{y}_{i(k)}\right) = \left( \theta_h - \theta_k \right) \sigma_{y_{i(h)}}
\]

\[
E_k\left(\bar{x}_{i(k)} - \bar{x}_{i(h)}\right) = \left( \theta_h - \theta_k \right) \sigma_{x_{i(k)}}
\]
The following notations in the following paragraph will be used in deriving the mean square errors of proposed estimators.

\[ |R|_{y|x} \] denotes the determinant of population correlation matrix of variables \( y_1, y_2, \ldots, y_q \) and \( x_q \). \( |R_{i|y|x}|_{v|x} \) denotes the determinant of \( i^{th} \) minor of \( |R|_{y|x} \) corresponding to the \( i^{th} \) element of \( \rho_{y|x} \). \( \rho^2_{x,y} \) denotes the multiple coefficient of determination of \( y \) on \( x_1, x_2, \ldots, x_r \) and \( x_r \). \( \rho^2_{x,y} \) denotes the multiple coefficient of determination of \( y \) on \( x_1, x_2, \ldots, x_q \) and \( x_q \). \( |R|_y \) denotes the determinant of population correlation matrix of variables \( x_{r+1}, x_{r+2}, \ldots, x_{r+s-1} \) and \( x_r \). \( |R|_x \) denotes the determinant of population correlation matrix of variables \( x_1, x_2, \ldots, x_{r-1} \) and \( x_r \). \( |R|_y \) denotes the determinant of the correlation matrix of \( y_1, y_2, \ldots, y_q \). \( |R|_{y|x} \) denotes the determinant of the correlation matrix of \( y_1, y_2, \ldots, y_r \) and \( \rho_{y|x} \) denotes the determinant of the minor corresponding to \( \rho_{y|x} \) of the correlation matrix of \( y_1, y_2, \ldots, y_q \) and \( x_r \), for \( (i \neq j) \). \( |R|_{y|x} \) denotes the determinant of the minor corresponding to \( \rho_{y|x} \) of the correlation matrix of \( y_1, y_2, \ldots, y_q \) and \( x_r \), for \( (i \neq j) \).

2.1. Result: 1

The following result will help in deriving the mean square errors of suggested estimators

\[
\frac{|R|_{y|x}}{|R|_x} = (1 - \rho^2_{x,y}), \quad \text{(Arora and Lal, 1989)}.
\]

3. Generalized Multi-Phase Multivariate Ratio Estimator for Partial Information Case

Let we have \( q \) auxiliary variable \( X_1, X_2, \ldots, X_q \) and population means for first \( r \) auxiliary variables are not known and for the rest \( q - r = s \) auxiliary variables are known. For estimating the mean vector of variables of interest \( Y_1, Y_2, \ldots, Y_p \), let \( \bar{y}_{(k)} \) denotes the sample mean of \( k^{th} \) study variable for \( k^{th} \) phase and \( \bar{x}_{(h)} \) and \( \bar{x}_{(k)} \) denotes the sample mean of \( j^{th} \) auxiliary variable for \( h^{th} \) and \( k^{th} \) phase respectively. The generalized multi-phase multivariate ratio estimator in the presence of, multi-auxiliary variables for partial information case can be suggested as:

\[
T_{hk(1xp)} = \left[ \bar{y}_{(k)} \prod_{i=1}^{q} \frac{\bar{x}_{(h)j}}{\bar{x}_{(k)j}} \prod_{j=r+1}^{s} \frac{\bar{x}_{(h)j}}{\bar{x}_{(k)j}} \frac{\bar{x}_{(h)j}}{\bar{x}_{(k)j}} \frac{\bar{x}_{(h)j}}{\bar{x}_{(k)j}} \gamma_{11} \right.
\]

\[
\left. \ldots \bar{y}_{(k)p} \prod_{i=1}^{q} \frac{\bar{x}_{(h)j}}{\bar{x}_{(k)j}} \prod_{j=r+1}^{s} \frac{\bar{x}_{(h)j}}{\bar{x}_{(k)j}} \frac{\bar{x}_{(h)j}}{\bar{x}_{(k)j}} \frac{\bar{x}_{(h)j}}{\bar{x}_{(k)j}} \right]
\]

or

\[
T_{hk(1xp)} = \left[ \bar{y}_1 + \bar{e}_{y10} \left( 1 + \sum_{i=1}^{s} \frac{\alpha_{i1}}{X_i} (\bar{x}_{y10} - \bar{x}_{y00}) \right) \right.
\]

\[
\left. \left( 1 - \sum_{j=r+1}^{s} \gamma_{1j} \bar{x}_{y0j} \right) \left( 1 - \sum_{j=r+1}^{s} \gamma_{1j} \bar{x}_{y0j} \right) \right]
\]
\[
(\tilde{Y}_p + \bar{e}_{y_{1p}}) \left(1 + \sum_{i=1}^{r} \frac{\alpha_{ip}}{X_i} (\bar{e}_{x_{1p}} - \bar{e}_{x_{ip}}) \right) \\
\cdots \left(1 - \sum_{i=r+1}^{r+s+q} \frac{\beta_{ip}}{X_i} \bar{e}_{x_{ip}} \right) \left(1 - \sum_{i=r+1}^{r+s+q} \frac{\gamma_{ip}}{X_i} \bar{e}_{x_{ip}} \right)
\]

or

\[
T_{hk}(1xp) = \left[ \begin{array}{c} \tilde{Y}_1 + \bar{e}_{y_{11}} \\ \tilde{Y}_2 + \bar{e}_{y_{12}} \\ \vdots \end{array} \right] \\
\cdots \left[ \begin{array}{c} \tilde{Y}_p + \bar{e}_{y_{1p}} \end{array} \right] \\
+ \left[ \begin{array}{c} \sum_{i=1}^{r} \frac{\alpha_{1i}}{X_i} (\bar{e}_{x_{11}} - \bar{e}_{x_{1i}}) \sum_{i=1}^{r} \frac{\alpha_{2i}}{X_i} (\bar{e}_{x_{12}} - \bar{e}_{x_{1i}}) \cdots \sum_{i=1}^{r} \frac{\alpha_{pi}}{X_i} (\bar{e}_{x_{1p}} - \bar{e}_{x_{1i}}) \end{array} \right] \\
- \left[ \begin{array}{c} \sum_{i=r+1}^{r+s+q} \frac{\beta_{1i}}{X_i} \bar{e}_{x_{11}} \sum_{i=r+1}^{r+s+q} \frac{\beta_{2i}}{X_i} \bar{e}_{x_{12}} \cdots \sum_{i=r+1}^{r+s+q} \frac{\beta_{pi}}{X_i} \bar{e}_{x_{1p}} \end{array} \right]
\]

or

\[
T_{hk}(1xp) = \left[ \begin{array}{c} \tilde{Y}_1 + \bar{e}_{y_{11}} \\ \tilde{Y}_2 + \bar{e}_{y_{12}} \\ \vdots \end{array} \right] \\
\cdots \left[ \begin{array}{c} \tilde{Y}_p + \bar{e}_{y_{1p}} \end{array} \right] \\
+ \left[ \begin{array}{c} \sum_{i=1}^{r} \frac{\alpha_{1i}}{X_i} (\bar{e}_{x_{11}} - \bar{e}_{x_{1i}}) \sum_{i=1}^{r} \frac{\alpha_{2i}}{X_i} (\bar{e}_{x_{12}} - \bar{e}_{x_{1i}}) \cdots \sum_{i=1}^{r} \frac{\alpha_{pi}}{X_i} (\bar{e}_{x_{1p}} - \bar{e}_{x_{1i}}) \end{array} \right] \\
- \left[ \begin{array}{c} \sum_{i=r+1}^{r+s+q} \frac{\beta_{1i}}{X_i} \bar{e}_{x_{11}} \sum_{i=r+1}^{r+s+q} \frac{\beta_{2i}}{X_i} \bar{e}_{x_{12}} \cdots \sum_{i=r+1}^{r+s+q} \frac{\beta_{pi}}{X_i} \bar{e}_{x_{1p}} \end{array} \right]
\]

or

\[
T_{hk}(1xp) = \left[ \begin{array}{c} \tilde{Y}_1 + \bar{e}_{y_{11}} \\ \tilde{Y}_2 + \bar{e}_{y_{12}} \\ \vdots \end{array} \right] \\
\cdots \left[ \begin{array}{c} \tilde{Y}_p + \bar{e}_{y_{1p}} \end{array} \right] \\
+ \left[ \begin{array}{c} \sum_{i=1}^{r} \frac{\alpha_{1i}}{X_i} (\bar{e}_{x_{11}} - \bar{e}_{x_{1i}}) \sum_{i=1}^{r} \frac{\alpha_{2i}}{X_i} (\bar{e}_{x_{12}} - \bar{e}_{x_{1i}}) \cdots \sum_{i=1}^{r} \frac{\alpha_{pi}}{X_i} (\bar{e}_{x_{1p}} - \bar{e}_{x_{1i}}) \end{array} \right]_{1xp} \\
\left[ \begin{array}{c} \tilde{y}_j \\ \tilde{x}_{0i} \end{array} \right]_{(xp)} \\
- \left[ \begin{array}{c} \bar{e}_{x_{1i+1}} \bar{e}_{x_{1i+2}} \cdots \bar{e}_{x_{1i+s}} \end{array} \right]_{1xp} \left[ \begin{array}{c} \tilde{y}_j \\ \tilde{x}_{0i} \end{array} \right]_{(xp)}
\]

or

\[
T_{hk}(1xp) = \tilde{y}_{1xp} + \tilde{d}_{1xi10}A_{1xp} - \tilde{d}_{1xi10}B_{1xp} - \tilde{d}_{1xi10}C_{1xp}.
\]

Where

\[
\tilde{d}_{1xi} = \left[ \begin{array}{c} \tilde{x}_{0i1} - \bar{x}_{0i1} \\ \tilde{x}_{0i2} - \bar{x}_{0i2} \\ \vdots \\ \tilde{x}_{0is} - \bar{x}_{0is} \end{array} \right]_{1xp}
\]

\[
\tilde{d}_{1xi10} = \left[ \begin{array}{c} \tilde{x}_{0i1} - \bar{x}_{0i1} \\ \tilde{x}_{0i2} - \bar{x}_{0i2} \\ \vdots \\ \tilde{x}_{0is} - \bar{x}_{0is} \end{array} \right]_{1xp}
\]

(3.2)
\begin{align*}
\bar{d}_n &= \left[ (x_{0i} - \bar{X}_{r+1}) (x_{0i} - \bar{X}_{r+2}) \cdots (x_{0i} - \bar{X}_{r+s}) \right] \\
&= \left[ \bar{a}_{x_{0i}} \bar{a}_{x_{0i}+1} \cdots \bar{a}_{x_{0i+s}} \right]_{1 \times 4} \\
\bar{d}_n &= \left[ (x_{ki} - \bar{X}_{r+1}) (x_{ki} - \bar{X}_{r+2}) \cdots (x_{ki} - \bar{X}_{r+s}) \right] \\
&= \left[ \bar{a}_{x_{ki2}} \bar{a}_{x_{ki2}+1} \cdots \bar{a}_{x_{ki2+r}} \right]_{1 \times 4},
\end{align*}

\begin{align*}
A &= \frac{\bar{Y}}{X_i} \bar{a}_{ij} \bigg|_{(x^p)} , \quad \text{for } i = 1, 2, \ldots, r, \ j = 1, 2, \ldots, p \\
B &= \frac{\bar{Y}}{X_i} \bar{b}_{ij} \bigg|_{(x^p)} , \quad \text{for } i = r + 1, r + 2, \ldots, r + s, \ j = 1, 2, \ldots, p.
\end{align*}

and

\begin{equation}
C = \frac{\bar{Y}}{X_i} \bar{y}_{ij} \bigg|_{(x^p)} , \quad \text{for } i = r + 1, r + 2, \ldots, r + s, \ j = 1, 2, \ldots, p.
\end{equation}

Letting, \( \bar{y} = \bar{Y} + \bar{d}_n \), where and \( \bar{d}_n = [ \bar{a}_{x_{0i}} \bar{a}_{x_{0i}+1} \cdots \bar{a}_{x_{0i+s}} ] \).

We can write (3.2) as:

\begin{equation}
T_{hh(1xp)} = \bar{Y} + \bar{d}_i + \bar{d}_{xh}A - \bar{d}_{xh}B - \bar{d}_{xh}C.
\end{equation}

We use information related to auxiliary variables from first and second phase both then the mean square error of \( T_{hh(1xp)} \) can be written as:

\begin{equation}
\Sigma_{T_{hh(1xp)}} = E_1 E_2 \left( T_{hh} - \bar{Y} \right) \left( T_{hh} - \bar{Y} \right) = E_1 E_2 \left( \bar{d}_i + \bar{d}_{xh}A - \bar{d}_{xh}B - \bar{d}_{xh}C \right) \left( \bar{d}_i + \bar{d}_{xh}A - \bar{d}_{xh}B - \bar{d}_{xh}C \right),
\end{equation}

We can write

\begin{equation}
E_1 E_2 \left( d'_{i} d_1 \right) = \theta_i \Sigma_i = \theta_i [\sigma_{y_iy_i}]_{(x^p)} , \quad \text{for } i = j, \ \sigma_{y_iy_i} = \sigma^2_{y_i}
\end{equation}

\begin{equation}
E_1 E_2 \left( d'_{i} d_{e} \right) = \theta_e \Sigma_e = \theta_e [\sigma_{y_iy_i}]_{(y^e)}
\end{equation}

\begin{equation}
E_1 E_2 \left( d'_{e} d_1 \right) = \theta_1 \Sigma_e = \theta_1 [\sigma_{y_iy_i}]_{(x^e)}
\end{equation}

\begin{equation}
E_1 E_2 \left( d'_{e} d_{e} \right) = \theta_e \Sigma_e = \theta_e [\sigma_{y_iy_i}]_{(y^e)}, \quad \text{for } i = j, \ \sigma_{y_iy_i} = \sigma^2_{y_i}
\end{equation}

\begin{equation}
E_1 E_2 \left( d'_{i} d_{h} \right) = \theta_{h} \Sigma_{e} = \theta_{h} [\sigma_{y_iy_i}]_{(y^h)} \quad \text{for } i = j, \ \sigma_{y_iy_i} = \sigma^2_{y_i}
\end{equation}

\begin{equation}
E_1 E_2 \left( d'_{i} d_{e} \right) = \theta_{e} \Sigma_{e} = \theta_{e} [\sigma_{y_iy_i}]_{(y^e)}, \quad \text{for } i = j, \ \sigma_{y_iy_i} = \sigma^2_{y_i}
\end{equation}

\begin{equation}
E_1 E_2 \left( d'_{i} d_{h} \right) = \theta_{h} \Sigma_{e} = \theta_{h} [\sigma_{y_iy_i}]_{(y^h)}, \quad \text{for } i = j, \ \sigma_{y_iy_i} = \sigma^2_{y_i}.
\end{equation}
Using above substitutions in expression of variance covariance matrix given in (3.3), we write:

\[
\Sigma_{T_{(xp)p}} = \theta_k \Sigma_{\gamma_{yp}} - \theta_k \Sigma_{\nu_{yp}} A_{(xp)} - \theta_k \Sigma_{\nu_{yp}} B_{(xp)} \\
+ (\theta_k - \theta_h) \Sigma_{\nu_{yp}} C_{(xp)} - \theta_k B_{(xp)} \Sigma_{\nu_{yp}} B_{(xp)} \\
+ \theta_k B_{(xp)} \Sigma_{\nu_{yp}} C_{(xp)} - \theta_k C_{(xp)} \Sigma_{\nu_{yp}} B_{(xp)} \\
+ \theta_k C_{(xp)} \Sigma_{\nu_{yp}} C_{(xp)} - (\theta_k - \theta_h) C_{(xp)} \Sigma_{\nu_{yp}} A_{(xp)} \\
+ (\theta_k - \theta_h) A_{(xp)} \Sigma_{\nu_{yp}} A_{(xp)} - (\theta_k - \theta_h) A_{(xp)} \Sigma_{\nu_{yp}} A_{(xp)} \\
+ (\theta_k - \theta_h) A_{(xp)} \Sigma_{\nu_{yp}} A_{(xp)}.  
\]

Differentiating (3.4) with respect to the matrices on unknown constants \(A, B\) and \(C\) and equating to zero, solving these equations for optimum values of \(A, B\) and \(C\) we get:

\[
A_{(xp)} = W^{-1}_{\nu_{yp}} \left( \Sigma_{\nu_{yp}} - \Sigma_{\nu_{yp}}^{-1} \Sigma'_{\nu_{yp}} \right)  
\]

\[
B_{(xp)} = \left( \Sigma_{\nu_{yp}} - \Sigma_{\nu_{yp}}^{-1} \Sigma'_{\nu_{yp}} \right) - \left( \Sigma_{\nu_{yp}} - \Sigma_{\nu_{yp}}^{-1} \Sigma'_{\nu_{yp}} \right) 
\]

and

\[
C_{(xp)} = \left( \Sigma_{\nu_{yp}} - \Sigma_{\nu_{yp}}^{-1} \Sigma'_{\nu_{yp}} \right) - \left( \Sigma_{\nu_{yp}} - \Sigma_{\nu_{yp}}^{-1} \Sigma'_{\nu_{yp}} \right).  
\]

Using the above expressions of \(A, B\) and \(C\) in (3.4), we can write the variance covariance matrix of \(T_{(xp)p}\) as:

\[
\Sigma_{T_{(xp)p}} = \theta_k \left( \Sigma_{\gamma_{yp}} - \Sigma_{\nu_{yp}}^{-1} \Sigma'_{\nu_{yp}} \right) \\
- (\theta_k - \theta_h) \Sigma_{\nu_{yp}} A_{(xp)} - (\theta_k - \theta_h) \Sigma_{\nu_{yp}} A_{(xp)} \\
W^{-1}_{\nu_{yp}} \left( \Sigma'_{\nu_{yp}} - \Sigma_{\nu_{yp}}^{-1} \Sigma'_{\nu_{yp}} \right),  
\]

where \(W^{-1}_{\nu_{yp}} = \left( \Sigma_{\nu_{yp}} - \Sigma_{\nu_{yp}}^{-1} \Sigma'_{\nu_{yp}} \right)^{-1}\) provided \(\Sigma_{\nu_{yp}}^{-1}\), \(W^{-1}_{\nu_{yp}}\) and \(\Sigma_{\nu_{yp}}^{-1}\) exist.

The variance covariance matrix in the form of variance of \(y_j\), covariances and correlation coefficients of \(x_i\) and \(y_j\) can be written as:

\[
\Sigma_{T_{(xp)p}} = \left[ \sigma_{y_i} \sigma_{y_j} \left[ \theta_k \rho_{y_i y_j} - \rho_{y_i y_j} \right] + \theta_k \rho_{y_i y_j} - \rho_{y_i y_j} \right] \right]_{xp}; \quad (i, j = 1, 2, \ldots, p)  
\]

for \(i = j, \sigma_{y_i} \rho_{y_i} = \sigma_{y_i}^2, \rho_{y_i y_j} = 1, \rho_{y_i y_j} = \rho_{y_i y_j}^2\) and \(\rho_{y_i y_j} = \rho_{y_i y_j}^2\).

In determinants of correlation matrices for \(|R|_k \neq 0\) and \(|R|_k \neq 0\), (3.9) can be written as:

\[
\Sigma_{T_{(xp)p}} = \left[ \sigma_{y_i} \sigma_{y_j} \left[ \theta_k \frac{|R|_{y_i y_j}}{|R|_{y_j y_j}} + \theta_k \left( \frac{|R|_{y_i y_j}}{|R|_{y_j y_j}} - \frac{|R|_{y_j y_j}}{|R|_{y_j y_j}} \right) \right] \right]_{xp}; \quad (i, j = 1, 2, \ldots, p),  
\]

for \(i = j, \sigma_{y_i} \rho_{y_j} = \sigma_{y_j}^2, \rho_{y_i y_j} = |R|_{y_i y_j}^2\) and \(|R|_{y_i y_j} = |R|_{y_j y_j}^2\).

3.1. Remark: 1

To develop generalized multivariate ratio estimator for two-phase sampling using multi- auxiliary variables for Partial Information Case, replace \(h\) by 1 and \(k\) by 2 in (3.1), we get the following estimator

\[
T_{12(xp)} = \left[ \bar{y}_{(2i)} \prod_{i=1}^{r} \left( \frac{\bar{x}_{(1i)}}{\bar{x}_{(2i)}} \right)^{a_{i} + a_{i} + q_{i} + q_{i}} \prod_{i=1}^{r} \left( \frac{\bar{x}_{(1j)}}{\bar{x}_{(2j)}} \right)^{b_{i} + b_{i} + q_{i} + q_{i}} \right] \gamma_i  
\]
The expressions of unknown matrices for which the mean square error of above estimator will be minimum are same as given in (3.5), (3.6) and (3.7). The expression for variance covariance matrix can be directly written from (3.8) just replacing \( h \) by 1 and \( k \) by 2 as:

\[
\Sigma_{T_{12}(pxp)} = \theta_2 \left( \frac{\Sigma_{y_{(yp)}}}{\Sigma_{x_{(xp)}}} - \Sigma_{x_{(xp)}} \Sigma_{y_{(yp)}} \Sigma_{x_{(xp)}}^{-1} \right) - (\theta_2 - \theta_1) \left( \frac{\Sigma_{x_{(xp)}}}{\Sigma_{y_{(yp)}}} - \Sigma_{y_{(yp)}} \Sigma_{x_{(xp)}}^{-1} \Sigma_{y_{(yp)}} \Sigma_{x_{(xp)}}^{-1} \Sigma_{y_{(yp)}} \Sigma_{x_{(xp)}}^{-1} \right)
\]

(3.12)

The variance covariance matrix in the form of variance of \( \gamma_i \), covariances and correlation coefficients of \( x_i \) and \( y_i \) is written as:

\[
\Sigma_{T_{12}(pxp)} = \left[ \sigma_{x_i} \sigma_{y_j} \left\{ \theta_2 \left( \rho_{y_i,y_j} - \rho_{y_i,y_j} \right) + \theta_1 \left( \rho_{y_i,y_j} - \rho_{y_i,y_j} \right) \right\} \right]_{pxp} \cdot \ (i, j = 1, 2, \ldots, p),
\]

(3.13)

for \( i = j \), \( \sigma_{x_i} \sigma_{y_j} = \sigma_i^2 \), \( \rho_{y_i,y_j} = 1 \), \( \rho_{y_i,y_j} = \rho_{y_i,y_j} \) and \( \rho_{y_i,y_j} = \rho_{y_i,y_j} \).

In determinants of correlation matrices for \( |R|^2_{y_i} \neq 0 \) and \( |R|^2_{y_i} \neq 0 \), (3.13) can be written as:

\[
\Sigma_{T_{12}(pxp)} = \left[ \sigma_{x_i} \sigma_{y_j} \left\{ \theta_2 \left| R_{y_i,y_j} \right| \left| R_{y_i,y_j} \right| + \theta_1 \left( \left| R_{y_i,y_j} \right| \left| R_{y_i,y_j} \right| \right) \right\} \right]_{pxp} \cdot \ (i, j = 1, 2, \ldots, p),
\]

(3.14)

for \( i = j \), \( \sigma_{x_i} \sigma_{y_j} = \sigma_i^2 \), \( \left| R_{y_i,y_j} \right| = \left| R_{y_i,y_j} \right| \) and \( \left| R_{y_i,y_j} \right| = \left| R_{y_i,y_j} \right| \).

3.2. Remark: 2

We can develop a univariate generalized ratio estimator for multiphase sampling using multi auxiliary variable for Partial Information Case if we put \( p = 1 \) in (3.1) as:

\[
T_{hk} = \bar{y}_{(k)} \prod_{i=1}^{r} \left( \frac{x_{(ki)}}{\bar{x}_{(ki)}} \right) \prod_{j=1}^{p} \left( \frac{x_{(kj)}}{\bar{x}_{(kj)}} \right) \prod_{j=1}^{p} \left( \frac{x_{(kj)}}{\bar{x}_{(kj)}} \right) ^{y_{(ki)}}
\]

(3.15)

The expression for vectors of unknown constants for which the mean square error will be minimum can be written from (3.5), (3.6) and (3.7) as:

\[
A_{(px1)} = W_{x_{(x)}}^{-1} \left( \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} - \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} \right)
\]

(3.16)

\[
B_{(px1)} = \left( \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} W_{x_{(x)}}^{-1} \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} \right) - \left( \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} W_{x_{(x)}}^{-1} \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} \right)
\]

(3.17)

and

\[
C_{(px1)} = \left( \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} + \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} W_{x_{(x)}}^{-1} \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} \right) - \left( \Sigma_{x_{(x)}} \Sigma_{x_{(x)}} W_{x_{(x)}}^{-1} \Sigma_{x_{(x)}} \right)
\]

(3.18)
The above expressions for unknown matrices can be written in determinants form as:

\[
\alpha_i = (-1)^{i+1} \frac{\bar{Y}_i}{\bar{X}_i} \left| \frac{\Sigma_{i}^{r}_{ik=1} \Sigma_{\bar{y}_{ik}(x)}^{T}}{\Sigma_{i}^{r}_{ik=1} \Sigma_{\bar{y}_{ik}(x)}^{T}} \right| = (-1)^{i+1} \beta_{y_{i}-\bar{y}_{i}}, \quad (i = 1, 2, \ldots, r) \tag{3.19}
\]

\[
\beta_i = (-1)^{i+1} \frac{\bar{Y}_i}{\bar{X}_i} \left\{ \frac{\Sigma_{i}^{r}_{ik=1} \Sigma_{\bar{y}_{ik}(x)}^{T}}{\Sigma_{i}^{r}_{ik=1} \Sigma_{\bar{y}_{ik}(x)}^{T}} \right\} = (-1)^{i+1} \left( \beta_{y_{i},1} - \beta_{y_{i},2} \right), \quad (i = r + 1, r + 2, \ldots, r + s) \tag{3.20}
\]

\[
\gamma_i = (-1)^{i+1} \frac{\bar{Y}_i}{\bar{X}_i} \left| \frac{\Sigma_{i}^{r}_{ik=1} \Sigma_{\bar{y}_{ik}(x)}^{T}}{\Sigma_{i}^{r}_{ik=1} \Sigma_{\bar{y}_{ik}(x)}^{T}} \right| = (-1)^{i+1} \beta_{y_{i} \cdot \bar{y}_{i}}, \quad (i = r + 1, r + 2, \ldots, r + s). \tag{3.21}
\]

The expression for mean square error can be directly written from (3.8) as:

\[
\text{MSE}(T_{hi}) = \theta_k \left( \sigma_y^2 - \Sigma_{y_{hi}(x)}^{T} \Sigma_{y_{hi}(x)}^{-1} \Sigma_{y_{hi}(x)}^{T} \right) - (\theta_h - \theta_k) \left( \Sigma_{y_{hi}(x)}^{T} \Sigma_{y_{hi}(x)}^{-1} \Sigma_{y_{hi}(x)}^{T} \Sigma_{y_{hi}(x)}^{-1} \Sigma_{y_{hi}(x)}^{T} \right) \tag{3.22}
\]

It can be written the form of multiple coefficient of determination as:

\[
\text{MSE}(T_{hi}) = \bar{Y}^2 \Sigma_{y_{hi}(x)}^{-1} \left[ \theta_h \left( 1 - \rho_{y_{hi},1}^2 \right) + \theta_k \left( \rho_{y_{hi},2}^2 - \rho_{y_{hi},3}^2 \right) \right]. \tag{3.23}
\]

3.3. Remark: 3

To develop a generalized univariate ratio estimator for two phase sampling using multi-auxiliary variables for Partial Information Case we put \( h = 1 \) and \( k = 2 \) in (3.15). The required estimator becomes

\[
T_{12} = \bar{Y} \prod_{i=1}^{r} \left( \frac{\bar{X}_{i}}{\bar{X}_{i2}} \right) \prod_{i=1}^{r} \left( \frac{\bar{X}_{i}}{\bar{X}_{i1}} \right) \prod_{i=1}^{r} \left( \frac{\bar{X}_{i}}{\bar{X}_{i2}} \right) \gamma_i \tag{3.24}
\]

The expression for vectors of unknown constants for which the mean square error will be minimum are same as given in (3.16), (3.17) and (3.18), and these expressions are also given in determinants of correlation matrices in (3.19), (3.20) and (3.21). The expression for mean square error can be written from (3.22) just by replacing \( h = 1 \) and \( k = 2 \) as:

\[
\text{MSE}(T_{12}) = \theta_k \left( \sigma_y^2 - \Sigma_{y_{hi}(x)}^{T} \Sigma_{y_{hi}(x)}^{-1} \Sigma_{y_{hi}(x)}^{T} \right) - (\theta_2 - \theta_k) \left( \Sigma_{y_{hi}(x)}^{T} \Sigma_{y_{hi}(x)}^{-1} \Sigma_{y_{hi}(x)}^{T} \Sigma_{y_{hi}(x)}^{-1} \Sigma_{y_{hi}(x)}^{T} \right) \tag{3.25}
\]

It can be written the form of multiple coefficient of determination as:

\[
\text{MSE}(T_{12}) = \bar{Y}^2 \Sigma_{y_{hi}(x)}^{-1} \left[ \theta_k \left( 1 - \rho_{y_{hi},1}^2 \right) + \theta_1 \left( \rho_{y_{hi},2}^2 - \rho_{y_{hi},3}^2 \right) \right]. \tag{3.26}
\]
3.4. Remark: 4

Generalized multivariate ratio estimator as suggested by Hanif et al. (2009) for multi-phase sampling using multi-auxiliary variables when information on all auxiliary variables is not available for population (No Information Case) can be developed by putting $\beta_i$'s and $\gamma_i$'s equals to zero in (3.1) as:

$$T_{hk(pxp)}' = \begin{bmatrix} \bar{y}_{h1} \\ \bar{y}_{h2} \\ \vdots \\ \bar{y}_{hp} \end{bmatrix} \prod_{i=1}^{r} \left( \begin{bmatrix} \bar{x}_{h1j} \\ \bar{x}_{h2j} \\ \vdots \\ \bar{x}_{hjp} \end{bmatrix} \right)^{a_i} \begin{bmatrix} \bar{y}_{k1} \\ \bar{y}_{k2} \\ \vdots \\ \bar{y}_{kp} \end{bmatrix} \prod_{i=1}^{r} \left( \begin{bmatrix} \bar{x}_{k1j} \\ \bar{x}_{k2j} \\ \vdots \\ \bar{x}_{kpj} \end{bmatrix} \right)^{a_i}.$$  \quad (3.27)

The expression of unknown matrix for which the mean square error will be minimum can be directly obtained by considering only those matrices from (3.5), (3.6) and (3.7) those includes only order $p \times r$ and $r \times r$ than we get the required matrix that is $\Sigma_{x_{kp}}^{-1} \Sigma_{y_{kp}}$. The variance covariance matrix can be obtained from (3.8) just by considering those matrices having order $p \times r$ and $r \times r$. The variance covariance matrix is

$$\Sigma_{T_{hk}(pxp)} = \theta_k \Sigma_{y_{kp}} - (\theta_k - \bar{\theta}_h) \Sigma_{y_{kp}}^\prime \Sigma_{x_{rk}}^{-1} \Sigma_{y_{rk}}.$$  \quad (3.28)

All special cases of estimator given in (3.27), in the case on no information and full information, have been discussed by Hanif et al. (2009).

4. Empirical Study of Newly Developed Estimators

It can be empirically investigated that the estimator constructed for full information case will be more efficient than the estimator developed for partial information case and the estimator for partial information case will be more efficient than the estimator for no information case. In the case of multiphase, the estimator will be less efficient by increasing the phases but cost will be reduced.

For empirical investigation we use determinants of variance covariance matrices/MSE. We use the data of 1998 census reports of province Punjab, Pakistan for four districts Jhang, Gujrat, Kasur and Sailkot. The detail of populations and variables description is given in Table 1 and 2 respectively of Appendix. We consider three variables of interests denoted by $Y$'s and five auxiliary variables denoted by $X$'s for computing the determinants of variance covariance matrices multivariate ratio estimators. For univariate ratio estimators we consider $Y_2$ as study variable and the same five auxiliary variables as considered in multivariate case. The necessary population parameters for computing variance covariance matrices/MSE's are given in Tables 3, 4, 5, 6 and 7. We calculate pair-wise determinant of variance covariance matrices/MSE for no information case because in this case two phases at a time can be used. For full information case we calculate variance covariance matrices/MSE for each five phase separately as it is admissible. The determinants of variance covariance matrices of multivariate ratio estimators for multiphase sampling using pair-wise phases for no information case are given in Tables 8 and 9, for partial information case, in Tables 10 and 11 and using each phase for full information case in Table 12. The mean square errors of univariate estimators for multiphase sampling using pair-wise phases for no information case are given in Tables 13 and 14 and for partial information case in Tables 15 and 16 and for full information case using each phase in Table 17.

From Tables 8, 9, 10, 11, and 12, we can say that the multivariate ratio estimators for full information case are more efficient than partial information case and estimators for partial information case are more efficient than no information case for each phase e.g. T2 is more efficient than T12, T3 is more efficient than T13 & T23 etc. and the same is true for univariate ratio estimators (see Tables 13, 14, 15, 16 and 17). Furthermore we can say for no information case and partial information case from Tables 8, 9, 10 and 11 that as we increase phases the efficiency decreases e.g. T12, is more efficient than T13 & T23 etc.
than all others, T13 is more efficient than all others except T12, T34 is more efficient than T35, T45 but less efficient than all others and so on, similarly the same argument can be made for univariate case from Tables Tables 13 and 14. Also for full information case the estimators become less efficient as we increase phases because the sample size decreases by increasing phases, it can be seen from Tables 10, 11, 15 and 16 for multivariate and univariate estimators respectively.

Appendix

Table 1: Detail of Populations

<table>
<thead>
<tr>
<th>Sr. No.</th>
<th>Source of Populations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Population census report of Jhang district (1998), Pakistan</td>
</tr>
<tr>
<td>2</td>
<td>Population census report of Gujrat district (1998), Pakistan</td>
</tr>
<tr>
<td>3</td>
<td>Population census report of Kasur (1998), Pakistan</td>
</tr>
<tr>
<td>4</td>
<td>Population census report of Sialkot district (1998), Pakistan</td>
</tr>
</tbody>
</table>

Table 2: Description of variables (Each variables is taken from Rural Locality)

<table>
<thead>
<tr>
<th>$Y_1$</th>
<th>$X_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_2$</td>
<td>$X_2$</td>
</tr>
<tr>
<td>$Y_3$</td>
<td>$X_3$</td>
</tr>
<tr>
<td>$Y_4$</td>
<td>$X_4$</td>
</tr>
</tbody>
</table>

Table 3: Parameters of populations

<table>
<thead>
<tr>
<th>District</th>
<th>$\bar{Y}_1$</th>
<th>$\sigma_1$</th>
<th>$\bar{Y}_2$</th>
<th>$\sigma_2$</th>
<th>$\bar{Y}_3$</th>
<th>$\sigma_3$</th>
<th>$\bar{C}_1$</th>
<th>$\bar{C}_2$</th>
<th>$\bar{C}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>29.70</td>
<td>5.62</td>
<td>860.11</td>
<td>17.07</td>
<td>897.71</td>
<td>19.87</td>
<td>0.27</td>
<td>0.59</td>
<td>0.51</td>
</tr>
<tr>
<td>Gujrat</td>
<td>57.53</td>
<td>5.62</td>
<td>1101.28</td>
<td>19.87</td>
<td>1102.54</td>
<td>31.39</td>
<td>0.14</td>
<td>0.48</td>
<td>0.49</td>
</tr>
<tr>
<td>Kasur</td>
<td>31.89</td>
<td>5.62</td>
<td>1393.20</td>
<td>19.87</td>
<td>1449.02</td>
<td>31.39</td>
<td>0.75</td>
<td>0.55</td>
<td>0.53</td>
</tr>
<tr>
<td>Sialkot</td>
<td>52.06</td>
<td>5.62</td>
<td>1058.74</td>
<td>19.87</td>
<td>998.22</td>
<td>31.39</td>
<td>0.15</td>
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Table 4: Parameters of populations (Cont. . .)

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<th>$\sigma_1$</th>
<th>$\bar{Y}_2$</th>
<th>$\sigma_2$</th>
<th>$\bar{Y}_3$</th>
<th>$\sigma_3$</th>
<th>$\bar{C}_1$</th>
<th>$\bar{C}_2$</th>
<th>$\bar{C}_3$</th>
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</thead>
<tbody>
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<td>511.91</td>
<td>17.07</td>
<td>459.84</td>
<td>19.87</td>
<td>0.27</td>
<td>0.59</td>
<td>0.51</td>
</tr>
<tr>
<td>Gujrat</td>
<td>8.36</td>
<td>5.62</td>
<td>533.04</td>
<td>19.87</td>
<td>537.24</td>
<td>31.39</td>
<td>0.14</td>
<td>0.48</td>
<td>0.49</td>
</tr>
<tr>
<td>Kasur</td>
<td>23.82</td>
<td>5.62</td>
<td>767.64</td>
<td>19.87</td>
<td>767.80</td>
<td>31.39</td>
<td>0.75</td>
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<td>0.53</td>
</tr>
<tr>
<td>Sialkot</td>
<td>7.64</td>
<td>5.62</td>
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<td>31.39</td>
<td>0.15</td>
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Table 5: Parameters of populations (Cont. . .)

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<th>$\sigma_1$</th>
<th>$\bar{Y}_2$</th>
<th>$\sigma_2$</th>
<th>$\bar{Y}_3$</th>
<th>$\sigma_3$</th>
<th>$\bar{C}_1$</th>
<th>$\bar{C}_2$</th>
<th>$\bar{C}_3$</th>
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<td>.733</td>
<td>.131</td>
<td>.460</td>
<td>.548</td>
<td>.185</td>
<td>.129</td>
<td>.428</td>
<td>.912</td>
</tr>
<tr>
<td>Gujrat</td>
<td>.984</td>
<td>.988</td>
<td>.092</td>
<td>.334</td>
<td>.543</td>
<td>.069</td>
<td>.103</td>
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<td>.941</td>
</tr>
<tr>
<td>Kasur</td>
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<td>.989</td>
<td>.299</td>
<td>.255</td>
<td>.352</td>
<td>.301</td>
<td>.250</td>
<td>.998</td>
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<td>Sialkot</td>
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Table 6: Parameters of populations (Cont. . .)

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<th>$\sigma_1$</th>
<th>$\bar{Y}_2$</th>
<th>$\sigma_2$</th>
<th>$\bar{Y}_3$</th>
<th>$\sigma_3$</th>
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<th>$\bar{C}_2$</th>
<th>$\bar{C}_3$</th>
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<tr>
<td>Jhang</td>
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<td>.732</td>
<td>.748</td>
<td>.559</td>
<td>.489</td>
<td>.416</td>
<td>.421</td>
<td>.317</td>
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<tr>
<td>Gujrat</td>
<td>.984</td>
<td>.933</td>
<td>.749</td>
<td>.487</td>
<td>.986</td>
<td>.954</td>
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<tr>
<td>Kasur</td>
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<td>.752</td>
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<td>.988</td>
<td>.792</td>
<td>.764</td>
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<td>.933</td>
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Table 7: Parameters of populations (Cont. . .)

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<th>District</th>
<th>$p_{x1x5}$</th>
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<th>$p_{x4x5}$</th>
<th>$p_{x5x5}$</th>
<th>$p_{x1x5}$</th>
<th>$p_{x2x5}$</th>
<th>$p_{x3x5}$</th>
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<th>$p_{x5x5}$</th>
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<tr>
<td>Jhang</td>
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<td>0.475</td>
<td>0.432</td>
<td>0.590</td>
<td>0.464</td>
<td>0.325</td>
<td>0.885</td>
<td></td>
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<tr>
<td>Gujrat</td>
<td>0.996</td>
<td>0.892</td>
<td>0.500</td>
<td>0.958</td>
<td>0.420</td>
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<td>0.505</td>
<td>0.996</td>
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<tr>
<td>Kasur</td>
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<td>0.798</td>
<td>0.764</td>
<td>0.614</td>
<td>0.896</td>
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</tr>
<tr>
<td>Sialkot</td>
<td>0.939</td>
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<td>0.985</td>
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<td>0.939</td>
<td>0.877</td>
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Table 8: Determinants of variance covariance matrices of multivariate ratio estimators for pair-wise phases (No Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>$T12$ ($h = 1, k = 2$)</th>
<th>$T13$ ($h = 1, k = 3$)</th>
<th>$T14$ ($h = 1, k = 4$)</th>
<th>$T15$ ($h = 1, k = 5$)</th>
<th>$T23$ ($h = 2, k = 3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>212279.97</td>
<td>1223241.60</td>
<td>7221747.65</td>
<td>46128460.86</td>
<td>3572866.63</td>
</tr>
<tr>
<td>Gujrat</td>
<td>95363.18</td>
<td>312081.54</td>
<td>1102996.85</td>
<td>4211396.91</td>
<td>1123210.67</td>
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<td>Kasur</td>
<td>203091.03</td>
<td>901801.71</td>
<td>3838230.04</td>
<td>16192403.14</td>
<td>2176260.22</td>
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<td>Sialkot</td>
<td>9555.27</td>
<td>41464.31</td>
<td>173806.26</td>
<td>723929.58</td>
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</table>

Table 9: Determinants of variance covariance matrices of multivariate ratio estimators for pair-wise phases (Partial Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>$T24$ ($h = 2, k = 4$)</th>
<th>$T25$ ($h = 2, k = 5$)</th>
<th>$T34$ ($h = 3, k = 4$)</th>
<th>$T35$ ($h = 3, k = 5$)</th>
<th>$T45$ ($h = 4, k = 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
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<tr>
<td>Gujrat</td>
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<td>11251537.17</td>
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<td>Kasur</td>
<td>8973735.68</td>
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<td>79389873.19</td>
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</tr>
<tr>
<td>Sialkot</td>
<td>462925.70</td>
<td>1897366.87</td>
<td>1256890.23</td>
<td>4554991.24</td>
<td>11150157.88</td>
</tr>
</tbody>
</table>

Table 10: Determinants of variance covariance matrices of multivariate ratio estimators for pair-wise phases (Full Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>$T1$ ($k = 1$)</th>
<th>$T2$ ($k = 2$)</th>
<th>$T3$ ($k = 3$)</th>
<th>$T4$ ($k = 4$)</th>
<th>$T5$ ($k = 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>1023.378901</td>
<td>27361.23032</td>
<td>351018.363</td>
<td>345390.739</td>
<td>30487480.83</td>
</tr>
<tr>
<td>Gujrat</td>
<td>27.15981853</td>
<td>367.7474988</td>
<td>3710.595857</td>
<td>33124.8694</td>
<td>279222.8025</td>
</tr>
<tr>
<td>Kasur</td>
<td>103.7803005</td>
<td>1306.215104</td>
<td>12804.97189</td>
<td>112839.0944</td>
<td>946342.2579</td>
</tr>
<tr>
<td>Sialkot</td>
<td>2.156156072</td>
<td>36.85754751</td>
<td>405.2293669</td>
<td>3755.836241</td>
<td>32256.39965</td>
</tr>
</tbody>
</table>

Table 11: Determinants of variance covariance matrices of multivariate ratio estimators for pair-wise phases (Partial Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>$T24$ ($h = 2, k = 4$)</th>
<th>$T25$ ($h = 2, k = 5$)</th>
<th>$T34$ ($h = 3, k = 4$)</th>
<th>$T35$ ($h = 3, k = 5$)</th>
<th>$T45$ ($h = 4, k = 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>138163.6</td>
<td>899963.2</td>
<td>5820643.1</td>
<td>40379695.2</td>
<td>2350013.1</td>
</tr>
<tr>
<td>Gujrat</td>
<td>1683.1</td>
<td>10844.4</td>
<td>738477</td>
<td>534079.5</td>
<td>18218.9</td>
</tr>
<tr>
<td>Kasur</td>
<td>247034.4</td>
<td>1500728.6</td>
<td>9605459.6</td>
<td>66188757.3</td>
<td>2573608.2</td>
</tr>
<tr>
<td>Sialkot</td>
<td>322.1</td>
<td>2207.0</td>
<td>15843.0</td>
<td>118825.5</td>
<td>3970.3</td>
</tr>
</tbody>
</table>

Table 12: Determinants of variance covariance matrices of multivariate ratio estimators for each phase (Full Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>$T1$ ($k = 1$)</th>
<th>$T2$ ($k = 2$)</th>
<th>$T3$ ($k = 3$)</th>
<th>$T4$ ($k = 4$)</th>
<th>$T5$ ($k = 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>1023.378901</td>
<td>27361.23032</td>
<td>351018.363</td>
<td>345390.739</td>
<td>30487480.83</td>
</tr>
<tr>
<td>Gujrat</td>
<td>27.15981853</td>
<td>367.7474988</td>
<td>3710.595857</td>
<td>33124.8694</td>
<td>279222.8025</td>
</tr>
<tr>
<td>Kasur</td>
<td>103.7803005</td>
<td>1306.215104</td>
<td>12804.97189</td>
<td>112839.0944</td>
<td>946342.2579</td>
</tr>
<tr>
<td>Sialkot</td>
<td>2.156156072</td>
<td>36.85754751</td>
<td>405.2293669</td>
<td>3755.836241</td>
<td>32256.39965</td>
</tr>
</tbody>
</table>
Table 13: MSE’s of univariate ratio estimators for pair-wise phases (No Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>T12 (h = 1, k = 2)</th>
<th>T13 (h = 1, k = 3)</th>
<th>T14 (h = 1, k = 4)</th>
<th>T15 (h = 1, k = 5)</th>
<th>T23 (h = 2, k = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>21227.9</td>
<td>1223241.6</td>
<td>7221747.6</td>
<td>46128460.8</td>
<td>35728666.6</td>
</tr>
<tr>
<td>Gujrat</td>
<td>9536.1</td>
<td>312081.5</td>
<td>1102996.8</td>
<td>4211396.9</td>
<td>1123210.6</td>
</tr>
<tr>
<td>Kasur</td>
<td>203091.0</td>
<td>901801.7</td>
<td>3838230.0</td>
<td>16192403.1</td>
<td>2176260.2</td>
</tr>
<tr>
<td>Sialkot</td>
<td>9555.27</td>
<td>41464.3</td>
<td>173806.26</td>
<td>723929.5</td>
<td>126175.4</td>
</tr>
</tbody>
</table>

Table 14: MSE’s of univariate ratio estimators for pair-wise phases (No Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>T24 (h = 2, k = 4)</th>
<th>T25 (h = 2, k = 5)</th>
<th>T34 (h = 3, k = 4)</th>
<th>T35 (h = 3, k = 5)</th>
<th>T45 (h = 4, k = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>1601134.8</td>
<td>79961159.68</td>
<td>38868782.02</td>
<td>158093587.7</td>
<td>2.60491E+11</td>
</tr>
<tr>
<td>Gujrat</td>
<td>3372979.2</td>
<td>11251537.17</td>
<td>10702183.50</td>
<td>30944428.16</td>
<td>93069748.63</td>
</tr>
<tr>
<td>Kasur</td>
<td>8973735.6</td>
<td>36805031.49</td>
<td>19922737.86</td>
<td>79389873.19</td>
<td>170075469.89</td>
</tr>
<tr>
<td>Sialkot</td>
<td>482925.7</td>
<td>1897366.87</td>
<td>1256890.23</td>
<td>4554991.24</td>
<td>7419672</td>
</tr>
</tbody>
</table>

Table 15: MSE’s of univariate ratio estimators for pair-wise phases (No Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>T12 (h = 1, k = 2)</th>
<th>T13 (h = 1, k = 3)</th>
<th>T14 (h = 1, k = 4)</th>
<th>T15 (h = 1, k = 5)</th>
<th>T23 (h = 2, k = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>652.99</td>
<td>980.5526</td>
<td>1635.678</td>
<td>2945.928</td>
<td>1795.189</td>
</tr>
<tr>
<td>Gujrat</td>
<td>309.4727</td>
<td>900.3699</td>
<td>2082.164</td>
<td>4445.753</td>
<td>624.3232</td>
</tr>
<tr>
<td>Kasur</td>
<td>378.5487</td>
<td>1044.532</td>
<td>2376.499</td>
<td>5040.433</td>
<td>771.9548</td>
</tr>
<tr>
<td>Sialkot</td>
<td>234.4814</td>
<td>682.5967</td>
<td>1578.827</td>
<td>3371.289</td>
<td>474.9672</td>
</tr>
</tbody>
</table>

Table 16: MSE’s of univariate ratio estimators for pair-wise phases (No Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>T24 (h = 2, k = 4)</th>
<th>T25 (h = 2, k = 5)</th>
<th>T34 (h = 3, k = 4)</th>
<th>T35 (h = 3, k = 5)</th>
<th>T45 (h = 4, k = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>2450.314</td>
<td>4079.586</td>
<td>5389.837</td>
<td>8648.382</td>
<td>4079.586</td>
</tr>
<tr>
<td>Gujrat</td>
<td>1806.117</td>
<td>1254.024</td>
<td>3617.613</td>
<td>2513.426</td>
<td>1254.024</td>
</tr>
<tr>
<td>Kasur</td>
<td>2105.922</td>
<td>1558.767</td>
<td>4222.701</td>
<td>3132.391</td>
<td>1558.767</td>
</tr>
<tr>
<td>Sialkot</td>
<td>1371.198</td>
<td>955.9386</td>
<td>2748.4</td>
<td>1917.882</td>
<td>955.9386</td>
</tr>
</tbody>
</table>

Table 17: MSE’s of univariate ratio estimators for each-wise phase (Full Information Case)

<table>
<thead>
<tr>
<th>District</th>
<th>T1 (k = 1)</th>
<th>T2 (k = 2)</th>
<th>T3 (k = 3)</th>
<th>T4 (k = 4)</th>
<th>T5 (k = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jhang</td>
<td>81.89064</td>
<td>245.6719</td>
<td>573.2344</td>
<td>1228.36</td>
<td>2538.61</td>
</tr>
<tr>
<td>Gujrat</td>
<td>213.5579</td>
<td>509.0065</td>
<td>1099.904</td>
<td>2281.698</td>
<td>4645.286</td>
</tr>
<tr>
<td>Kasur</td>
<td>251.1011</td>
<td>584.0929</td>
<td>1250.076</td>
<td>2582.043</td>
<td>5245.977</td>
</tr>
<tr>
<td>Sialkot</td>
<td>142.167</td>
<td>366.2247</td>
<td>814.34</td>
<td>1710.571</td>
<td>3503.032</td>
</tr>
</tbody>
</table>

References


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