On Almost Continuity

ERDAL EKICI
Department of Mathematics, Canakkale Onsekiz Mart University, Terzioglu Campus, 17020 Canakkale, Turkey
e-mail: eekici@comu.edu.tr

ABSTRACT. A new class of functions is introduced in this paper. This class is called almost \( \delta \)-precontinuity. This type of functions is seen to be strictly weaker than almost precontinuity. By using \( \delta \)-preopen sets, many characterizations and properties of the said type of functions are investigated.

1. Introduction

The notion of \( \delta \)-preopen set was introduced by Raychaudhuri and Mukherjee [20] in 1993. We introduce here a new type of functions strictly weaker than almost continuity. We call this the almost \( \delta \)-precontinuous functions. We investigate properties of such functions.

Throughout the present paper, spaces mean topological spaces and \( f : (X, \tau) \to (Y, \sigma) \) (or simply \( f : X \to Y \)) denotes a function \( f \) of a space \( (X, \tau) \) into a space \( (Y, \sigma) \). Let \( S \) be a subset of a space \( X \). The closure and the interior of \( S \) are denoted by \( \text{cl}(S) \) and \( \text{int}(S) \), respectively.

A subset \( S \) of a space \( X \) is said to be regular open [23] if \( S = \text{int}(%5Ctext{cl}(S)) \) and \( \delta \)-open [24] if for each \( x \in S \), there exists a regular open set \( W \) such that \( x \in W \subset S \).

A subset \( S \) of a space \( X \) is said to be \( \alpha \)-open [13] (resp. semi-open [8], preopen [10], \( \gamma \)-open [7], \( \beta \)-open [1] or semi-preopen [2]) if \( S \subset \text{int}(\text{cl}(S)) \) (resp. \( S \subset \text{cl}(\text{int}(S)), S \subset \text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S)), S \subset \text{cl}(\text{int}(S)))\).

The complement of a regular open set is said to be regular closed [23]. The complement of a semiopen set is said to be semiclosed [5]. The intersection of all semiclosed sets containing a subset \( A \) of \( X \) is called the semi-closure [5] of \( A \) and is denoted by \( s\text{-cl}(A) \). The union of all semiopen sets contained in a subset \( A \) of \( X \) is called the semi-interior of \( A \) and is denoted by \( s\text{-int}(A) \). A point \( x \in X \) is called a \( \delta \)-cluster (resp. \( \theta \)-cluster) point of \( A \) [24] if \( A \cap \text{int}(\text{cl}(U)) \neq \emptyset \) (resp. \( A \cap \text{cl}(U) \neq \emptyset \)) for each open set \( U \) containing \( x \). The set of all \( \delta \)-cluster (resp. \( \theta \)-cluster) points of \( A \) is called the \( \delta \)-closure (resp. \( \theta \)-closure) of \( A \) and is denoted by \( \delta\text{-cl}(A) \) (resp. \( \theta\text{-cl}(A) \)). If \( \delta\text{-cl}(A) = A \) (resp. \( \theta\text{-cl}(A) = A \)), then \( A \) is said to be \( \delta \)-closed (resp. \( \theta \)-closed)....

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θ-closed). The complement of a δ-closed (resp. θ-closed) set is said to be δ-open (resp. θ-open).

A subset $S$ of a topological space $X$ is said to be δ-preopen [20] iff $S \subset \text{int}(\delta\text-cl}(S))$. The complement of a δ-preopen set is called a δ-preclosed set [20]. The union (resp. intersection) of all δ-preopen (resp. δ-preclosed) sets, each contained in (resp. containing) a set $S$ in a topological space $X$ is called the δ-preinterior (resp. δ-preclosure) of $S$ and it is denoted by $\delta\text{-pint}(S)$ (resp. $\delta\text{-pcl}(S)$) [20].

The family of all δ-preopen (resp.regular open, preopen, δ-open, α-open, semi-open, δ-open) sets of a space $X$ will be denoted by $\delta\text{PO}(X)$ (resp. $\text{RO}(X)$, $\text{PO}(X)$, $\beta\text{O}(X)$, $\alpha\text{O}(X)$, $\text{SO}(X)$, $\delta\text{O}(X)$). The family of all δ-preclosed (resp. regular closed, δ-closed) sets in a space $X$ is denoted by $\delta\text{PC}(X)$ (resp. $\text{RC}(X)$, $\delta\text{C}(X)$). The family of all δ-preopen (resp.regular open, δ-open) sets containing a point $x \in X$ will be denoted by $\delta\text{PO}(X, x)$ (resp. $\text{RO}(X, x)$, $\delta\text{O}(X, x)$).

**Lemma 1** (Raychaudhuri and Mukherjee [20]). Let $A$ be a subset of a space $X$. Then

1. $\delta - \text{pcl}(X \setminus A) = X \setminus \delta - \text{pint}(A),$
2. $x \in \delta - \text{pcl}(A)$ if and only if $A \cap U \neq \emptyset$ for each $U \in \delta\text{PO}(X, x),$
3. $A$ is δ-preclosed in $X$ if and only if $A = \delta - \text{pcl}(A),$
4. $\delta - \text{pcl}(A)$ is δ-preclosed in $X$.

**Lemma 2** (Noiri [17], [18]). For a subset of a space $Y$, the following hold:

1. $\alpha - \text{cl}(V) = \text{cl}(V)$ for every $V \in \beta\text{O}(Y),$
2. $p - \text{cl}(F) = \text{cl}(V)$ for every $V \in \text{SO}(Y),$
3. $s - \text{cl}(V) = \text{int}(\text{cl}(V))$ for every preopen set $V$ of a space $X$.

2. Characterizations

**Definition 3.** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be almost δ-precontinuous if for each $x \in X$ and each $V \in \text{RO}(Y)$ containing $f(x)$, there exists $U \in \delta\text{PO}(X)$ containing $x$ such that $f(U) \subset V$.

**Definition 4.** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be R-map [4] (resp. almost continuous [21], almost α-continuous [16], almost precontinuous [11], δ-continuous [14]) if $f^{-1}(V) \in \text{RO}(X)$ (resp. $f^{-1}(V) \in \tau$, $f^{-1}(V) \in \alpha\text{O}(X)$, $f^{-1}(V) \in \text{PO}(X)$, $f^{-1}(V) \in \delta\text{O}(X)$) for every $V \in \text{RO}(Y)$.

**Remark 5.** The following implications hold:

al. contin. $\Rightarrow$ al. α-contin. $\Rightarrow$ al. precontin. $\Rightarrow$ al. δ-precontin.

The converses are not true in general.

**Example 6.** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Let $f : X \rightarrow X$ be a function defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then, $f$ is almost δ-precontinuous but not almost precontinuous.
The other examples can be seen in [11], [16].

**Theorem 7.** For a function \( f : (X, \tau) \rightarrow (Y, \sigma) \), the following are equivalent:

1. \( f \) is almost \( \delta \)-precontinuous;
2. for each \( x \in X \) and each \( V \in \sigma \) containing \( f(x) \), there exists \( U \in \delta PO(X) \) containing \( x \) such that \( f(U) \subset \text{int}(\text{cl}(V)) \);
3. \( f^{-1}(F) \in \delta PC(X) \) for every \( F \in RC(Y) \);
4. \( f^{-1}(V) \in \delta PO(X) \) for every \( V \in RO(Y) \).
5. \( f(\delta - \text{pcl}(A)) \subset \delta - \text{cl}(f(A)) \) for every subset \( A \) of \( X \);
6. \( \delta - \text{pcl}(f^{-1}(B)) \subset f^{-1}(\delta - \text{cl}(B)) \) for every subset \( B \) of \( Y \);
7. \( f^{-1}(F) \in \delta PC(X) \) for every \( \delta \)-closed set \( F \) of \( (Y, \sigma) \);
8. \( f^{-1}(V) \in \delta PO(X) \) for every \( \delta \)-open set \( V \) of \( (Y, \sigma) \);
9. \( \delta - \text{pcl}(f^{-1}(\text{cl}(\text{int}(\text{cl}(B)))))) \subset f^{-1}(\text{cl}(B)) \) for every subset \( B \) of \( Y \);
10. \( \delta - \text{pcl}(f^{-1}(\text{cl}(F)))) \subset f^{-1}(F) \) for every closed set \( F \) of \( Y \);
11. \( \delta - \text{pcl}(f^{-1}(\text{cl}(V)))) \subset f^{-1}(\text{cl}(V)) \) for every open set \( V \) of \( Y \);
12. \( f^{-1}(V) \subset \delta - \text{pint}(f^{-1}(s - \text{cl}(V))) \) for every open set \( V \) of \( Y \);
13. \( f^{-1}(V) \subset \text{int}(\delta - \text{cl}(f^{-1}(s - \text{cl}(V)))) \) for every open set \( V \) of \( Y \);
14. \( f^{-1}(V) \subset \delta - \text{pint}(f^{-1}(\text{int}(\text{cl}(V)))) \) for every open set \( V \) of \( Y \);
15. \( f^{-1}(V) \subset \text{int}(\delta - \text{cl}(f^{-1}(\text{int}(\text{cl}(V))))) \) for every open set \( V \) of \( Y \);
16. \( \delta - \text{pcl}(f^{-1}(V)) \subset f^{-1}(\text{cl}(V)) \) for each \( V \in \beta O(Y) \);
17. \( \delta - \text{pcl}(f^{-1}(V)) \subset f^{-1}(\text{cl}(V)) \) for each \( V \in SO(Y) \);
18. \( f^{-1}(V) \subset \delta - \text{pint}(f^{-1}(\text{int}(\text{cl}(V)))) \) for each \( V \in PO(Y) \);
19. \( \delta - \text{pcl}(f^{-1}(V)) \subset f^{-1}(\alpha - \text{cl}(V)) \) for each \( V \in \beta O(Y) \);
20. \( \delta - \text{pcl}(f^{-1}(V)) \subset f^{-1}(p - \text{cl}(V)) \) for each \( V \in SO(Y) \);
21. \( f^{-1}(V) \subset \delta - \text{pint}(f^{-1}(s - \text{cl}(V))) \) for each \( V \in PO(Y) \).

**Proof.** (1)\( \Rightarrow \)(2). Let \( x \in X \) and \( V \in \sigma \) containing \( f(x) \). We have \( \text{int}(\text{cl}(V)) \in RO(Y) \). Since \( f \) is almost \( \delta \)-precontinuous, then there exists \( U \in \delta PO(X, x) \) such that \( f(U) \subset \text{int}(\text{cl}(V)) \).

(2)\( \Rightarrow \)(1). Obvious.

(3)\( \Leftrightarrow \)(4). Obvious.

(1)\( \Rightarrow \)(4). Let \( x \in X \) and \( V \in RO(Y, f(x)) \). Since \( f \) is almost \( \delta \)-precontinuous, then there exists \( U_x \in \delta PO(X, x) \) such that \( f(U_x) \subset V \). We have \( U_x \subset f^{-1}(V) \). Thus, \( f^{-1}(V) = \bigcup U_x \in \delta PO(X) \).
(4)⇒(1). Obvious.
(1)⇒(5). Let A be a subset of X. Since $\delta - \text{cl}(f(A))$ is $\delta$-closed in Y, it is denoted by $\cap \{ F_i : F_i \in RC(Y), i \in I \}$, where I is an index set. By (1)⇒(3), we have $A \subset f^{-1}(\delta - \text{cl}(f(A))) = \cap \{ f^{-1}(F_i) : i \in I \} \in \delta PC(X)$ and hence $\delta - \text{pcl}(A) \subset f^{-1}(\delta - \text{cl}(f(A)))$. Therefore, we obtain $f(\delta - \text{pcl}(A)) \subset \delta - \text{cl}(f(A))$.

(5)⇒(6). Let $B$ be a subset of Y. We have $f(\delta - \text{pcl}(f^{-1}(B))) \subset \delta - \text{cl}(f(f^{-1}(B))) \subset \delta - \text{cl}(B)$ and hence $\delta - \text{pcl}(f^{-1}(B)) \subset f^{-1}(\delta - \text{cl}(B))$.

(6)⇒(7). Let $F$ be any $\delta$-closed set of $(Y, \sigma)$. We have $\delta - \text{pcl}(f^{-1}(F)) \subset f^{-1}(\delta - \text{cl}(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is $\delta$-precontinuous in $(X, \tau)$.

(7)⇒(8). Let $V$ be any $\delta$-open set of $(Y, \sigma)$. We have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V) \subset \delta PC(X)$ and hence $f^{-1}(V) \subset \delta PO(X)$.

(8)⇒(1). Let $V$ be any regular open set of $(Y, \sigma)$. Since $V$ is $\delta$-open in $(Y, \sigma)$, $f^{-1}(V) \in \delta PO(X)$ and hence, by (1)⇒(4), $f$ is almost $\delta$-precontinuous.

(1)⇒(9). Let $B$ be any subset of Y. Assume that $x \in X \setminus f^{-1}(\text{cl}(B))$. Then $f(x) \in Y \setminus \text{cl}(B)$ and there exists an open set $V$ containing $f(x)$ such that $V \cap B = \emptyset$; hence $\text{int}(cV) \cap \text{cl}(\text{int}(\text{cl}(B))) = \emptyset$. Since $f$ is almost $\delta$-precontinuous, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset \text{int}(\text{cl}(V))$. Therefore, we have $U \cap f^{-1}(\text{cl}(\text{int}(\text{cl}(B)))) = \emptyset$ and hence $x \in X \setminus \delta - \text{pcl}(f^{-1}(\text{cl}(\text{int}(\text{cl}(B)))))$. Thus we obtain $\delta - \text{pcl}(f^{-1}(\text{cl}(\text{int}(\text{cl}(B))))) \subset f^{-1}(\text{cl}(B))$.

(9)⇒(10). Let $F$ be any closed set of Y. Then we have

$$\delta - \text{pcl}(f^{-1}(\text{cl}(\text{int}(F)))) = \delta - \text{pcl}(f^{-1}(\text{cl}(\text{cl}(F))))$$

$$\subset f^{-1}(\text{cl}(F)) = f^{-1}(F).$$

(10)⇒(11). For any open set $V$ of Y, $\text{cl}(V)$ is regular closed in Y and we have

$$\delta - \text{pcl}(f^{-1}(\text{cl}(V))) = \delta - \text{pcl}(f^{-1}(\text{cl}(\text{cl}(V)))) \subset f^{-1}(\text{cl}(V)).$$

(11)⇒(12). Let $V$ be any open set of Y. Then $Y \setminus \text{cl}(V)$ is open in Y and by using Lemma 2 we have

$$X \setminus \delta - \text{pint}(f^{-1}(s - \text{cl}(V)))$$

$$= \delta - \text{pcl}(f^{-1}(Y \setminus \text{int}(\text{cl}(V)))) \subset f^{-1}(\text{cl}(Y \setminus \text{cl}(V))) \subset X \setminus f^{-1}(V).$$

Therefore, we obtain $f^{-1}(V) \subset \delta - \text{pint}(f^{-1}(s - \text{cl}(V)))$.

(12)⇒(13). Let $V$ be any open set of Y. We obtain $f^{-1}(V) \subset \delta - \text{pint}(f^{-1}(s - \text{cl}(V))) \subset \text{int}(\delta - \text{cl}(f^{-1}(s - \text{cl}(V))))$.

(13)⇒(1). Let $x$ be any point of X and $V$ any open set of Y containing $f(x)$. Then $x \in f^{-1}(\text{int}(\text{cl}(V))) \subset \text{int}(\delta - \text{cl}(f^{-1}(s - \text{cl}(\text{int}(\text{cl}(V)))))) = \text{int}(\delta - \text{cl}(f^{-1}(\text{int}(\text{cl}(V)))))).$ Thus, $f^{-1}(\text{int}(\text{cl}(V))) \in \delta PO(X)$. Take $U = f^{-1}(\text{int}(\text{cl}(V)))$. We obtain $x \in U$ and $f(U) \subset \text{int}(\text{cl}(V))$. Therefore, $f$ is almost $\delta$-precontinuous.

(12)⇔(14) and (13)⇔(15). Obvious.

(1)⇒(16). Let $V$ be any $\beta$-open set of Y. It follows from [2, Theorem 2.4] that $\text{cl}(V)$ is regular closed in Y. Since $f$ is almost $\delta$-precontinuous, by (1)⇒(3), $f^{-1}(\text{cl}(V))$ is $\delta$-preclosed in X. Therefore, we obtain $\delta - \text{pcl}(f^{-1}(V)) \subset f^{-1}(\text{cl}(V))$. 

(16)⇒(17). This is obvious since \( SO(Y) \subset \beta O(Y) \).

(17)⇒(1). Let \( F \) be any regular closed set of \( Y \). Then \( F \) is semi-open in \( Y \) and hence \( \delta - \text{pcl}(f^{-1}(F)) \subset f^{-1}(\text{cl}(F)) = f^{-1}(F) \). This shows that \( f^{-1}(F) \) is \( \delta \)-preclosed. Therefore, by (1)⇔(3), \( f \) is almost \( \delta \)-precontinuous.

(1)⇒(18). Let \( V \) be any preopen set of \( Y \). Then \( V \subset \text{int}(\text{cl}(V)) \) and \( \text{int}(\text{cl}(V)) \) is regular open in \( Y \). Since \( f \) is almost \( \delta \)-precontinuous, by (1)⇔(4), \( f^{-1}(\text{int}(\text{cl}(V))) \) is \( \delta \)-preopen in \( X \) and hence we obtain that \( f^{-1}(V) \subset f^{-1}(\text{int}(\text{cl}(V))) \subset \delta - \text{pint}(f^{-1}(\text{cl}(V))) \).

(18)⇒(1). Let \( V \) be any regular open set of \( Y \). Then \( V \) is preopen and \( f^{-1}(V) \subset \delta - \text{pint}(f^{-1}(\text{cl}(V))) = \delta - \text{pint}(f^{-1}(V)) \). Therefore, \( f^{-1}(V) \) is \( \delta \)-preopen in \( X \) and hence, by (1)⇔(4), \( f \) is almost \( \delta \)-precontinuous.

(16)⇔(19), (17)⇔(20), (18)⇔(21). Obvious.

\[\text{Lemma 8 (Raychaudhuri and Mukherjee [20])}. \text{ A set } S \text{ in } X \text{ is } \delta \text{-preopen if and only if } S \cap G \in \delta PO(X) \text{ for every } \delta \text{-open set } G \text{ of } X.\]

\[\text{Lemma 9 (Raychaudhuri and Mukherjee [20])}. \text{ Let } A \text{ and } X_0 \text{ be subsets of a space } (X, \tau). \text{ If } A \in \delta PO(X) \text{ and } X_0 \in \delta O(X), \text{ then } A \cap X_0 \in \delta PO(X_0).\]

\[\text{Theorem 10. If } f : (X, \tau) \to (Y, \sigma) \text{ is almost } \delta \text{-precontinuous and } A \text{ is } \delta \text{-open in } (X, \tau), \text{ then the restriction } f \mid A : (A, \tau_A) \to (Y, \sigma) \text{ is almost } \delta \text{-precontinuous.}\]

Proof. Let \( V \) be any regular open set of \( Y \). By Theorem 7, we have \( f^{-1}(V) \in \delta PO(X_0) \) and hence \( (f \mid A)^{-1}(V) \cap A \in \delta PO(A) \) by Lemma 9. Thus, it follows that \( f \mid A \) is almost \( \delta \)-precontinuous.

\[\text{Lemma 11 (Raychaudhuri and Mukherjee [20])}. \text{ Let } A \text{ and } X_0 \text{ be subsets of a space } (X, \tau). \text{ If } A \in \delta PO(X_0) \text{ and } X_0 \in \delta O(X), \text{ then } A \in \delta PO(X).\]

\[\text{Theorem 12. Let } f : (X, \tau) \to (Y, \sigma) \text{ be a function and } \{U_i : i \in I\} \text{ a cover of } X \text{ by } \delta \text{-open sets of } (X, \tau). \text{ If } f \mid U_i : (U_i, \tau_{U_i}) \to (Y, \sigma) \text{ is almost } \delta \text{-precontinuous for each } i \in I, \text{ then } f \text{ is almost } \delta \text{-precontinuous.}\]

Proof. Let \( V \) be any regular open set of \( (Y, \sigma) \). Then, we have \( f^{-1}(V) = X \cap f^{-1}(V) = \cap \{U_i \cap f^{-1}(V) : i \in I\} = \cup \{(f \mid U_i)^{-1}(V) : i \in I\}. \)

Since \( f \mid U_i \) is almost \( \delta \)-precontinuous, \( (f \mid U_i)^{-1}(V) \in \delta PO(U_i) \) for each \( i \in I \). By Lemma 11, for each \( i \in I \), \( (f \mid U_i)^{-1}(V) \) is \( \delta \)-preopen in \( X \) and hence \( f^{-1}(V) \) is \( \delta \)-preopen in \( X \). Therefore, \( f \) is almost \( \delta \)-precontinuous.

\[\text{Theorem 13. Let } f : (X, \tau) \to (Y, \sigma) \text{ be a function and } g : (X, \tau) \to (X \times Y, \tau \times \sigma) \text{ the graph function defined by } g(x) = (x, f(x)) \text{ for every } x \in X. \text{ Then } g \text{ is almost } \delta \text{-precontinuous if and only if } f \text{ is almost } \delta \text{-precontinuous.}\]

Proof. \( (\Rightarrow) \). Let \( x \in X \) and \( V \in RO(Y) \) containing \( f(x) \). Then, we have \( g(x) = (x, f(x)) \in X \times V \in RO(X \times Y) \). Since \( g \) is almost \( \delta \)-precontinuous, there exists a \( \delta \)-preopen set \( U \) of \( X \) containing \( x \) such that \( g(U) \subset X \times V \). Therefore, we obtain \( f(U) \subset V \) and hence \( f \) is almost \( \delta \)-precontinuous.
Let $x \in X$ and $W$ be a regular open set of $X \times Y$ containing $g(x)$. There exist $U_1 \in RO(X)$ and $V \in RO(Y)$ such that $(x, f(x)) \in U_1 \times V \subset W$. Since $f$ is almost $\delta$-precontinuous, there exists $U_2 \in \delta P O(X)$ such that $x \in U_2$ and $f(U_2) \subset V$. Put $U = U_1 \cap U_2$, then we obtain $x \in U \in \delta P O(X)$ and $g(U) \subset U_1 \times V \subset W$. This shows that $g$ is almost $\delta$-precontinuous.

**Definition 14.** The $\delta$-prefrontier of a subset $A$ of $X$, denoted by $\delta - pfr(A)$, is defined by $\delta - pfr(A) = \delta - pcl(A) \cap \delta - pcl(X \setminus A) = \delta - pcl(A) \setminus \delta - pint(A)$.

**Theorem 15.** The set of all points $x$ of $X$ at which a function $f : X \rightarrow Y$ is not almost $\delta$-precontinuous is identical with the union of the $\delta$-prefrontiers of the inverse images of regular open sets containing $f(x)$.

**Proof.** Let $x$ be a point of $X$ at which $f$ is not almost $\delta$-precontinuous. Then, there exists a regular open set $V$ of $Y$ containing $f(x)$ such that $U \cap (X \setminus f^{-1}(V)) \neq \emptyset$ for every $U \in \delta P O(X, x)$. Therefore, we have $x \in \delta - pcl(X \setminus f^{-1}(V)) = X \setminus \delta - pint(f^{-1}(V))$ and $x \in f^{-1}(V)$. Thus, we obtain $x \in \delta - pfr(f^{-1}(V))$.

Conversely, suppose that $f$ is almost $\delta$-precontinuous at $x \in X$ and let $V$ be a regular open set containing $f(x)$. Then there exists $U \in \delta P O(X, x)$ such that $U \subset f^{-1}(V)$; hence $x \in \delta - pint(f^{-1}(V))$. Therefore, it follows that $x \in X \setminus \delta - pfr(f^{-1}(V))$. This completes the proof. \qed

**Definition 16.** A space $X$ is said to be $\delta$-pre-$T_2$ [6] if for any distinct points $x, y$ of $X$, there exist disjoint $\delta$-preopen sets $U, V$ of $X$ such that $x \in U$ and $y \in V$.

**Definition 17.** A function $f : X \rightarrow Y$ is said to be weakly $\delta$-precontinuous if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \delta P O(X, x)$ such that $f(U) \subset cl(V)$.

**Theorem 18.** If for each pair of distinct points $x_1$ and $x_2$ in a space $X$, there exists a function $f$ of $X$ into a Hausdorff space $Y$ such that

1. $f(x_1) \neq f(x_2)$,
2. $f$ is weakly $\delta$-precontinuous at $x_1$ and
3. almost $\delta$-precontinuous at $x_2$,

then $X$ is $\delta$-pre-$T_2$.

**Proof.** Since $Y$ is Hausdorff, there exist open sets $V_1$ and $V_2$ of $Y$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$; hence $cl(V_1) \cap int(cl(V_2)) = \emptyset$. Since $f$ is weakly $\delta$-precontinuous at $x_1$, there exists $U_1 \in \delta P O(X, x_1)$ such that $f(U_1) \subset cl(V_1)$. Since $f$ is almost $\delta$-precontinuous at $x_2$, there exists $U_2 \in \delta P O(X, x_2)$ such that $f(U_2) \subset int(cl(V_2))$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$. This shows that $X$ is $\delta$-pre-$T_2$. \qed

Let $f : X \rightarrow Y$ be a function. The subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of $f$ and is denoted by $G(f)$.

**Definition 19.** A function $f : X \rightarrow Y$ has a $(\delta_p, r)$-graph if for each $(x, y) \in X \times Y \setminus G(f)$, there exist $U \in \delta P O(X, x)$ and a regular open set $V$ of $Y$ containing...
\[ \text{Lemma 20. A function } f : X \rightarrow Y \text{ has a } (\delta_p, r)\text{-graph if and only if for each } (x, y) \in X \times Y \text{ such that } y \neq f(x), \text{ there exist a } \delta\text{-preopen set } U \text{ and a regular open set } V \text{ containing } x \text{ and } y, \text{ respectively, such that } f(U) \cap V = \emptyset. \]

\[ \text{Theorem 21. If } f : X \rightarrow Y \text{ is an almost } \delta\text{-precontinuous function and } Y \text{ is Hausdorff, then } f \text{ has a } (\delta_p, r)\text{-graph.} \]

Proof. Let \((x, y) \in X \times Y \text{ such that } y \neq f(x)\). Then there exist open sets \(V \subseteq Y\) such that \(y \in V\), \(f(x) \in W\) and \(V \cap W = \emptyset\); hence \(\text{int(cl}(V)) \cap \text{int(cl}(W)) = \emptyset\). Since \(f\) is almost \(\delta\)-precontinuous, there exists \(U \subseteq \delta PO(X, x)\) such that \(f(U) \subseteq \text{int(cl}(V))\). This implies that \(f(U) \cap \text{cl}(V) = \emptyset\). Therefore, \(f\) has a \((\delta_p, r)\)-graph. \(\square\)

\[ \text{Definition 22. A space } X \text{ is said to be } \delta\text{-precompact [6] if every } \delta\text{-preopen cover of } X \text{ has a finite subcover.} \]

\[ \text{Theorem 23. If } f : (X, \tau) \rightarrow (Y, \sigma) \text{ has a } (\delta_p, r)\text{-graph, then } f(K) \text{ is } \delta\text{-closed in } (Y, \sigma) \text{ for each subset } K \text{ which is } \delta\text{-precompact relative to } (X, \tau). \]

Proof. Suppose that \(y \notin f(K)\). Then \((x, y) \notin G(f)\) for each \(x \in K\). Since \(G(f)\) is \((\delta_p, r)\)-graph, there exist \(U_x \subseteq \delta PO(X)\) containing \(x\) and a regular open set \(V_x\) of \(Y\) containing \(y\) such that \(f(U_x) \cap V_x = \emptyset\). The family \(\{U_x : x \in K\}\) is a cover of \(K\) by \(\delta\)-preopen sets. Since \(K\) is \(\delta\)-precompact relative to \((X, \tau)\), there exists a finite subset \(K_0\) of \(K\) such that \(K \subseteq \bigcup\{U_x : x \in K_0\}\). Set \(V = \bigcap\{V_x : x \in K_0\}\). Then \(V\) is a regular open set in \(Y\) containing \(y\). Therefore, we have \(f(K) \cap V \subseteq \bigcup_{x \in K_0} f(U_x) \cap V \subseteq \bigcup_{x \in K_0} [f(U_x) \cap V] = \emptyset\). It follows that \(y \notin \text{cl}(f(K))\). Therefore, \(f(K)\) is \(\delta\)-closed in \((Y, \sigma)\). \(\square\)

\[ \text{Corollary 24. If } f : (X, \tau) \rightarrow (Y, \sigma) \text{ is an almost } \delta\text{-precontinuous function and } Y \text{ is Hausdorff, then } f(K) \text{ is } \delta\text{-closed in } (Y, \sigma) \text{ for each subset } K \text{ which is } \delta\text{-precompact relative to } (X, \tau). \]

\[ \text{Theorem 25. If } f : X \rightarrow Y \text{ is almost } \delta\text{-precontinuous, } g : X \rightarrow Y \text{ is } \delta\text{-continuous and } Y \text{ is Hausdorff, then the set } \{x \in X : f(x) = g(x)\} \text{ is } \delta\text{-preclosed in } X. \]

Proof. Let \(A = \{x \in X : f(x) = g(x)\}\) and \(x \in X \setminus A\). Then \(f(x) \neq g(x)\). Since \(Y\) is Hausdorff, there exist open sets \(V \subseteq Y\) and \(W \subseteq Y\) such that \(f(x) \in V\), \(g(x) \in W\) and \(V \cap W = \emptyset\); hence \(\text{int(cl}(V)) \cap \text{int(cl}(W)) = \emptyset\). Since \(f\) is almost \(\delta\)-precontinuous, there exists \(G \subseteq \delta PO(X, x)\) such that \(f(G) \subseteq \text{int(cl}(V))\). Since \(g\) is \(\delta\)-continuous, there exists an \(\delta\)-open set \(H\) of \(X\) containing \(x\) such that \(g(H) \subseteq \text{int(cl}(W))\). Now, put \(U = G \cap H\), then \(U \subseteq \delta PO(X, x)\) and \(f(U) \cap g(U) \subseteq \text{int(cl}(V)) \cap \text{int(cl}(W)) = \emptyset\). Therefore, we obtain \(U \cap A = \emptyset\) and hence \(x \in X \setminus \delta\) - \(\text{pcl}(A)\). This shows that \(A\) is \(\delta\)-preclosed in \(X\). \(\square\)

\[ \text{Theorem 26. If } f_1 : X_1 \rightarrow Y \text{ is weakly } \delta\text{-precontinuous, } f_2 : X_2 \rightarrow Y \text{ is almost } \delta\text{-precontinuous and } Y \text{ is Hausdorff, then the set } \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\} \]
is $\delta$-preclosed in $X_1 \times X_2$.

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = f(x_2)\}$ and $A = \{(x_1, x_2) \in (X_1 \times X_2) \setminus A\}$. Then $f(x_1) \neq f(x_2)$ and there exist open sets $V_1$ and $V_2$ of $Y$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Hence, $\text{cl}(V_1) \cap \text{int}(\text{cl}(V_2)) = \emptyset$. Since $f_1$ (resp., $f_2$) is weakly $\delta$-continuous (resp., almost $\delta$-precontinuous), there exists $U_1 \in \delta\text{PO}(X_1, x_1)$ such that $f_1(U_1) \subset \text{cl}(V_1)$ (resp., $U_2 \in \delta\text{PO}(X_2, x_2)$ such that $f_2(U_2) \subset \text{int}((\text{cl}(V_2)))$. Therefore, we obtain $(x_1, x_2) \in U_1 \times U_2 \subset (X_1 \times X_2) \setminus A$ and $U_1 \times U_2 \in \delta\text{PO}(X_1 \times X_2)$. This shows that $A$ is $\delta$-preclosed in $X_1 \times X_2$. \hfill $\square$

Let $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ be any two families of spaces with the same index set $I$. For each $i \in I$, let $f_i : X_i \rightarrow Y_i$ be a function. The product space $\prod_{i \in I} X_i$ will be denoted by $\prod X_i$, and the product function $\prod f_i : \prod X_i \rightarrow \prod Y_i$ is simply denoted by $f : \prod X_i \rightarrow \prod Y_i$.

**Theorem 27.** If a function $f : X \rightarrow \prod Y_i$ is almost $\delta$-precontinuous, then $p_i \circ f : X \rightarrow Y_i$ is almost $\delta$-precontinuous for each $i \in I$, where $p_i$ is the projection of $\prod Y_i$ onto $Y_i$.

Proof. Let $V_i$ be any regular open set of $Y_i$. Since $p_i$ is continuous open, it is an R-map and hence $p_i^{-1}(V_i) \in \text{RO}(\prod Y_i)$. By Theorem 7, $f^{-1}(p_i^{-1}(V_i)) = (p_i \circ f)^{-1}(V_i) \in \delta\text{PO}(X)$. This shows that $p_i \circ f$ is almost $\delta$-precontinuous for each $i \in I$. \hfill $\square$

**Theorem 28.** The product function $f : \prod X_i \rightarrow \prod Y_i$ is almost $\delta$-precontinuous if and only if $f_i : X_i \rightarrow Y_i$ is almost $\delta$-precontinuous for each $i \in I$.

Proof. (Necessity). Let $k$ be an arbitrarily fixed index and $V_k$ any regular open set of $Y_k$. Then $\prod Y_j \times V_k$ is regular open in $\prod Y_i$, where $j \in I$ and $j \neq k$, and hence $f^{-1}(\prod Y_j \times V_k) = \prod Y_j \times f_k^{-1}(V_k)$ is $\delta$-preopen in $\prod X_i$. Thus, $f_k^{-1}(V_k)$ is $\delta$-preopen in $X_k$ and hence $f_k$ is almost $\delta$-precontinuous.

(Sufficiency). Let $\{x_i\}$ be any point of $\prod X_i$ and $W$ any regular open set of $\prod Y_i$ containing $f(\{x_i\})$. There exists a finite subset $I_0$ of $I$ such that $V_k \in \text{RO}(Y_k)$ for each $k \in I_0$ and $\{f_i(x_i)\} \in \prod \{V_k : k \in I_0\} \times \prod \{Y_j : j \notin I \setminus I_0\} \subset W$. For each $k \in I_0$, there exists $U_k \in \delta\text{PO}(X_k)$ containing $x_k$ such that $f_k(U_k) \subset V_k$. Thus, $U = \prod \{U_k : k \in I_0\} \times \prod \{X_j : j \in I \setminus I_0\}$ is a $\delta$-preopen set of $\prod X_i$ containing $\{x_i\}$ and $f(U) \subset W$. This shows that $f$ is almost $\delta$-precontinuous. \hfill $\square$

3. Functions

**Definition 29.** Let $(X, \tau)$ be a topological space. The collection of all regular open sets forms a base for a topology $\tau_s$. It is called the semiregularization. In case when $\tau = \tau_s$, the space $(X, \tau)$ is called semi-regular [23].

**Theorem 30.** Let $(X, \tau)$ be a semi-regular space. Then a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is almost precontinuous if and only if it is almost $\delta$-precontinuous.

**Definition 31.** A function $f : X \rightarrow Y$ is said to be $\delta$-almost continuous [20] if for
each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in \delta PO(X, x) \) such that \( f(U) \subset V \).

**Definition 32.** A function \( f : X \to Y \) is said to be \( \delta \)-preirresolute if for each \( x \in X \) and each \( \delta \)-preopen set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in \delta PO(X, x) \) such that \( f(U) \subset V \).

**Definition 33.** A function \( f : X \to Y \) is said to be almost \( \delta \)-precontinuous if for

**Theorem 34.** If \( f : X \to Y \) is an almost \( \delta \)-preopen and weakly \( \delta \)-precontinuous function, then \( f \) is almost \( \delta \)-precontinuous.

**Proof.** Let \( x \in X \) and let \( V \) be an open set of \( Y \) containing \( f(x) \). Since \( f \) is weakly \( \delta \)-precontinuous, there exists \( U \in \delta PO(X, x) \) such that \( f(U) \subset \text{cl}(V) \). Since \( f \) is almost \( \delta \)-preopen, \( f(U) \subset \text{int}(\text{cl}(f(U))) \subset \text{int}(\text{cl}(V)) \) and hence \( f \) is almost \( \delta \)-precontinuous. \( \Box \)

**Definition 35.** A space \( X \) is said to be

(1) almost regular [22] if for any regular closed set \( F \) of \( X \) and any point \( x \in X \backslash F \) there exist disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( F \subset V \).

(2) semi-regular if for any open set \( U \) of \( X \) and each point \( x \in U \) there exists a regular open set \( V \) of \( X \) such that \( x \in V \subset U \).

**Theorem 36.** If \( f : X \to Y \) is a weakly \( \delta \)-precontinuous function and \( Y \) is almost regular, then \( f \) is almost \( \delta \)-precontinuous.

**Proof.** Let \( x \in X \) and let \( V \) be any open set of \( Y \) containing \( f(x) \). By the almost regularity of \( Y \), there exists a regular open set \( G \) of \( Y \) such that \( f(x) \in G \subset \text{cl}(G) \subset \text{int}(\text{cl}(V)) \) [22, Theorem 2.2]. Since \( f \) is weakly \( \delta \)-precontinuous, there exists \( U \in \delta PO(X, x) \) such that \( f(U) \subset \text{cl}(G) \subset \text{int}(\text{cl}(V)) \). Therefore, \( f \) is almost \( \delta \)-precontinuous. \( \Box \)

**Theorem 37.** If \( f : X \to Y \) is an almost \( \delta \)-precontinuous function and \( Y \) is semi-
regular, then \( f \) is \( \delta \)-almost continuous.

**Proof.** Let \( x \in X \) and let \( V \) be any open set of \( Y \) containing \( f(x) \). By the semi-
regularity of \( Y \), there exists a regular open set \( G \) of \( Y \) such that \( f(x) \in G \subset V \). Since \( f \) is almost \( \delta \)-precontinuous, there exists \( U \in \delta PO(X, x) \) such that \( f(U) \subset \text{int}(\text{cl}(G)) = G \subset V \) and hence \( f \) is \( \delta \)-almost continuous. \( \Box \)

**Theorem 38.** Let \( f : X \to Y \) and \( g : Y \to Z \) be functions. Then the following hold:

1. If \( f \) is almost \( \delta \)-precontinuous and \( g \) is an \( R \)-map, then the composition \( g \circ f : X \to Z \) is almost \( \delta \)-precontinuous.

2. If \( f \) is \( \delta \)-preirresolute and \( g \) is almost \( \delta \)-precontinuous, the composition \( g \circ f : X \to Z \) is almost \( \delta \)-precontinuous.

**Definition 39.** A function \( f : X \to Y \) is said to be faintly \( \delta \)-precontinuous if for
each \( x \in X \) and each \( \theta \)-open set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in \delta PO(X, x) \) such that \( f(U) \subset V \).

**Theorem 40.** Let \( f : X \rightarrow Y \) be a function. Suppose that \( Y \) is regular. Then, the following properties are equivalent:

1. \( f \) is \( \delta \)-almost continuous,
2. \( f^{-1}(\delta - \text{cl}(B)) \) is \( \delta \)-preclosed in \( X \) for every subset \( B \) of \( Y \),
3. \( f \) is almost \( \delta \)-precontinuous,
4. \( f \) is weakly \( \delta \)-precontinuous,
5. \( f \) is faintly \( \delta \)-precontinuous.

**Proof.** (1)\( \Rightarrow \) (2). Since \( \delta - \text{cl}(B) \) is closed in \( Y \) for every subset \( B \) of \( Y \), \( f^{-1}(\delta - \text{cl}(B)) \) is \( \delta \)-preclosed in \( X \).

(2)\( \Rightarrow \) (3). For any subset \( B \) of \( Y \), \( f^{-1}(\delta - \text{cl}(B)) \) is \( \delta \)-preclosed in \( X \) and hence we have \( \delta - \text{pcl}(f^{-1}(B)) \subset \delta - \text{pcl}(f^{-1}(\delta - \text{cl}(B))) = f^{-1}(\delta - \text{cl}(B)) \). It follows that \( f \) is almost \( \delta \)-precontinuous

(3)\( \Rightarrow \) (4). This is obvious.

(4)\( \Rightarrow \) (5). Let \( A \) be any subset of \( X \). Let \( x \in \delta - \text{pcl}(A) \) and \( V \) be any open set of \( Y \) containing \( f(x) \). There exists \( U \in \delta PO(X, x) \) such that \( f(U) \subset \text{cl}(V) \). Since \( x \in \delta - \text{pcl}(A) \), we have \( U \cap A \neq \emptyset \) and hence \( f(U) \cap f(A) \subset \text{cl}(V) \cap f(A) \).

Therefore, we have \( f(x) \in \theta - \text{cl}(f(A)) \) and hence \( f(\delta - \text{pcl}(A)) \subset \theta - \text{cl}(f(A)) \).

Let \( B \) be any subset of \( Y \). We have \( f(\delta - \text{pcl}(f^{-1}(B))) \subset \theta - \text{cl}(B) \) and \( \delta - \text{pcl}(f^{-1}(B)) \subset f^{-1}(\theta - \text{cl}(B)) \). It follows that \( \delta - \text{pcl}(f^{-1}(F)) \subset f^{-1}(\theta - \text{cl}(F)) = f^{-1}(F) \).

Therefore \( f^{-1}(F) \) is \( \delta \)-preclosed in \( X \) and hence \( f \) is faintly \( \delta \)-precontinuous.

(5)\( \Rightarrow \) (1). Let \( V \) be any open set of \( Y \). Since \( Y \) is regular, \( V \) is \( \theta \)-open in \( Y \). By the faint \( \delta \)-precontinuity of \( f \), \( f^{-1}(V) \) is \( \delta \)-preopen in \( X \). Therefore, \( f \) is \( \delta \)-almost continuous. \( \square \)

Recall that a space \((X, \tau)\) is said to be (1) submaximal \([3]\) if every dense subset of \( X \) is open in \( X \), (2) extremally disconnected \([3, 15]\) if \( \text{cl}(U) \in \tau \) for every \( U \in \tau \).

**Definition 41.** A function \( f : X \rightarrow Y \) is said to be faintly continuous \([9]\) (resp. faintly semi-continuous \([19]\), faintly precontinuous \([19]\), faintly \( \beta \)-continuous \([12]\), \([9]\), faintly \( \alpha \)-continuous \([12]\)) if \( f^{-1}(V) \) is open (resp. semi-open, preopen, \( \beta \)-open, \( \alpha \)-open) in \( X \) for each \( \theta \)-open set \( V \) of \( Y \).

**Theorem 42.** If \((X, \tau)\) is submaximal extremally disconnected semi-regular and \((Y, \sigma)\) is regular, then the following are equivalent for a function \( f : (X, \tau) \rightarrow (Y, \sigma) :\)

1. \( f \) is faintly continuous,
2. \( f \) is faintly \( \alpha \)-continuous,
(3) $f$ is faintly semi-continuous,
(4) $f$ is faintly precontinuous,
(5) $f$ is faintly $\delta$-precontinuous,
(6) $f$ is faintly $\gamma$-continuous,
(7) $f$ is faintly $\beta$-continuous,
(8) $f$ is $\delta$-almost continuous,
(9) $f$ is almost $\delta$-precontinuous,
(10) $f$ is weakly $\delta$-precontinuous.

References


