Commutative Semigroups whose Proper Homomorphic Images are All of Smaller Cardinality

RALPH P. TUCCI
Department of Mathematics and Computer Science, Loyola University New Orleans, New Orleans, LA 70118, U.S.A.
e-mail: tucci@loyno.edu

Abstract. We characterize those commutative semigroups $S$ such that every non-isomorphic homomorphic image of $S$ has smaller cardinality than $S$. We also characterize commutative groups with the same property.

In [3] Kaplansky posed the following problem for an infinite commutative group $G$: Show that every proper (not isomorphic) homomorphic image of $G$ is finite if and only if $G$ is an infinite cyclic group. In [2] Jensen and Miller characterized all infinite commutative semigroups $S$ such that every proper homomorphic image of $S$ is finite; they called such semigroups homomorphically finite or HF semigroups. In this note we characterize those infinite commutative semigroups $S$ such that every proper homomorphic image of $S$ is of smaller cardinality than $S$. We call such semigroups $H$-smaller. Surprisingly, the $H$-smaller semigroups are precisely those in Jensen and Miller’s Theorem. As part of the proof of this fact we also generalize the exercise in Kaplansky by showing that, if $G$ is an infinite commutative group, then every proper homomorphic image of $G$ is of smaller cardinality than $G$ if and only if $G$ is an infinite cyclic group.

For any semigroup $S$ let $S^0$, $S^1$, and $S^{0,1}$ denote $S$ with zero adjoined, $S$ with identity adjoined, and $S$ with both zero and identity adjoined, respectively. The group of integers is denoted $\mathbb{Z}$. The symbol $\mathbb{N}'$ stands for any subsemigroup of $(\mathbb{N},+)$, the semigroup of positive integers under addition. We now state Jensen and Miller’s theorem.

Theorem 1 [2, Theorem 3]. Let $S$ be an infinite commutative semigroup. Then every proper homomorphic image of $S$ is finite if and only if $S$ is either $\mathbb{Z}$, $\mathbb{Z}^0$, $\mathbb{N}'$, $(\mathbb{N}')^0$, $(\mathbb{N}')^1$, or $(\mathbb{N}')^{0,1}$.

We let $|X|$ denote the cardinality of $X$ for any set $X$. Throughout the rest of this note $S$ will denote an infinite commutative $H$-smaller semigroup. Our result follows easily from the following lemmas, which are taken almost without change from [2].

Received November 5, 2004.
2000 Mathematics Subject Classification: 20M14, 20M15.
Key words and phrases: semigroup, commutative, homomorphic image, cardinality.
Lemma 2. If $I$ is a nonzero ideal of $S$, then $|I| = |S|$.

Proof. If $I \neq 0$, then $|S/I| < |S|$. But $S = (S \setminus I) \cup I$ so that $|S| = |S \setminus I| + |I| = |S/I| + |I|$, which implies that $|S| = |I|$.

Lemma 3.

(a) If $S$ has no zero, then $S$ embeds in a group.

(b) If $S$ has a zero, then $S \setminus \{0\}$ embeds in a group.

Proof. It suffices to show that $S$ or $S \setminus \{0\}$ is cancellative. First we show that if $0 \in S$ then $S$ has no nonzero nilpotent elements. Let $s \in S$ be nilpotent of index $n$. Then the ideal $s^{n-1}S^1 = \{s^{n-1}t \mid t \in S^1\} sS^1$ satisfies $|s^{n-1}S^1| \leq |S^1/sS^1| < |S|$. By Lemma 2 it follows that $s = 0$.

Now we show that if $0 \in S$ then $S \setminus 0$ is closed under multiplication. Let $s, t \in S$ with $st = 0$, and assume that $s \neq 0$. Then $(sS \cap tS)^2 = 0$ so that $sS \cap tS = 0$. Hence $tS$ embeds in $S/sS$ so $|tS| < |S|$, and Lemma 2 implies that $tS = 0$. In particular, $t^2 = 0$, so by the previous paragraph $t = 0$.

Finally we show that $S$ or $S \setminus 0$ is cancellative. Let $0 \neq s \in S$ and define the congruence $\rho_s$ by the following: if $a, b \in S$ then $a\rho sb$ if and only if $as = bs$. By the previous paragraphs $sS \neq 0$. Then $|S| = |sS| = |S/\rho_s|$ so that $\rho_s$ is the identity congruence, and hence $a = b$.

Lemma 4. The group of quotients of $S$ or $S \setminus \{0\}$ is countable.

Proof. Let $G$ be the group of quotients of $S$ or $S \setminus \{0\}$. We first show that $G$ is $H$-smaller. Clearly, $|G| = |S|$. Let $\rho$ be a congruence on $G$ which is not 1-1. Suppose that $\frac{a}{b} \rho \frac{c}{d}$ for distinct $\frac{a}{b}, \frac{c}{d} \in G$. Then $((\frac{a}{b})bd)\rho((\frac{c}{d})bd)$; i.e., $ad$ be and ad $\neq bc$. Thus, $\rho$ is not 1-1 on $S$ so that $|S/\rho| < |S|$, and hence $|G/\rho| < |G|$.

It is now easy to see that $G$ is countable. Let $g \in G$ be any non-identity element and let $K = \langle g \rangle$ be the group generated by $g$. Then $|G| = |K||G/K|$ and $|G/K| < |G|$, so $|G| = |K|$ where $K$ is countable.

Corollary 5. Let $G$ be an infinite commutative group. Then $G$ is $H$-smaller if and only if $G \cong \mathbb{Z}$.

Proof. This follows from the proof of the previous lemma.

Theorem 6. Let $S$ be an infinite commutative semigroup. Then the following are equivalent:

(1) $S$ is $H$-smaller;

(2) $S$ is HF;

(3) $S$ is one of the following: $\mathbb{Z}$, $\mathbb{Z}^0$, $\mathbb{N}^0$, $(\mathbb{N}'^0, (\mathbb{N}')^1$, or $(\mathbb{N}')^{0,1}$.

Proof. (1) $\Rightarrow$ (2). By Lemma 4, $S$ is countable and $H$-smaller, and hence HF.

(2) $\Rightarrow$ (1). This follows by definition.

(2) $\Leftrightarrow$ (3). This is Theorem 1, Jensen and Miller’s Theorem.
Acknowledgement. The author would like to thank the referee for his comments, which substantially improved this paper.

References

