Weighted Sharing of Two Sets

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Abstract. Using the notion of weighted sharing of sets we improve two results of H. X. Yi on uniqueness of meromorphic functions.

1. Introduction, definitions and results

Let $f$ and $g$ be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f$ and $g$ have the same set of $a$-points with same multiplicities then we say that $f$ and $g$ share the value $a$ CM (counting multiplicities). If we do not take the multiplicities into account, $f$ and $g$ are said to share the value $a$ IM (ignoring multiplicities).

Let $S$ be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a \in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$.

In the paper we denote by $S_1$ and $S_2$ the following sets $S_1 = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$ and $S_2 = \{\infty\}$, where $\omega = \cos \frac{2\pi}{n} + isin \frac{2\pi}{n}$ and $n$ is a positive integer.

Yi ([6], [8]), Song-Li ([5]) and other authors investigate the problem of uniqueness of two meromorphic functions $f$, $g$ for which $E_f(S_i) = E_g(S_i)$ or $\overline{E}_f(S_i) = \overline{E}_g(S_i)$, where $i = 1, 2$.

In 1997 H. X. Yi and L. Z. Yang proved the following two results.

**Theorem A ([10]).** Let $f$ and $g$ be two nonconstant meromorphic functions such that $E_f(S_1) = E_g(S_1)$ and $\overline{E}_f(S_2) = \overline{E}_g(S_2)$. If $n \geq 6$ then one of the following hold:

1. $f \equiv tg$,
2. $f.g \equiv s$,

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where \( s^n = 1 \) and \( 0, \infty \) are lacunary values of \( f \) and \( g \).

**Theorem B ([10])**. Let \( f \) and \( g \) be two nonconstant meromorphic functions such that \( E_f(S_1) = E_g(S_1) \) and \( E_f(S_2) = E_g(S_2) \). If \( n \geq 10 \) then \( f \) and \( g \) satisfy (1) or (2).

In the paper, we investigate the possibility of improving Theorem A and B by relaxing the nature of sharing the sets. To this end we employ the idea of weighted sharing of values and sets introduced in [2], [3] which measures how close a shared value is to being shared IM or to being shared CM. In the following definition we explain this notion.

**Definition 1 ([2], [3])**. Let \( k \) be a nonnegative integer or infinity. For \( a \in \mathbb{C} \cup \{ \infty \} \) we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \), where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(a; f) = E_k(a; g) \), we say that \( f, g \) share the value \( a \) IM or CM if and only if \( f, g \) share \( (a, 0) \) or \( (a, \infty) \) respectively.

**Definition 2 ([3])**. Let \( S \) be a set of distinct elements of \( \mathbb{C} \cup \{ \infty \} \) and \( k \) be a nonnegative integer or \( \infty \). We denote by \( E_f(S, k) \) the set \( \cup_{a \in S} E_k(a; f) \).

Clearly \( E_f(S) = E_f(S, \infty) \) and \( \overline{E_f}(S) = \overline{E_f}(S, 0) \).

We now state the main results of the paper.

**Theorem 1**. If \( E_f(S_1, 2) = E_g(S_1, 2), E_f(S_2, 0) = E_g(S_2, 0) \) and \( n \geq 6 \) then \( f, g \) satisfy one of (1) and (2).

**Theorem 2**. If \( E_f(S_1, 0) = E_g(S_1, 0), E_f(S_2, 3) = E_g(S_2, 3) \) and \( n \geq 10 \) then \( f, g \) satisfy one of (1) and (2).

Though for the standard definitions and notations of the value distribution theory we refer to [1], we now explain some notations which are used in the paper.

**Definition 3 ([2], [3])**. We denote by \( N(r, a; f) = 1 \) the counting function of simple \( a \)-points of \( f \).

**Definition 4 ([2], [3])**. If \( s \) is a positive integer, we denote by \( \overline{N}(r, a; f) \geq s \) the reduced counting function of those \( a \) points of \( f \) whose multiplicities are not less than \( s \).

**Definition 5 ([2], [3])**. Let \( f, g \) share a value \( a \) IM. We denote by \( \overline{N}_*(r, a; f, g) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicities differ from the multiplicities of the corresponding \( a \)-points of \( g \).

Clearly \( \overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) \).

**Definition 6 ([10])**. Let \( f, g \) share a value \( a \) IM. Let \( z_0 \) be an \( a \)-point of \( f \) with
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multiplicity $p$ and an $a$-point of $g$ with multiplicity $q$. We denote by $N_L(r, a; f)$ the reduced counting function of those $a$-points of $f$ where $p > q$ and by $N_L^{(1)}(r, a; f)$ the counting function of those $a$-points of $f$ where $p = q = 1$. Also by $N_L^{(2)}(r, a; f)$ we denote the counting function of those $a$-points of $f$ where $p = q \geq 2$.

Clearly $N_*(r, a; f, g) = N_L(r, a; f) + N_L(r, a; g)$.

**Definition 7.** Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f|g = b)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are $b$-points of $g$.

**Definition 8.** Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f|g \not= b_1, b_2, \ldots, b_q)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are not the $b_i$-points of $g$ for $i = 1, 2, \ldots, q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let $F$ and $G$ be two nonconstant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by $H$ and $V$ the following two functions

$$H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} - \frac{2G'}{G-1},$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

**Lemma 1 ([10]).** If $F, G$ share $(1, 0)$ and $H \not= 0$ then

$$N_L^{(1)}(r, 1; F) \leq N(r, H) + S(r, F) + S(r, G).$$

**Lemma 2 ([4]).** The following holds

$$N(r, 0; F|F \not= 0) \leq N(r, \infty; F) + N(r, 0; F) + S(r, F).$$

**Lemma 3.** If $F$ and $G$ share $(1, 0)$ then

$$T(r, F) \leq N_L^{(1)}(r, 1; F) + 2N(r, 0; F) + 2N(r, \infty; F) + N(r, 0; G) + N(r, \infty; G) - 2N_0(r, 0; F') - N_0(r, 0; G') + S(r, F) + S(r, G),$$

where $N_0(r, 0; F')$ is the counting function of those zeros of $F'$ which are not the zeros of $F(F - 1)$ and $N_0(r, 0; G')$ is similarly defined.
Proof. In view of Lemma 2 we get
\[
\overline{N}(r, 1; F) = N_1^1(r, 1; F) + \overline{N}_E(r, 1; F) + N(r, 1; G)
\]
\[
\leq N_1^1(r, 1; F) + \overline{N}(r, 1; F \geq 2) + \overline{N}(r, 1; G \geq 2)
\]
\[
\leq N_1^1(r, 1; F) + N(r, 0; F') F = 1) + N(r, 0; G') G = 1)
\]
\[
\leq N_1^1(r, 1; F) + N(r, 0; F') F \neq 0) + N(r, 0; G') G \neq 0)
\]
\[
- N_0(r, 0; F') - N_0(r, 0; G')
\]
\[
\leq N_1^1(r, 1; F) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 0; G)
\]
\[
+ \overline{N}(r, \infty; G) - N_0(r, 0; F') - N_0(r, 0; G') + S(r, F)
\]
\[
+ S(r, G).
\]
So by the second fundamental theorem we obtain
\[
T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 1; F) - N_0(r, 0; F') + S(r, F)
\]
\[
\leq N_1^1(r, 1; F) + 2 \overline{N}(r, 0; F) + 2 \overline{N}(r, \infty; F) + \overline{N}(r, 0; G)
\]
\[
+ \overline{N}(r, \infty; G) - 2 N_0(r, 0; F') - N_0(r, 0; G') + S(r, F)
\]
\[
+ S(r, G).
\]
This proves the lemma. \(\square\)

Lemma 4. If \(F, G\) share \((1, 0), (\infty, 0)\) and \(H \neq 0\) then
\[
N(r, H) \leq \overline{N}(r, 0; F \geq 2) + \overline{N}(r, 0; G \geq 2) + \overline{N}_*(r, 1; F, G)
\]
\[
+ \overline{N}_*(r, \infty; F, G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'),
\]
where \(\overline{N}_0(r, 0; F')\) is the reduced counting function of those zeros of \(F'\) which are not the zeros of \(F(F - 1)\) and \(\overline{N}_0(r, 0; G')\) is similarly defined.

Proof. We can easily verify that possible poles of \(H\) occur at

(i) multiple zeros of \(F\) and \(G\),

(ii) those poles of \(F\) and \(G\) whose multiplicities are distinct from the multiplicities of the corresponding poles of \(G\) and \(F\) respectively,

(iii) those 1-points of \(F\) and \(G\) whose multiplicities are distinct from the multiplicities of the corresponding 1-points of \(G\) and \(F\) respectively,

(iv) zeros of \(F'\) which are not the zeros of \(F(F - 1)\),

(v) zeros of \(G'\) which are not zeros of \(G(G - 1)\).

Since \(H\) has only simple poles, the lemma follows from above. This proves the lemma. \(\square\)
Lemma 5 ([7]). If \( H \equiv 0 \) then \( T(r, G) = T(r, F) + O(1) \). Also if \( H \equiv 0 \) and
\[
\limsup_{r \to \infty} \frac{N(r,0;F) + N(r,\infty;F) + N(r,0;G) + N(r,\infty;G)}{T(r,F)} < 1
\]
where \( I \subseteq (0,\infty) \) is a set of infinite linear measure, then \( F \equiv G \) or \( F.G \equiv 1 \).

Remark 1. Let \( F = f^n \) and \( G = g^n \), where \( n \geq 5 \) is an integer. If \( H \equiv 0 \) then from Lemma 5 it follows that \( f \) and \( g \) satisfy one of (1) and (2).

Lemma 6 ([9]). If \( F, G \) share \((\infty,0)\) and \( V \equiv 0 \) then \( F \equiv G \).

Lemma 7. Let \( F = f^n, G = g^n \) and \( V \not\equiv 0 \). If \( f, g \) share \((\infty,k)\), where \( 0 \leq k < \infty \), then the poles of \( F \) and \( G \) are the zeros of \( V \) and
\[
(nk+n-1)N(r,\infty;f) \geq k + 1 \leq N(r,\infty;V) + S(r,f) + S(r,g).
\]

Proof. Since \( f, g \) share \((\infty,k)\), it follows that \( F, G \) share \((\infty,nk)\) and so a pole of \( F \) with multiplicity \( p \geq nk+1 \) is a pole of \( G \) with multiplicity \( r \geq nk+1 \) and vice-versa. Noting that \( F \) and \( G \) have no pole of multiplicity \( q \) where \( nk < q < nk+n \), we get from the definition of \( V \)
\[
(nk+n-1)N(r,\infty;f) \geq k + 1 = (nk+n-1)N(r,\infty;f) \geq nk + n
\]
\[
\leq N(r,0;V)
\]
\[
\leq N(r,\infty;V) + S(r,f) + S(r,g).
\]

Lemma 8. Let \( F = f^n, G = g^n \) and \( V \not\equiv 0 \). If \( f,g \) share \((\infty,k)\), where \( 0 \leq k < \infty \), and \( F, G \) share \((1,0)\) then
\[
(nk+n-1)N(r,\infty;f) \geq k + 1 \leq 2N(r,0;f) + 2N(r,0;g) + 2N(r,\infty;f)
\]
\[
-N(r,0;f) \neq 0, 1, \omega, \cdots, \omega^{n-1}
\]
\[
-N(r,0;g) \neq 0, 1, \omega, \cdots, \omega^{n-1}
\]
\[
+S(r,f) + S(r,g).
\]
Proof. From the definition of $V$ and Lemma 2 it follows that
\[
N(r, \infty; V) \leq N(r, 0; F) + N(r, 0; G) + N(r, 1; F, G)
\]
\[
\leq N(r, 0; F) + N(r, 0; G) + N(r, 1; F, G)
\]
\[
\leq N(r, 0; f) + N(r, 0; g) + N(r, 1; F \geq 2) + N(r, 1; G \geq 2)
\]
\[
\leq N(r, 0; f) + N(r, 0; g) + N(r, 0; F \neq 0) + N(r, 0; G \neq 0)
\]
\[
- N_0(r, 0; F') - N_0(r, 0; G')
\]
\[
\leq 2 N(r, 0; f) + 2 N(r, 0; g) + 2 N(r, \infty; f) - N_0(r, 0; F')
\]
\[
- N_0(r, 0; G') + S(r, f) + S(r, g).
\]
Noting that $N_0(r, 0; F') = N(r, 0; f \neq 0, 1, \omega, \cdots, \omega^{n-1})$ and $N_0(r, 0; G') = N(r, 0; g \neq 0, 1, \omega, \cdots, \omega^{n-1})$, the lemma follows from above and Lemma 7. This proves the lemma.

**Lemma 9.** Let $F = f^n$, $G = g^n$ and $V \neq 0$. If $f, g$ share $(\infty, 0)$ and $F, G$ share $(1, k)$, where $1 \leq k \leq \infty$, then
\[
(n - 1 - \frac{1}{k}) N(r, \infty; f) \leq \frac{k+1}{k} N(r, 0; f) + N(r, 0; g)
\]
\[
- \frac{1}{k} N(r, 0; f \neq 0, 1, \omega, \cdots, \omega^{n-1}) + S(r, f) + S(r, g).
\]

Proof. From the definition of $V$ and Lemma 2 we get
\[
N(r, \infty; V) \leq N(r, 0; F) + N(r, 0; G) + N(r, 1; F, G)
\]
\[
\leq N(r, 0; f) + N(r, 0; g) + N(r, 1; F \geq k + 1)
\]
\[
\leq N(r, 0; f) + N(r, 0; g) + \frac{1}{k} N(r, 0; F = 1)
\]
\[
\leq N(r, 0; f) + N(r, 0; g) + \frac{1}{k} N(r, 0; F \neq 0) - \frac{1}{k} N_0(r, 0; F')
\]
\[
\leq \frac{k+1}{k} N(r, 0; f) + N(r, 0; g) + \frac{1}{k} N(r, \infty; f)
\]
\[
- \frac{1}{k} N(r, 0; f \neq 0, 1, \omega, \cdots, \omega^{n-1}) + S(r, f).
\]
Combining this with Lemma 7 and noting that $f, g$ share $(\infty, 0)$, the lemma is proved. This proves the lemma.

**Lemma 10 ([2]).** If $F, G$ share $(1, 2)$ then
\[
N_0(r, 0; G') + N(r, 1; G \geq 2) + N_*(r, 1; F, G)
\]
\[
\leq N(r, 0; G) + N(r, \infty; G) + S(r, G).
\]
3. Proofs of the theorems

Proof of Theorem 1. Let $F = f^n$, $G = g^n$ and $f$, $g$ do not satisfy (1). Since $E_f(S_1, 2) = E_g(S_1, 2)$ and $E_f(S_2, 0) = E_g(S_2, 0)$, it follows that $F$, $G$ share $(1, 2)$ and $(\infty, 0)$. If possible, we suppose that $H \not\equiv 0$. Then by the second fundamental theorem, Lemma 1, 4 and 10 and noting that $N_E^{1/3}(r; 1; F) = N(r; 1; F) = 1$ we obtain

\begin{align*}
3. \quad & T(r, F) \leq \frac{N(r, \infty; F)}{N(r, 0; F)} + \frac{N(r, 1; F)}{N(r, 0; F)} - N_{0}(r, 0; F') + S(r, F) \\
& \leq \frac{N(r, \infty; f)}{N(r, 0; f)} + \frac{N(r, 0; f)}{N(r, 0; f') + N(r, 0; G)} + \frac{N_{0}(r, 0; F) + N_{0}(r, 0; G') - N_{0}(r, 0; F') + S(r, F)}{2} \\
& \leq 2 \frac{N(r, \infty; f) + 2 N(r, 0; f) + N(r, 0; g) + N(r, 0; G)}{2} + S(r, F) + S(r, G) \\
& = 3 \frac{N(r, \infty; f) + 2 N(r, 0; f) + 2 N(r, 0; g) + S(r, f) + S(r, g)}{2}.
\end{align*}

Since $F \not\equiv G$ we get by Lemma 6 that $V \not\equiv 0$. So by Lemma 9 for $k = 2$ we get from (3)

\begin{align*}
4. \quad & n T(r, f) \leq \frac{6}{2n - 3} \left[ \frac{3}{2} N(r, 0; f) + N(r, 0; g) \right] + 2 N(r, 0; f) \\
& + 2 N(r, 0; g) + S(r, f) + S(r, g) \\
& \leq \frac{4n + 3}{2n - 3} T(r, f) + \frac{4n}{2n - 3} T(r, g) + S(r, f) + S(r, g).
\end{align*}

Similarly we obtain

\begin{align*}
5. \quad & n T(r, g) \leq \frac{4n}{2n - 3} T(r, f) + \frac{4n + 3}{2n - 3} T(r, g) + S(r, f) + S(r, g).
\end{align*}

Adding (4) and (5) we get

\[ \frac{2n^2 - 11n - 3}{2n - 3} \{ T(r, f) + T(r, g) \} \leq S(r, f) + S(r, g), \]

which is a contradiction for $n \geq 6$.

Hence $H \equiv 0$ and so by Lemma 5 and Remark 1 the theorem is proved. This proves the theorem. \(\square\)

Proof of Theorem 2. Let $F = f^n$, $G = g^n$ and $f$, $g$ do not satisfy (1). Since $E_f(S_1, 0) = E_g(S_1, 0)$ and $E_f(S_2, 3) = E_g(S_2, 3)$, it follows that $F$, $G$ share $(1, 0)$ and $(\infty, 3n)$. If possible, we suppose that $H \not\equiv 0$. Then by Lemmas 1, 2, 3 and 4
we get
\[ T(r, F) \leq N_E(1, r; F) + 2 N(r, 0; f) + 2 N(r, \infty; f) + N(r, 0; g) + N(r, \infty; g) \]
\[ -2 N(0, r; f) - N(0, r; G) + S(r, F) + S(r, G) \]
\[ \leq N(r, 0; f) + N(r, 0; g) + N(r, \infty; f) \geq 4 + N^*(r, 1; F, G) + 2 N(r, 0; f) \]
\[ + 2 N(r, \infty; f) + N(r, 0; g) + N(r, \infty; g) + S(r, f) + S(r, g) \]
\[ \leq 3 N(r, 0; f) + 2 N(r, 0; g) + 3 N(r, \infty; f) + N(r, \infty; f) \geq 4 \]
\[ + N(r, 1; F) \geq 2 + N(r, 1; G) \geq 2 + S(r, f) + S(r, g) \]
\[ \leq 3 N(r, 0; f) + 2 N(r, 0; g) + 3 N(r, \infty; f) + N(r, \infty; f) \geq 4 \]
\[ + N(r, 0; F) \neq 0 + N(r, 0; G) \neq 0 + S(r, f) + S(r, g) \]
\[ \leq 4 N(r, 0; f) + 3 N(r, 0; g) + 5 N(r, \infty; f) + N(r, \infty; f) \geq 4 \]
\[ + S(r, f) + S(r, g). \]

Since \( F \neq G \), by Lemma 6 we get \( V \neq 0 \). So by Lemma 8 for \( k = 3 \) we get from above
\[ (6) \ n T(r, f) \leq 4 N(r, 0; f) + 3 N(r, 0; g) + 5 N(r, \infty; f) \]
\[ + \frac{1}{4n - 1} \{ 2 N(r, 0; f) + 2 N(r, 0; g) + 2 N(r, \infty; f) \} \]
\[ + S(r, f) + S(r, g) \]
\[ \leq (4 + \frac{2}{4n - 1}) N(r, 0; f) + (3 + \frac{2}{4n - 1}) N(r, 0; g) \]
\[ + \frac{2}{n - 3} \{ 5 + \frac{2}{4n - 1} \} \{ N(r, 0; f) + N(r, 0; g) \} \]
\[ + S(r, f) + S(r, g) \]
\[ \leq \{ 4 + \frac{42n - 12}{(n - 3)(4n - 1)} \} T(r, f) + \{ 3 + \frac{42n - 12}{(n - 3)(4n - 1)} \} T(r, g) \]
\[ + S(r, f) + S(r, g). \]

Similarly we obtain
\[ (7) \ n T(r, g) \leq \{ 3 + \frac{42n - 12}{(n - 3)(4n - 1)} \} T(r, f) + \{ 4 + \frac{42n - 12}{(n - 3)(4n - 1)} \} T(r, g) \]
\[ + S(r, f) + S(r, g). \]

Adding (6) and (7) we get
\[ \{ n - 7 - \frac{84n - 24}{(n - 3)(4n - 1)} \} \{ T(r, f) + T(r, g) \} \leq S(r, f) + S(r, g), \]
which is a contradiction for \( n \geq 10 \).

Hence \( H \equiv 0 \) and so by Lemma 5 and Remark 1 the theorem follows. This proves the theorem. □
References