On Direct Sums of Lifting Modules and Internal Exchange Property

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Abstract. Let $R$ be a ring with identity and let $M = M_1 \oplus M_2$ be an amply supplemented $R$-module. Then it is proved that $M_i$ has $(D_1)$ and is $M_j$-cojective for $i \neq j$, $i = 1, 2$, if and only if for any coclosed submodule $X$ of $M$, there exist $M_i' \leq M_1$ and $M_2' \leq M_2$ such that $M = X \oplus M_1' \oplus M_2'$.

1. Introduction

Throughout this paper all rings will have an identity and all modules will be unital left $R$-modules. $N \leq M(N|M)$ will mean $N$ is a submodule (a direct summand) of the module $M$. For $M = \bigoplus_{i \in I} M_i$ and $K \subseteq I$, $M(K) = \bigoplus_{i \in K} M_i$.

A module is extending (or satisfies $(C_1)$) if every submodule is essential in a direct summand. Dually, a module $M$ is called a lifting module (or satisfies $(D_1)$), if for any submodule $N$ of $M$, there exist a direct summand $K$ of $M$ such that $K \leq N$ and $N/K \ll M/K$, equivalently, for any submodule $N$ of $M$ there exist submodules $K_1, K_2$ of $M$ such that $M = K_1 \oplus K_2$, $K_1 \leq N$ and $N \cap K_2 \ll K_2$. Lifting modules generalize discrete and quasi-discrete ones; they have been studied extensively (see, for examples, [1], [2], [6], [8], [9]) but many questions remain unresolved.

An open problem is to find sensible necessary and sufficient conditions for the direct sum of lifting modules to be lifting. If $M_1$ and $M_2$ are relatively projective, quasi-projective and $(D_1)$-modules then $M = M_1 \oplus M_2$ is a $(D_1)$-module ([9, Theorem 9]). Let $M = \bigoplus_{i=1}^n M_i$ be a finite direct sum of relatively projective modules $M_i$. Then $M$ is lifting if and only if $M$ is an amply supplemented and $M_i$ is lifting for all $1 \leq i \leq n$([1, Corollary 2.9]). However, it is not a sufficient condition for a finite direct sum of lifting modules to be a lifting module. Let $p$ be a prime integer and $M$ denote the $\mathbb{Z}$-module, $(\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z})$. Then $M$ is a lifting module and $\mathbb{Z}/p\mathbb{Z}$ is not $\mathbb{Z}/p^2\mathbb{Z}$-projective (see [9, Example 4]).

In this paper we consider when the direct sum of two lifting modules is lifting. In [3] the authors claim that for any closed submodule $X$ of $M = M_1 \oplus M_2$, $M$ decomposes as $M = X \oplus M_1' \oplus M_2'$ with $M_i' \leq M_i$, if and only if $M_i$ has $(C_1)$ and

Received November 19, 2003, and, in revised form, October 2, 2004.
2000 Mathematics Subject Classification: 16D10, 16D99.
Key words and phrases: lifting module, $M$-cojective module, $M$-cojective module.
is \( M_j \)-jective for \( i \neq j \). Dually, we prove that if \( M = M_1 \oplus M_2 \) be an amply supplemented \( R \)-module. Then it is proved that \( M_i \) has \((D_1)\) and is \( M_j \)-jective for \( i \neq j \) if and only if for any coclosed submodule \( X \) of \( M \), there exist \( M'_1 \leq M_1 \) and \( M'_2 \leq M_2 \) such that \( M = X \oplus M'_1 \oplus M'_2 \).

2. Preliminaries

Let \( M \) be a module and \( S \leq M \). \( S \) is called small in \( M \) (denoted by \( S \ll M \)) if for any \( T \leq M \), \( S + T = M \) implies \( T = M \). For \( N, L \leq M, N \) is a supplement of \( L \) in \( M \) if \( N + L = M \) with \( N \cap L \ll N \). Following [7], a module \( M \) is called supplemented if every submodule of \( M \) has a supplement in \( M \). On the other hand, the module \( M \) is amply supplemented if, for any submodules \( A, B \) of \( M \) with \( M = A + B \) there exists a supplement \( P \) of \( A \) in \( M \) such that \( P \leq B \). Following [10], the module \( M \) is called a weakly supplemented module if for each submodule \( A \) of \( M \) there exists a submodule \( B \) of \( M \) such that \( M = A + B \) and \( A \cap B \ll M \).

Let \( M \) be a module and \( B \leq A \leq M \). If \( A/B \ll M/B \), then \( B \) is called a coessential submodule of \( A \) in \( M \). A submodule \( A \) of \( M \) is called coclosed if \( A \) has no proper coessential submodule. Also, we will call \( B \) an coclosure(or an s-closure) of \( B \) in \( M \), if \( B \) is a coessential submodule of \( A \) and \( B \) is coclosed in \( M \).

Let \( M \) be a module. Then by [8, Proposition 4.8], \( M \) is lifting if and only if \( M \) is amply supplemented and every supplement submodule of \( M \) is a direct summand.

We list a few lemmas for later use.

**Lemma 2.1.** Let \( M \) be a module and \( N \leq M \). Consider the following conditions:

1. \( N \) is a supplement submodule of \( M \);
2. \( N \) is coclosed in \( M \);
3. For all \( X \leq N \), \( X \ll M \) implies \( X \ll N \).

Then \( (1) \Rightarrow (2) \Rightarrow (3) \) hold. If \( M \) is a weakly supplemented module then \( (3) \Rightarrow (1) \) holds.

**Proof.** [1, Lemma 1.1]. \( \square \)

**Lemma 2.2.** Let \( M = M_1 \oplus M_2 \) and \( N, L \leq M_1 \). If \( N \) is a supplement of \( L \) in \( M_1 \), then \( N \oplus M_2 \) is a supplement of \( L \) in \( M \).

**Proof.** Let \( N \) be a supplement of \( L \) in \( M_1 \). Then \( M_1 = N + L \) and \( N \cap L \ll N \). It is easy to see that \( M = (N \oplus M_2) + L \) and \( (N \oplus M_2) \cap L = N \cap L \ll N \). Thus \( N \oplus M_2 \) is a supplement of \( L \) in \( M \). \( \square \)

**Lemma 2.3.** Let \( K \leq L \leq M \). If \( K \) is coclosed in \( M \), then \( K \) is coclosed in \( L \) and the converse is true if \( L \) is coclosed in \( M \).

**Proof.** [11, Lemma 2.6]. \( \square \)

**Definition 2.4 ([3]).** Let \( M = \oplus_{i \in I} M_i \) be a direct sum of submodules \( M_i \). Then...
we say that the decomposition $M = \oplus_{i \in I} M_i$ is exchangeable if for any direct summand $N$ of $M$ we have $M = (\oplus_{i \in I} M'_i) \oplus N$ with $M'_i \leq M_i$.

3. *Cojective modules

Let $A$ and $B$ be modules. Following [5], $B$ is called $A$-ojective if any diagram

$$
\begin{array}{c}
X > \\
\phi
\end{array}
\begin{array}{c}
\rightarrow
\downarrow
\rightarrow
\downarrow
\end{array}
\begin{array}{c}
A
\end{array}
$$

can be embedded in a diagram

$$
\begin{array}{c}
X > \\
\phi
\end{array}
\begin{array}{c}
\rightarrow
\downarrow
\rightarrow
\downarrow
\end{array}
\begin{array}{c}
A = A_1 \oplus A_2
\end{array}
\begin{array}{c}
\phi_1
\phi_2
\end{array}
\begin{array}{c}
B_1 \oplus B_2
\end{array}
$$

such that $\phi_2$ is a monomorphism and for $x = a_1 + a_2$ and $\phi(x) = b_1 + b_2$ one has $b_1 = \phi_1(a_1)$ and $a_2 = \phi_2(b_2)$.

Mohamed and Müller named it ojectivity in honor of Oshiro and they characterize it in [3] as follows:

**Theorem 3.1.** Let $M = A \oplus B$. Then the following are equivalent:

1. $B$ is $A$-ojective;
2. For any complement $C$ of $B$, $M$ decomposes as $M = C \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$.

According to this characterization, Mohamed and Müller give the following dual definition in [4, Definition 2.3].

**Definition 3.2.** Let $A$, $B$ be left $R$-modules. We say $B$ is $A$-*cojective* if for any supplement $C$ of $A$ in $A \oplus B$, $A \oplus B$ decomposes as $A \oplus B = C \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$. If $B$ is $A$-*cojective* and $A$ is $B$-*cojective*, we say that $A$ and $B$ are mutually *cojective*.

As supplements need not exist, *cojectivity is not the precise dual of ojectivity. The precise dual of ojectivity as follows (See, [4]):

Let $A$ and $B$ be modules. $A$ is $B$-cojective if any diagram

$$
\begin{array}{c}
A
\end{array}
\begin{array}{c}
\phi
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
X \prec \pi
\end{array}
\begin{array}{c}
B
\end{array}
$$
can be embedded in a diagram
\[
\begin{array}{c}
A & = & A_1 & \oplus & A_2 \\
\varphi & \downarrow & \varphi_1 & \downarrow & \varphi_2 \\
X & \prec & B = B_1 & \oplus & B_2
\end{array}
\]

such that \( \varphi_2 \) is onto, \( \pi \varphi_1 = \varphi|A_1 \) and \( \varphi \varphi_2 = \pi|B_2 \).

In [4, Theorem 2.8], Mohamed and Müller give the following characterization of cojectivity:

Let \( M = A \oplus B \). Then \( A \) is \( B \)-cojective if and only if whenever \( M = N + B \), we have \( M = N' \oplus A' \oplus B' = N' + B \) with \( N' \leq N, A' \leq A \) and \( B' \leq B \). Therefore if \( A \) is \( B \)-cojective, then \( A \) is \( B \)-cojective (See, [4, Proposition 2.9]).

**Proposition 3.3.** Let \( M = M_1 \oplus M_2 \). If \( M_1 \) is \( M_2 \)-projective, then \( M_1 \) is \( M_2 \)-cojective.

**Proof.** Let \( N \) be a supplement of \( M_2 \) in \( M \). Then \( M = N + M_2 \) and \( N \cap M_2 \ll N \).
Since \( M_1 \) is \( M_2 \)-projective, by [1, Lemma 2.5], there exists a submodule \( N' \) of \( N \) such that \( M = N' \oplus M_2 \). Clearly \( N = N' \oplus (N \cap M_2) \). Hence \( N = N' \) and \( M = N + M_2 \). Thus \( M_1 \) is \( M_2 \)-cojective. \( \square \)

Let \( M \) be a module. Consider the following condition:

\((D_3) \) For every direct summands \( K, L \) of \( M \) with \( M = K + L, K \cap L \) is a direct summand of \( M \).

Following [8], if the module \( M \) is lifting and has \( (D_3) \) then it is called a quasi-discrete module.

Let \( M_1 \) and \( M_2 \) be modules. Following [1], the module \( M_1 \) is small \( M_2 \)-projective if every homomorphism \( f : M_1 \to M_2/A \), where \( A \) is a submodule of \( M_2 \) and \( Im f \ll M_2/A \), can be lifted to a homomorphism \( f : M_1 \to M_2 \).

**Proposition 3.4.** Let \( M = M_1 \oplus M_2 \) be an amply supplemented module with \((D_3) \).
If \( M_1 \) is \( M_2 \)-cojective, then \( M_1 \) is small \( M_2 \)-projective.

**Proof.** Let \( N \) be a submodule of \( M \) such that \((N + M_1)/N \ll M/N \). Then \( M = N + M_2 \). Since \( M \) is amply supplemented there exists a submodule \( N' \) of \( M \) such that \( N' \leq N, M = N' + M_2 \) and \( N' \cap M_2 \ll N' \), that is, \( N' \) is a supplement of \( M_2 \) in \( M \). Since \( M_1 \) is \( M_2 \)-cojective, \( M = N' \oplus M_1 \oplus M_2' \) with \( M_2' \leq M_2 \). By \((D_3) \), \( N' \cap M_2 \) is a direct summand of \( M \), and so \( M = N' \oplus M_2 \). By [1, Lemma 2.4], \( M_1 \) is small \( M_2 \)-projective. \( \square \)

**Proposition 3.5.** Let \( A_1 |A \) and \( B_1 |B \). If \( B \) is \( A \)-cojective, then \( B_1 \) is \( A_1 \)-cojective.

**Proof.** Write \( M = A \oplus B, A = A_1 \oplus A_2 \) and \( B = B_1 \oplus B_2 \).

(1) First we prove that \( B_1 \) is \( A \)-cojective. Write \( N = A \oplus B_1 \), and let \( X \) be a supplement of \( A \) in \( N \). By Lemma 2.2, \( X \oplus B_2 \) is a supplement of \( A \) in \( M \).

As \( B \) is \( A \)-cojective, \( M = X \oplus B_2 \oplus A' \oplus B' \) with \( A' \leq A \) and \( B' \leq B \). Hence
$N = X \oplus A' \oplus (N \cap (B_2 \oplus B'))$. The result now follows if we show $N \cap (B_2 \oplus B') \leq B_1$. Indeed, $N \cap (B_2 \oplus B') = (A \oplus B_1) \cap (B_2 \oplus B') \leq (A \oplus B_1) \cap B = B_1$.

(2) Next we prove that $B$ is $A_1$-cojective. Write $L = A_1 \oplus B$, and let $Y$ be a supplement of $A_1$ in $L$. By Lemma 2.1, it is easy to see that $A_1$ is a supplement of $Y$ in $M$. Then $A$ is a supplement of $Y$ in $M$ by Lemma 2.2. Again by Lemma 2.1, $Y$ is a supplement of $A$ in $M$. As $B$ is $A$-cojective, $M = Y \oplus A'' \oplus B''$ with $A'' \leq A$ and $B'' \leq B$. Hence $L = Y \oplus B' \oplus (L \cap A')$. It remains to show that $L \cap A' \leq A_1$. Let $a'' \in L \cap A'$. Then $a'' = a_1 + b$ with $a_1 \in A_1$ and $b \in B$. Hence $b = a'' - a_1 \in A \cap B = 0$, and so $a'' = a_1 \in A_1$.

(3) Our proposition follows from (1) and (2). \hfill \square

**Lemma 3.6.** Let $M = A \oplus B$ where $A$ is $B$-cojective and $B$ has $(D_1)$. If $X$ is a coclosed submodule of $M$ with $M = X + B$, then $M$ decomposes as $M = X \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$.

**Proof.** Let $M = X + B$. Since $B$ has $(D_1)$, there exists a direct summand $B_1$ of $B$ such that $B = B_1 \oplus B_2$ and $B_1 \leq X \cap B, X \cap B_2 \leq B_2$. Now $M = A \oplus B_1 \oplus B_2$. Write $N = A \oplus B_2$. Then $X = B_1 \cap X_1$, where $X_1 = X \cap N$. Hence $M = X + B = X_1 + B_1 + B_2$, and so $N = X_1 + B_2$. Clearly $X_1 \cap B_2 = X \cap B_2 \leq B_2$. Then $B_2$ is a supplement of $X_1$ in $N$. Now $X_1$ is a coclosed submodule of $X$, and $X$ is a coclosed submodule of $M$. It then follows by Lemma 2.3 that $X_1$ is coclosed in $N$. It is easy to see that $X_1$ is a supplement of $B_2$ in $N$. Now $A$ is $B_2$-cojective, by Proposition 3.5. Now we get $N = X_1 \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B_2$. Hence $M = N \oplus B_1 = X_1 \oplus B_1 \oplus A' \oplus B_2 = X \oplus A' \oplus B_2'$. \hfill \square

We prove here the dual of the result of [3, Theorem 10].

**Theorem 3.7.** Let $M = M_1 \oplus M_2$ be an amply supplemented module. Then $M_i$ has $(D_1)$ and is $M_j$-cojective for $i \neq j$ if and only if for any coclosed submodule $X$ of $M$, we have $M = X \oplus M_i' \oplus M_2$ with $M_i' \leq M_i$.

**Proof.** The sufficiency follows from Definition 3.2 and from the fact that $(D_1)$ is inherited by summands.

Conversely, suppose that $M_i$ has $(D_1)$ and is $M_j$-cojective for $i \neq j$. Let $X$ be a coclosed submodule of $M$. It is easy to see that $M/X$ is amply supplemented, and so $(X + M_j)/X$ has a coclosure in $M/X$ by [1, Proposition 1.5], that is, there exists a coclosed submodule $N/X$ of $M/X$ such that $N/X \leq (X + M_j)/X$ and $(X + M_1)/N \leq M_i/N$. By [1, Lemma 1.4], $N$ is coclosed in $M$. As $M = N + M_2$, we get by Lemma 3.6 that $M = N \oplus M_1' \oplus M_2$ with $M_i' \leq M_i$. Write $N_1 = M_1' \oplus M_2$. Note that $X = N \cap (X + N_1)$ and $M = N + (X + N_1)$, and so $M/X = N/X \oplus (X + N_1)/X$. Therefore $(X + N_1)/X$ is coclosed in $M/X$. Again by [1, Lemma 1.4], $X + N_1$ is coclosed in $M$. Moreover, $M = (X + N_1) + M_1$. Again by Lemma 3.6, $M = (X + N_1) \oplus M_1'' \oplus M_2''$ with $M_i'' \leq M_i$. Write $N_2 = M_1'' \oplus M_2''$. Hence $N_1 = (X + N_1) \cap (N_1 + N_2)$ and $N \cap (X + N_1) \cap (N_1 + N_2) = X \cap (N_1 + N_2) = 0$. So $M = X \oplus (N_1 + N_2) = X \oplus (M_1' + M_2' + M_1'' + M_2'') = X \oplus M_1' \oplus M_2$ where $M_1' = M_1' + M_1'$ and $M_2'' = M_2' + M_2'$. \hfill \square
In view of Definition 2.4, Theorem 3.7 may be reformulated as follows:

**Theorem 3.8.** Let $M = M_1 \oplus M_2$ be an amply supplemented module. Then $M$ has $(D_1)$ and the decomposition is exchangeable if and only if, for $i = 1, 2$, $M_i$ has $(D_1)$ and is $M_j$-cojective for $i \neq j$.

By analogy with the proof of [3, Theorem 11], we have

**Theorem 3.9.** Let $n \geq 2$ be an integer and let $M = \bigoplus_{i=1}^{n} M_i$ be an amply supplemented module. Then the following are equivalent:

1. $M$ has $(D_1)$ and the decomposition is exchangeable;
2. The $M_i$ have $(D_1)$, and $M_1 \oplus \cdots \oplus M_{i-1}$ and $M_i$ are mutually $^*$cojective, for $2 \leq i \leq n$;
3. The $M_i$ have $(D_1)$, and $M(I)$ is $M(J)$-cojective for any disjoint nonempty subset $I$ and $J$ of $\{1, 2, \cdots n\}$.

4. Semi-discrete modules

**Definition 4.1 ([3]).** Let $\mu$ be a cardinal number. A module $M$ is said to have the $\mu$-internal exchange property if any decomposition $M = \bigoplus_{i \in I} M_i$ with $|I| \leq \mu$, is exchangeable.

**Definition 4.2.** We call a module $M$ semi-discrete if $M$ has $(D_1)$ and the 2-internal exchange property.

Thus, if $M$ is an amply supplemented module, then $M$ is semi-discrete if and only if for any coclosed submodule $C$ of $M$ and any decomposition $M = A \oplus B$, we have $M = C \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$.

It is well known that a discrete module has the exchange (hence the internal exchange) property, and so is semi-discrete module. However, it is not known whether a quasi-discrete module has the internal exchange property. Let $M$ be a quasi-discrete module. In [8, Corollary 4.19], it is proved that if every hollow summand of $M$ has a local endomorphism ring, then $M$ has exchange property, and so these modules are semi-discrete.

In [8, Lemma 4.23], it is noticed that if $M$ is a quasi-discrete module, then for every decomposition $M = A \oplus B$, $A$ and $B$ are mutually projective. The following is analogue for semi-discrete modules.

**Proposition 4.3.** Let $M$ be any module. $M$ is semi-discrete if and only if $M$ has $(D_1)$ and for every decomposition $M = A \oplus B$, $A$ and $B$ are mutually $^*$cojective.

**Proof.** The result follows from Theorem 3.7 and from the fact that $(D_1)$ is inherited by summands.

A module $M = M_1 \oplus \cdots \oplus M_n$ is quasi-discrete if and only if $M_i$ is quasi-discrete and $M_j$-projective for all $i \neq j$([1, Theorem 2.13]). For $n = 2$ the following is an
analogous result for semi-discrete modules.

**Theorem 4.4.** Let \( M = M_1 \oplus M_2 \) be an amply supplemented module. Then \( M \) is semi-discrete if and only if \( M_i \) is semi-discrete and \( M_j \)-cojective for \( i \neq j \).

*Proof.* The necessity follows from [3, Proposition 15] and Proposition 4.3. The sufficiency is analogous with the proof of [3, Theorem 19]. \( \square \)

**Corollary 4.5.** Let \( n \geq 2 \) be an integer and let \( M = \oplus_{i=1}^{n} M_i \) be an amply supplemented module. Then the following are equivalent:

1. \( M \) is semi-discrete;
2. The \( M_i \) are semi-discrete, and \( M_1 \oplus \cdots \oplus M_{i-1} \) and \( M_i \) are mutually \( \ast \)-cojective, for \( 2 \leq i \leq n \);
3. Every \( M_i \) is semi-discrete, and \( M(I) \) is \( M(J) \)-\( \ast \)-cojective for any disjoint nonempty subset \( I \) and \( J \) of \( \{1, 2, \cdots n\} \).

*Proof.* Theorem 4.4 and induction. \( \square \)

**Proposition 4.6.** Let \( M \) be a quasi-projective module. Then the following are equivalent:

1. \( M \) is supplemented;
2. \( M \) is amply supplemented;
3. \( M \) is lifting;
4. \( M \) is quasi-discrete;
5. \( M \) is discrete;
6. \( M \) is semi-discrete.

*Proof.* (5) \( \Rightarrow \) (6) \( \Rightarrow \) (3) are clear. Now the result follows by [12, Proposition 2.2]. \( \square \)

**Theorem 4.7.** For any ring \( R \) the following are equivalent:

1. \( R \) is a left perfect ring;
2. Every quasi-projective left \( R \)-module is semi-discrete.

*Proof.* This is clear by Proposition 4.6 and [12, Theorem 2.6]. \( \square \)

**Theorem 4.8.** For any ring \( R \) the following are equivalent:

1. \( R \) is a left perfect ring;
2. Every projective left \( R \)-module is semi-discrete.
Proof. This is clear by Proposition 4.6 and [12, Theorem 2.7]. □

**Corollary 4.9.** Let $M$ be a quasi-projective module such that $M = \oplus_{i=1}^{n} M_i$ is a finite direct sum of submodules $M_i$, $(1 \leq i \leq n)$. Then $M$ is semi-discrete if and only if $M_i$ $(1 \leq i \leq n)$ is semi-discrete.

**Proof.** Necessity is clear. Conversely, suppose that each $M_i$ is semi-discrete. Then, by [12, Proposition 2.8], $M$ is lifting. Hence $M$ is semi-discrete by Proposition 4.6. □

**Acknowledgements.** I would like to express my gratefulness to Professor Liu Zhongkui and the referee for their invaluable suggestions and comments, which certainly improved the presentation of the paper.

**References**


